Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 31, 2006, 329–348

ON CONFORMAL REPRESENTATIONS OF THE INTERIOR OF AN ELLIPSE

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Abstract. We consider the conformal mappings f and g of the unit disk onto the inside of an ellipse with foci at ± 1 so that $f(0)=0$, $f'(0)>0$, $g(0)=-1$ and $g'(0)>0$. The main purpose of this article is to show positivity of the Taylor coefficients of f and g about the origin. To this end, we use a special relation between f and g and the fact that f satisfies a second-order linear ODE. Some applications are given to the class of k-uniformly convex functions.

1. Introduction

If a univalent function $f(z)=a_0+a_1z+a_2z^2+\cdots$ in the unit disk $\mathbf{D}=\{z\in\mathbf{C}:|z|<1\}$ has non-negative Taylor coefficients about the origin, various sharp estimates can be easily deduced. For example, one can show the sharp inequalities

$$(1.1)\quad |f(z)-a_0-a_1z-\cdots-a_kz^k|\le f(|z|)-a_0-a_1|z|-\cdots-a_k|z|^k,\quad |z|<1,$$

and

$$(1.2)\qquad |f^{(k)}(z)|\le f^{(k)}(|z|),\quad |z|<1,$$

for $k=0,1,2,\dots$.

As one immediately sees, necessary conditions for a univalent function f to have non-negative Taylor coefficients about the origin are that $f(0)\ge 0$, $f'(0)>0$ and that the image domain $\Omega=f(\mathbf{D})$ is symmetric in the real axis. Note that these conditions imply that the relation $\overline{f(\bar z)}=f(z)$ holds, and hence, all coefficients are real. It is also necessary that the farthest point of $\partial\Omega$ from $f(0)$ is the right-most point of $\overline{\Omega}\cap\mathbf{R}$. Conversely, under these conditions, however, it seems to be difficult to give a sufficient geometric condition for positivity of the Taylor coefficients. For instance, convexity of Ω is not sufficient. In fact, for constants

2000 Mathematics Subject Classification: Primary 30C20; Secondary 30C45, 33E05.

The second author was partially supported by the Ministry of Education, Grant-in-Aid for Encouragement of Young Scientists (B), 14740100.

$0 < c < 1$ and $N < \alpha < N+1$ with $c\alpha \le 1$ and N being a positive integer, the function

$$f(z) = (1+cz)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} (cz)^n$$

is univalent in $\mathbf{D}$ and has convex image because

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) = 1 + (\alpha-1)\operatorname{Re}\frac{cz}{1+cz} > 1 - (\alpha-1)\frac{c}{1-c} \ge 0.$$

Since

$$f^{(k)}(z) = c^k \alpha(\alpha-1)\cdots(\alpha-k+1)(1+cz)^{\alpha-k},$$

we observe that (1.2) is fulfilled by $k = 0, 1, \ldots, N$ but not by $k = N+1$. Note that one can deduce (1.1) for k from (1.2) for $k+1$ by repeated integrations.

In this paper, we show non-negativity of the Taylor coefficients of specific conformal mappings of the unit disk onto an ellipse. Let E_ξ be the ellipse given by

$$\left(\frac{u}{\cosh \xi}\right)^2 + \left(\frac{v}{\sinh \xi}\right)^2 = 1$$

and let D_ξ be the interior of E_ξ for $\xi > 0$. Note that E_ξ has foci at 1 and -1 and that an arbitrary ellipse is similar to E_ξ for some ξ. We prove the following two results.

Theorem 1.1. *Let f_ξ be the conformal mapping of the unit disk onto the interior D_ξ of the ellipse E_ξ determined by $f_\xi(0) = 0$ and $f'_\xi(0) > 0$. Then f_ξ has positive odd Taylor coefficients about the origin.*

Theorem 1.2. *Let g_ξ be the conformal mapping of the unit disk onto the interior D_ξ of the ellipse E_ξ determined by $g_\xi(0) = -1$ and $g'_\xi(0) > 0$. Then g_ξ has positive Taylor coefficients about the origin except for the first one.*

Since D_ξ is invariant under the rotation by angle π about the origin, f_ξ is an odd univalent function and is of the form

$$f_\xi(z) = a\big(z + A_1 z^3 + A_2 z^5 + \cdots\big), \quad a = f'_\xi(0) > 0.$$

In particular, there exists a univalent function φ with $\varphi(0) = 0$, $\varphi'(0) > 0$ such that $\big(f_\xi(z)\big)^2 = \varphi(z^2)$ (see [4]). Then

$$\varphi(z) = a^2 z \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} A_m A_{n-m}\right) z^n,$$

where we set $A_0 = 1$. Therefore, Theorem 1.1 yields that φ has positive Taylor coefficients about the origin except for the first one. We will show the relation

$$2\varphi = g_{2\xi} + 1$$

in Section 2 (see Theorem 2.1). In this way, Theorem 1.2 follows from Theorem 1.1.

An explicit form of f_ξ was first given by Schwarz as early as in 1869 and is well known nowadays. There is, however, less awareness of that f_ξ satisfies a second-order homogeneous linear ordinary differential equation (see Section 3). Using this ODE, we obtain linear recurrence relations between three successive coefficients A_{n-1}, A_n and A_{n+1}. It is still difficult to show positivity of the coefficients. The final stroke will be made by a theory of continued fractions, which will be presented in Section 4.

We apply Theorem 1.2 to the study of k-uniformly convex functions introduced by the first author and Wiśniowska [8]. Indeed, the present article grew out of a part of the first author's habilitation [6] which summarizes the study of that class. See Section 5 for details.

Acknowledgements. The authors are grateful to the referee for careful reading of the manuscript and helpful suggestions.

2. Conformal representation of the interior of an ellipse

We begin with introduction of special functions involving elliptic integrals. Let $\mathbf{K}(z,t)$ and $\mathbf{K}(t)$ be the normal and complete elliptic integrals of the first kind, respectively, i.e.,

$$\mathbf{K}(z,t)=\int_0^z\frac{dx}{\sqrt{(1-x^2)(1-t^2x^2)}}$$

and $\mathbf{K}(t)=\mathbf{K}(1,t)$ for $0<t<1$. Note that Jacobi's elliptic function $\operatorname{sn}(\,\cdot\,,t)$ is defined as the inverse function of $\mathbf{K}(\,\cdot\,,t)$ with $\operatorname{sn}(0,t)=0$, where our notation $\mathbf{K}(z,t)$ and $\operatorname{sn}(w,t)$ may not agree with the traditional one. It is well known that $\mathbf{K}(\,\cdot\,,t)$ maps the upper half plane conformally onto the rectangle with vertices at $\pm\mathbf{K}(t)$ and $\pm\mathbf{K}(t)+i\mathbf{K}(t')$, where $t'=\sqrt{1-t^2}$ (see, for instance, [11, Chapter VI, Section 3]). Since the interval $[-1,1]$ is mapped to the interval $[-\mathbf{K}(t),\mathbf{K}(t)]$, the function $\mathbf{K}(\,\cdot\,,t)$ can be continued analytically to the slit domain $\mathbf{C}\setminus\big((-\infty,-1]\cup[1,+\infty)\big)$ by the Schwarz reflection principle. In what follows, the function $\mathbf{K}(\,\cdot\,,t)$ will be understood in this way.

The quantity

$$\mu(t)=\frac{\pi}{2}\cdot\frac{\mathbf{K}(t')}{\mathbf{K}(t)},\quad t'=\sqrt{1-t^2}\,,$$

is known as the modulus of the Grötzsch ring $\mathbf{D}\setminus[0,t]$ for $0<t<1$. Note that $\mu(t)$ decreases from $+\infty$ to 0 when t moves from 0 to 1. For details, see [3].

We are now in a position to present an explicit expression of the function f_ξ. Choose a number $s\in(0,1)$ so that $\mu(s)=2\xi$. Then the formula

$$f_\xi(z)=\sin\left[\frac{\pi}{2\mathbf{K}(s)}\mathbf{K}\big(z/\sqrt{s}\,,s\big)\right] \tag{2.1}$$

can be deduced. Note that the inverse function is given by

$$z = \sqrt{s}\,\mathrm{sn}\big((2\mathbf{K}(s)/\pi)\arcsin w, s\big)$$

as is shown by [11, p. 296, (51)][1] (see also [13]).

Let us give an outline of the proof of (2.1) for the reader's convenience. Recall that the function $\mathbf{K}(z,s)$ maps the upper half-plane conformally onto the rectangle $\{u+iv : -\mathbf{K}(s) < u < \mathbf{K}(s),\ 0 < v < \mathbf{K}(s')\}$, where $s' = \sqrt{1-s^2}$. Since

$$\mathbf{K}\big(1/(s\bar{z}), s\big) = \overline{\mathbf{K}(z,s)} + i\mathbf{K}(s')$$

holds, the upper half of the disk $|z| < 1/\sqrt{s}$ is mapped conformally onto the rectangle $\{u+iv : -\mathbf{K}(s) < u < \mathbf{K}(s),\ 0 < v < \mathbf{K}(s')/2\}$. Therefore, the function $\big(\pi/\big(2\mathbf{K}(s)\big)\big)\mathbf{K}(z/\sqrt{s}, s)$ maps the upper half of the unit disk onto the rectangle $R = \{u+iv : -\pi/2 < u < \pi/2,\ 0 < v < \xi\}$. On the other hand, since

$$\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y, \tag{2.2}$$

the function $\sin z$ maps R onto the upper half of D_ξ. In this way, we see that the function $\sin\big(\big(\pi/\big(2\mathbf{K}(s)\big)\big)\mathbf{K}\big(z/\sqrt{s}, s\big)\big)$ maps the upper half of the unit disk onto the upper half of D_ξ. By the Schwarz reflection principle, we obtain the expression in (2.1).

Since $f_\xi\big(-\sqrt{s}\big) = -1$, the function g_ξ can be expressed by

$$g_\xi(z) = f_\xi\left(\frac{z-\sqrt{s}}{1-\sqrt{s}\,z}\right).$$

This formula is, however, not convenient to compute the Taylor coefficients of g_ξ about the origin. This is a motivation of deduction of the following formula.

Theorem 2.1. *For $\xi > 0$, the relation $g_{2\xi}(z) = 2\big(f_\xi\big(\sqrt{z}\big)\big)^2 - 1$ holds for $|z| < 1$.*

As was indicated, Theorem 1.2 immediately follows from Theorem 1.1 by means of this identity. See also Theorem 5.1 for a related result.

The representations f_ξ and g_ξ also give explicit values of the hyperbolic density of the domain D_ξ. Recall that the hyperbolic density ϱ_D of a simply connected domain D with $\#(\mathbf{C}\setminus D) \ge 2$ is defined by $\varrho_D(w_0) = 1/|f'(0)|$ for a conformal mapping f of the unit disk $\mathbf{D}$ onto D with $f(0) = w_0$.

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[1] We remark that there is a confusion on p. 296 of Nehari's book. Since $c = \pi K'/(2K)$ in his notation, the norm $q = e^{-K'/K}$ should be given by $e^{-2c/\pi}$ instead of e^{-2c}.

Corollary 2.2. *Let* $\xi = \mu(s)/2$ *for* $s \in (0,1)$. *Then*

$$\varrho_{D_\xi}(0) = \frac{2\sqrt{s}}{\pi}\,\mathbf{K}(s) \quad \textit{and} \quad \varrho_{D_{2\xi}}(-1) = \varrho_{D_{2\xi}}(1) = \frac{2s}{\pi^2}\big(\mathbf{K}(s)\big)^2.$$

Proof. Since $f_\xi'(0) = \pi/\big(2\sqrt{s}\,\mathbf{K}(s)\big)$ and $g_{2\xi}'(0) = 2\big(f_\xi'(0)\big)^2$ we obtain the required relations. □

In order to prove Theorem 2.1, we recall some facts about Chebyshev polynomials. We first consider the conformal mapping J of $\mathbf{D}$ onto $\widehat{\mathbf{C}} \setminus [-1,1]$ defined by $J(z) = (z + z^{-1})/2$. Since

$$J(e^{-\xi+i\eta}) = \cosh\xi\cos\eta - i\sinh\xi\sin\eta,$$

the circle $|z| = e^{-\xi}$ is mapped by J onto the ellipse E_ξ for $\xi > 0$ and the radial segment $(0, e^{i\eta})$ is mapped by J into the branch H_η of a hyperbola given by

$$\left(\frac{u}{\cos\eta}\right)^2 - \left(\frac{v}{\sin\eta}\right)^2 = 1, \quad u\cos\eta > 0, \tag{2.3}$$

for $\eta \in \mathbf{R}$ with $(2/\pi)\eta \notin \mathbf{Z}$. Note that these conic sections have the common foci at -1 and 1.

Let T_n be the Chebyshev polynomial of degree n, i.e., $T_n(\cos\theta) = \cos(n\theta)$. Then it is well known that the n-fold mapping $z \mapsto z^n$ is conjugate to T_n under J, in other words,

$$J(z^n) = T_n\big(J(z)\big)$$

holds for $|z| < 1$. In particular, one can see that the ellipse E_ξ is mapped by T_n onto $E_{n\xi}$ in an n-to-one fashion and that the branch H_η of a hyperbola is mapped by T_n bijectively to $H_{n\eta}$.

Applying the above argument to $T_2(w) = 2w^2 - 1$, we obtain the following. We recall that D_ξ is the interior of the ellipse E_ξ.

Lemma 2.3. *The Chebyshev polynomial* $T_2(w) = 2w^2 - 1$ *maps* D_ξ *onto* $D_{2\xi}$. *Also,* T_2 *maps the domain bounded by* H_η *and* $H_{\pi-\eta}$ *onto the connected component of* $\mathbf{C} \setminus H_{2\eta}$ *containing* -1. *Both are two-sheeted branched covering projections.*

On the basis of the above lemma, we can prove Theorem 2.1.

Proof of Theorem 2.1. By Lemma 2.3, the composed function $T_2 \circ f_\xi$ is a two-sheeted covering projection of $\mathbf{D}$ onto $D_{2\xi}$ which sends the origin to the focus -1 of $E_{2\xi}$. Since $T_2 \circ f_\xi$ is even, the function $g(z) = (T_2 \circ f_\xi)\big(\sqrt{z}\big)$ is single-valued and analytic in $\mathbf{D}$. By construction, g is conformal and satisfies $g'(0) > 0$, and therefore, $g = g_{2\xi}$. Thus the theorem has been proved. □

The same reasoning yields a relation between conformal mappings onto domains bounded by hyperbolas. Let F_η be the conformal mapping of $\mathbf{D}$ onto the domain bounded by H_η and $H_{\pi-\eta}$ which are given by (2.3) with $F_\eta(0)=0$ and $F_\eta'(0)>0$ for $0<\eta<\pi/2$. We also let G_η be the conformal mapping of $\mathbf{D}$ onto the left component of $\mathbf{C}\setminus H_\eta$ with $G_\eta(0)=-1$, $G_\eta'(0)>0$ for $0<\eta<\pi$, $\eta\neq\pi/2$. We define $G_{\pi/2}$ as the limit of G_η as $\eta\to\pi/2$, that is, $G_{\pi/2}(z)=(z-1)/(z+1)$. Then we obtain the following.

Proposition 2.4. *Let* $\eta\in(0,\pi/2)$. *Then* $G_{2\eta}(z)=2\big(F_\eta(\sqrt{z}\,)\big)^2-1$ *for* $|z|<1$.

In view of (2.2) we see that the function $\sin z$ maps the parallel strip $|\operatorname{Re} z|<a$ conformally onto the domain $\{u+iv:(u/\sin a)^2-(v/\cos a)^2<1\}$ for $0<a<\pi/2$. Noting that the function $\arctan z$ maps the unit disk onto the strip $|\operatorname{Re} w|<\pi/2$, we have the expression

$$F_\eta(z)=\sin\left(\left(2-\frac{4\eta}{\pi}\right)\arctan z\right)\quad\text{for } 0<\eta<\pi/2. \tag{2.4}$$

3. A linear ODE satisfied by the conformal representation f_ξ

It is a noteworthy fact that the conformal representations f_ξ and F_η, which are given in (2.1) and (2.4), respectively, satisfy simple second-order linear ordinary differential equations (ODE).

Indeed, if we write $b=2-4\eta/\pi$, we have $F_\eta'(z)=b(1+z^2)^{-1}\cos(b\arctan z)$. Differentiating both sides of $(1+z^2)F_\eta'(z)=b\cos(b\arctan z)$, we see that $w=F_\eta$ satisfies the differential equation

$$(1+z^2)^2w''+2z(1+z^2)w'+b^2w=0.$$

Similarly, one can check that the function $w=f_\xi(z)$ satisfies the differential equation

$$(1-2Mz^2+z^4)w''-2z(M-z^2)w'+cw=0 \tag{3.1}$$

in $\mathbf{D}$, where $M=(s+s^{-1})/2$, $c=\pi^2/\big(4s\big(\mathbf{K}(s)\big)^2\big)$ and $s\in(0,1)$ is chosen so that $\mu(s)=2\xi$.

Let $w=f(z)$ be the solution to the differential equation (3.1) with the initial conditions $f(0)=0$ and $f'(0)=1$. Note that f_ξ can be written in the form $f_\xi=f_\xi'(0)f$, and hence, positivity of the Taylor coefficients of f_ξ is equivalent to that of f.

As was seen in the introduction, $f(z)$ has the Taylor expansion of the form

$$f(z)=\sum_{n=0}^{\infty}A_nz^{2n+1},$$

with $A_0 = 1$. Substituting the above expansion to the equation (3.1), we obtain the following recurrence relations for the coefficients A_n:

$$(3.2)\qquad (2n+2)(2n+3)A_{n+1} - \{2M(2n+1)^2 - c\}A_n + 2n(2n-1)A_{n-1} = 0$$

for $n \geq 0$; here we have set $A_{-1} = 0$.

Since the image $f(\mathbf{D})$ is bounded by an ellipse and, in particular, convex, $|A_n| \leq 1$ holds for every $n \geq 1$ (see [4, p. 45]). As was explained in the introduction, all the coefficients A_n are real. However, we have no *a priori* information about the sign of A_n.

We take a closer look at the recurrence formula (3.2). We now transform the sequence by

$$B_n = (2n+1)A_n.$$

Then $B_{-1} = 0$, $B_0 = 1$ and the relation (3.2) turns to

$$(3.3)\qquad (n+1)B_{n+1} - \left\{M(2n+1) - \frac{c}{2(2n+1)}\right\}B_n + nB_{n-1} = 0$$

for $n \geq 0$. We further set

$$E_n = \frac{B_n}{B_{n-1}}$$

for $n \geq 0$. Here, we adopt the convention $E_n = \infty$ when B_{n-1} happens to be zero. Thus, for instance, $E_0 = \infty$. By dividing both sides of (3.3) by $(n+1)B_n$, we obtain

$$E_{n+1} = \frac{M(2n+1)}{n+1} - \frac{c}{(2n+1)(2n+2)} - \frac{n}{n+1} \cdot \frac{1}{E_n}.$$

By letting

$$(3.4)\qquad p_n = \frac{n}{n+1}, \quad q_n = \frac{M(2n-1)}{n} - \frac{c}{2n(2n-1)},$$

the last relation can be rewritten in the form

$$(3.5)\qquad E_{n+1} = q_{n+1} - \frac{p_n}{E_n}, \quad n = 0, 1, 2, \ldots.$$

We recall that the constants M and c are given by

$$M = \frac{1}{2}\left(s + \frac{1}{s}\right) \quad \text{and} \quad c = \frac{\pi^2}{4s\big(\mathbf{K}(s)\big)^2}.$$

We remark that $c = \big(f_\xi'(0)\big)^2 = 1/\big(\varrho_{D_\xi}(0)\big)^2$ (cf. Corollary 2.2). For a later use, we give estimates of c.

Lemma 3.1. *The quantity $c = \pi^2/\big(4s(\mathbf{K}(s))^2\big)$ satisfies the double inequality*

$$\frac{1}{s} - s \le c \le \frac{1}{s} - \frac{s}{2}.$$

Proof. It is easily checked that the required inequality is equivalent to

$$\frac{\pi}{2\sqrt{1-s^2/2}} \le \mathbf{K}(s) \le \frac{\pi}{2\sqrt{1-s^2}} \tag{3.6}$$

for $0 < s < 1$. First, using the inequality $\sqrt{1-s^2x^2} \ge \sqrt{1-s^2}$ for $0 < x < 1$, we obtain

$$\mathbf{K}(s) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-s^2x^2)}} \le \frac{1}{\sqrt{1-s^2}} \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2\sqrt{1-s^2}}\,.$$

To show the other part, we need another technique. We first express $\mathbf{K}(s)$ in the form

$$\mathbf{K}(s) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-s^2\sin^2\theta}} = \int_0^{\pi/4} \left(\frac{1}{\sqrt{1-s^2\sin^2\theta}} + \frac{1}{\sqrt{1-s^2\cos^2\theta}} \right) d\theta.$$

Note here the inequality

$$(1-s^2\sin^2\theta)(1-s^2\cos^2\theta) = 1 - s^2 + \frac{s^4}{4}\sin^2(2\theta) \le \left(1 - \frac{s^2}{2}\right)^2.$$

We now use the inequality $1/X + 1/Y \ge 2/\sqrt{XY}$ for $X, Y > 0$ to deduce

$$\mathbf{K}(s) \ge \int_0^{\pi/4} \frac{2}{\sqrt[4]{(1-s^2\sin^2\theta)(1-s^2\cos^2\theta)}}\, d\theta \ge \frac{\pi}{2\sqrt{1-s^2/2}}\,. \qquad \square$$

Remark. Matti Vuorinen told us that the inequalities

$$\frac{2}{1+\sqrt{1-s^2}} \le \frac{2}{\pi}\mathbf{K}(s) \le \frac{1}{\sqrt[4]{1-s^2}}\,,$$

which are better than (3.6), are known (see [3, 4.6(3)]). See also [2, Section 7.4] for a different kind of inequalities and related references. The authors thank him for the above information.

Let us explain the difficulty of the recurrence relations (3.5). First note that $p_n \to 1$ and $q_n \to s + s^{-1}$ as $n \to \infty$. Thus, one can think that the dynamical system $E_{n+1} = q_{n+1} - p_{n+1}/E_n$ converges to the autonomous system $E^\circ_{n+1} = s + s^{-1} - 1/E^\circ_n$ as $n \to \infty$. As is easily observed, the linear fractional transformation

$f(x) = s + s^{-1} - x^{-1}$ has the attracting fixed point $x = s^{-1}$ and the repelling fixed point $x = s$. On the other hand, in reality, a numerical computation suggests that $E_n \to s$ as $n \to \infty$. Therefore, usual methods of approximation and even induction arguments seem to fail to show positivity of E_n. Therefore, we have to take a different approach.

By (3.5), we can express E_n in terms of a continued fraction:

$$E_n = q_n - \cfrac{p_{n-1}}{q_{n-1} - \cfrac{p_{n-2}}{\ddots - \cfrac{p_1}{q_1}}} = q_n - \frac{p_{n-1}}{q_{n-1}} - \frac{p_{n-2}}{q_{n-2}} - \cdots - \frac{p_1}{q_1}.$$

We define the double sequence $q_{m,n}$ for $1 \le m \le n$ by induction of $n - m$. Fix a positive integer n. First we set

$$q_{n,n} = q_n.$$

Suppose that $q_{n,n}, q_{n-1,n}, \ldots, q_{m+1,n}$ have already been defined for $1 \le m < n$. Then, we set

$$q_{m,n} = q_m - \frac{p_m}{q_{m+1,n}}. \tag{3.7}$$

In this way, we can define $q_{n,n}, \ldots, q_{1,n}$. Then we can restate positivity of E_n in terms of $q_{m,n}$.

Lemma 3.2. *Let n be a positive integer. Then $E_m > 0$ holds for each m with $1 \le m \le n$ if and only if $q_{m,n} > 0$ for each m with $1 \le m \le n$.*

Proof. We first assume that $q_{m,n} > 0$ for all $1 \le m \le n$. Then $q_{1,n} > 0$ implies

$$E_1 = q_1 > \frac{p_1}{q_{2,n}}.$$

In particular, $E_1 > 0$ (though this is implied by Lemma 3.1). Since $q_{2,n} > 0$ by assumption, we obtain

$$q_{2,n} = q_2 - \frac{p_2}{q_{3,n}} > \frac{p_1}{E_1},$$

which is equivalent to

$$E_2 = q_2 - \frac{p_1}{E_1} > \frac{p_2}{q_{3,n}}.$$

In particular, we observe $E_2 > 0$. We now use $q_{3,n} > 0$ to see

$$q_{3,n} = q_3 - \frac{p_3}{q_{4,n}} > \frac{p_2}{E_2}.$$

We repeat this procedure to get finally

$$q_{n,n} = q_n > \frac{p_{n-1}}{E_{n-1}},$$

which yields

$$E_n = q_n - \frac{p_{n-1}}{E_{n-1}} > 0.$$

The converse can be seen by tracing back the above. □

At this stage, we collect some elementary properties of $q_{m,n}$. When we regard $q_{m,n}$ as a function of s in $(0,1)$, we sometimes write $q_{m,n}(s)$ to indicate the argument s. We also write $q_{m,n}(1) = \lim_{s\to 1} q_{m,n}(s)$ if the limit exists. In particular, we have $q_m(1) = q_{m,m}(1) = 2 - 1/m$ because $c = c(s) \to 0$ when $s \to 1$. We first prepare the following lemma.

Lemma 3.3. *Let m be a positive integer. The function $q_m(s)$ is positive in $0 < s < 1$ and the inequality $q_m(s) \geq q_m(1)$ holds for $0 < s \leq (2m-2)/(2m-1)$.*

Remark. If the inequality $q_m(s) \geq q_m(1)$ held for all $0 < s < 1$, the proof of positivity of E_n would be simpler. Unfortunately, this is not the case.

Proof. By Lemma 3.1, we have $c < s^{-1} - s/2$. Thus, we obtain

$$q_m \geq q_1 = \frac{s + s^{-1}}{2} - \frac{c}{2} > \frac{3s}{4} > 0.$$

It is easily verified that the condition $q_m(s) \geq q_m(1) = 2 - 1/m$ is equivalent to $(2/\pi)(1-s)\mathbf{K}(s) \geq 1/(2m-1)$. Since $\mathbf{K}(s) \geq \pi/2$, the condition $s \leq 1 - 1/(2m-1)$ is enough to ensure the inequality $q_m(s) \geq q_m(1)$. □

The following result is readily shown by (reverse) induction on m.

Lemma 3.4. *Let $a_1, \ldots, a_n; x_1, \ldots, x_n$ and $x'_1, \ldots, x'_n$ be positive numbers with $x_m \leq x'_m$ for $m = 1, \ldots, n$. Define y_m by reverse induction: $y_n = x_n$ and $y_m = x_m - a_{m+1}/y_{m+1}$ for $m < n$. Similarly, set $y'_n = x'_n$ and $y'_m = x'_m - a_{m+1}/y'_{m+1}$ for $m < n$. Further suppose that $y_m > 0$ for all $1 \leq m \leq n$. Then $y_m \leq y'_m$ and, in particular, $y'_m > 0$ for all m.*

With the aid of the above lemma, we can now show the following.

Lemma 3.5. *Let m and n be integers with $2 \leq m \leq n$. Then the quantity $q_{m,n} = q_{m,n}(s)$ is positive for $0 < s \leq (2m-2)/(2m-1)$.*

Proof. We first show that $q_{m,n}(1) > 0$ for $1 \leq m \leq n$. In view of Lemma 3.2, it is enough to see that $E_n(1)$ is positive for each $n \geq 1$. The solution $w = f(z)$ with $f(0) = 0$, $f'(0) = 1$ to the equation (3.1) corresponding to the case when $s = 1$ is nothing but the function $\operatorname{arctanh} z$. Clearly, this has positive odd Taylor coefficients about the origin, and therefore, the inequality $E_n(1) > 0$ follows. Lemmas 3.3 and 3.4 now yield the inequality $q_{m,n}(s) \geq q_{m,n}(1) > 0$ for $0 < s \leq (2m-2)/(2m-1)$ and for $2 \leq m \leq n$. □

The next simple fact will be a key to the proof of Theorem 1.1.

Lemma 3.6. *Let n and n_0 be integers with $2 \le n_0 \le n$. Suppose that $q_{m,n} > 0$ holds for every m with $n_0 \le m \le n$. Then $q_{m,n-1} > q_{m,n}$ holds for every m with $n_0 - 1 \le m \le n-1$.*

Proof. We shall show $q_{m,n-1} > q_{m,n}$ by reverse induction on m. For $m = n-1$, the inequality holds because $q_{n-1,n} = q_{n-1,n-1} - p_{n-1}/q_n$. We now assume that $q_{m,n-1} > q_{m,n}$ holds for some m with $n_0 \le m \le n-1$. Note now that $q_{m,n} > 0$ by assumption. Since

$$q_{m-1,n-1} - q_{m-1,n} = -\frac{p_{m-1}}{q_{m,n-1}} + \frac{p_{m-1}}{q_{m,n}} = \frac{p_{m-1}(q_{m,n-1} - q_{m,n})}{q_{m,n-1}q_{m,n}},$$

we obtain $q_{m-1,n-1} > q_{m-1,n}$. This procedure can be continued up to $m-1 = n_0 - 1$. □

At this stage, we can show that $\lim_{n\to\infty} q_{m,n}$ always exists.

Proposition 3.7. *For each $m \ge 1$, the sequence $q_{m,n}$ has a limit in $\widehat{\mathbf{C}}$ when $n \to \infty$.*

Proof. Since $q_{m,n}$ and $q_{m+1,n}$ are related by a Möbius transformation described in (3.7), if $q_{m,n}$ has a limit for *some* m then $q_{m,n}$ does for *all* m. By Lemma 3.5, for a fixed s, there exists an integer N such that $q_{m,n} > 0$ whenever $N \le m \le n$. Lemma 3.6 now implies that $q_{m-1,n}$ is monotone decreasing with respect to n. In particular, $q_{m-1,n}$ has a limit as $n \to \infty$ for $m \ge N-1$. □

We denote by $q_{m,\infty}$ the limit of $q_{m,n}$ as $n \to \infty$. In order to find a value of $q_{m,\infty}$, we employ the general theory of continued fractions, which will be explained in the next section.

4. A continued fraction approach

In order to apply the general theory of continued fractions to our problem, we recall some notions and results in the theory based on the work of L. Jacobsen (Lorentzen) and W. J. Thron [5].

Let $\{T_n\}$ be a sequence of Möbius maps. The sequence is said to be *restrained* if there exist sequences $\{u_n\}$ and $\{v_n\}$ of points in the Riemann sphere $\widehat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ such that

$$\liminf_{n\to\infty} d(u_n, v_n) > 0$$

and that

$$\lim_{n\to\infty} d\big(T_n(u_n), T_n(v_n)\big) = 0,$$

where $d(z,w)$ denotes the chordal distance between z and w, namely, $d(z,w) = |z-w|/\sqrt{(1+|z|^2)(1+|w|^2)}$. Note that the asymptotic behaviour of $\{T_n(u_n)\}$

is unique in the sense that $d\big(T_n(u_n), T_n(u_n')\big) \to 0$ for any other pair of sequences $\{u_n'\}$ and $\{v_n'\}$ satisfying $\liminf d(u_n', v_n') > 0$ and $\lim d\big(T_n(u_n'), T_n(v_n')\big) = 0$ (see [5, Theorem 2.1]). Sometimes we say that $\{T_n\}$ is restrained with $\{u_n\}$ when we want to indicate the associated sequence. A sequence $\{w_n\}$ is said to be *exceptional* with respect to the restrained sequence $\{T_n\}$ with $\{u_n\}$ if

$$\limsup_{n\to\infty} d\big(T_n(w_n), T_n(u_n)\big) > 0.$$

Among several interesting results in [5], the following will be made use of in the present paper.

Lemma 4.1 (Proposition 2.4 in [5]). *Let $\{T_n\}$ be a sequence of Möbius maps which is restrained with $\{u_n\}$. Suppose that $\liminf_{n\to\infty} d\big(T_n(u_n), \infty\big) > 0$. Then for an exceptional sequence $\{w_n\}$*

$$\liminf_{n\to\infty} d\big(w_n, T_n^{-1}(\infty)\big) = 0.$$

We now return to our problem. Let

$$R_m(z) = q_m - \frac{p_m}{z}, \quad S_m = R_1 \circ \cdots \circ R_m, \quad \text{and} \quad T_m = S_m^{-1}$$

for $m \ge 1$, where p_m and q_m are given by (3.4). Then, by definition,

$$q_{m,n} = (R_m \circ \cdots \circ R_n)(\infty).$$

In particular,

$$q_{1,n} = S_n(\infty) = S_{n+1}(0). \tag{4.1}$$

Also, by noting the relation $R_m^{-1}(w) = p_m/(q_m - w)$, we observe

$$T_n(0) = \frac{p_n}{E_n}. \tag{4.2}$$

We now claim that our $\{T_n\}$ is restrained. More concretely, we show the following.

Lemma 4.2. *For a fixed $s \in (0,1)$, there exists a non-empty open interval $I = I(s)$ in $\mathbf{R}$ such that $T_n(x) \to s$ for every $x \in I$.*

Proof. Since $q_n \to s + s^{-1}$ (> 2) as $n \to \infty$, one can take an integer N so that $q_n > 2$ for all $n \ge N$. Let $\alpha_n = 1/\sqrt{p_n} = \sqrt{(n+1)/n}$ and choose $t_n \in (0,1)$ for $n \ge N$ so that $\alpha_n q_n = t_n + 1/t_n$, and thus,

$$\frac{1}{t_n} = \frac{\alpha_n q_n + \sqrt{(\alpha_n q_n)^2 - 4}}{2}. \tag{4.3}$$

We now investigate the asymptotic behaviour of t_n as $n \to \infty$. Since

$$\alpha_n = 1 + \frac{1}{2n} - \frac{1}{8n^2} + \frac{1}{16n^3} + O(n^{-4})$$

as $n \to \infty$, we have

$$\alpha_n q_n = s + \frac{1}{s} - \frac{2c + 3(s + s^{-1})}{8} \cdot \frac{1}{n^2} + \frac{s + s^{-1} - 2c}{8} \cdot \frac{1}{n^3} + O(n^{-4})$$

and

$$\alpha_{n+1} q_{n+1} - \alpha_n q_n = \frac{2c + 3(s + s^{-1})}{4} \cdot \frac{1}{n^3} + O(n^{-4}) \tag{4.4}$$

as $n \to \infty$. In particular, $\alpha_{n+1} q_{n+1} > \alpha_n q_n$ and hence $t_{n+1} < t_n$ for sufficiently large n. It is obvious that $t_n \to s$ as $n \to \infty$. Also, by (4.3) and (4.4), we obtain $1/t_{n+1} - 1/t_n = O(n^{-3})$. If we set $s_n = t_n/\alpha_n$, then

$$\frac{1}{s_{n-1}} - \frac{1}{s_n} = \frac{\alpha_{n-1} - \alpha_n}{t_{n-1}} + \alpha_n \left(\frac{1}{t_{n-1}} - \frac{1}{t_n} \right) = \frac{1}{2sn^2} + o(n^{-2})$$

as $n \to \infty$. In particular, $s_{n-1} < s_n$ for sufficiently large n. We replace N by a larger number if necessary so that $t_{n+1} < t_n$ and $s_{n-1} < s_n$ hold for all $n \geq N$. In particular, we have $t_n \leq \tau$ for $n \geq N$, where we set $\tau = t_N (< 1)$. Since $1/\alpha_n - s_n \to 1 - s (> 0)$ as $n \to \infty$, we may further assume that the inequality

$$\tau \left(\frac{1}{\alpha_n} - s_n \right) < \frac{1}{\alpha_{n+1}} - s_{n+1} \tag{4.5}$$

holds for each $n \geq N$. In the rest of the proof, we consider only integers n with $n \geq N$.

It is easy to check that s_n is a fixed point of the map $U_n = R_n^{-1}$. Furthermore,

$$U_n(x) - U_n(s_n) = \frac{p_n(x - s_n)}{(q_n - x)(q_n - s_n)} = \frac{s_n}{q_n - x}(x - s_n).$$

When $s_n < x < 1/\alpha_n$, one has $0 < s_n/(q_n - x) = t_n/(\alpha_n q_n - \alpha_n x) \leq \tau/(2-1) = \tau$, and therefore

$$0 < U_n(x) - U_n(s_n) \leq \tau(x - s_n). \tag{4.6}$$

For $x_N \in \mathbf{R}$, we set $x_n = (U_n \circ \cdots \circ U_{N+1})(x_N)$ for $n > N$. Then, by (4.6) and (4.5), if $s_n < x_n < 1/\alpha_n$ we have

$$\begin{aligned} 0 < x_{n+1} - s_{n+1} &= U_{n+1}(x_n) - U_{n+1}(s_{n+1}) \leq \tau(x_n - s_{n+1}) < \tau(x_n - s_n) \\ &< \tau \left(\frac{1}{\alpha_n} - s_n \right) < \frac{1}{\alpha_{n+1}} - s_{n+1}, \end{aligned}$$

and therefore, $s_{n+1} < x_{n+1} < 1/\alpha_{n+1}$. By induction, we have $s_n < x_n < 1/\alpha_n$ for all $n \geq N$ if we take x_N from $(s_N, 1/\alpha_N)$. Therefore, the interval $I = S_N\big((s_N, 1/\alpha_N)\big)$ works for the assertion. □

If we choose a pair of distinct points x_0 and x_0' from the interval I in the last lemma, then $d\big(T_n(x_0), T_n(x_0')\big) \to d(s,s) = 0$ as $n \to \infty$. In particular, $\{T_n\}$ is a restrained sequence with constant sequence $\{x_0\}$. With the aid of this fact, we can now show the following.

Lemma 4.3. *We have* $q_{1,\infty} = 0$.

Proof. Fix a number $x_0 \in I$. Then $T_n(x_0) \to s$ as $n \to \infty$. In particular, $d\big(T_n(x_0), \infty\big) \to d(s, \infty) > 0$. Next we observe the relation $T_n^{-1}(\infty) = S_n(\infty) = q_{1,n}$ by (4.1). Suppose now that the constant sequence $\{0\}$ were not exceptional with respect to $\{T_n\}$. That would mean $T_n(0) \to s$ when $n \to \infty$. Now we recall the relation (4.2). Since $E_n = B_n/B_{n-1}$ and $B_n = (2n+1)A_n$, we would have

$$\lim_{n\to\infty} \frac{A_n}{A_{n-1}} = \lim_{n\to\infty} \frac{B_n}{p_n B_{n-1}} = \lim_{n\to\infty} \frac{1}{T_n(0)} = \frac{1}{s} > 1,$$

which would violate the boundedness of the sequence $\{A_n\}$. Thus we have concluded that the constant sequence $\{0\}$ is exceptional with respect to $\{T_n\}$. Lemma 4.1 now yields

$$\liminf_{n\to\infty} d(0, q_{1,n}) = \liminf_{n\to\infty} d\big(0, T_n^{-1}(\infty)\big) = 0,$$

which implies that 0 is a limit point of the convergent sequence $\{q_{1,n}\}$ (see Proposition 3.7). Thus $q_{1,\infty} = \lim_{n\to\infty} q_{1,n}$ must be 0. □

Proof of Theorem 1.1. We now show the inequality $q_{m,n} > 0$ for all $m \le n$. This implies $E_n > 0$ for all $n \ge 1$ by Lemma 3.2, and thus Theorem 1.1 follows.

For each $s \in (0,1)$, by Lemma 3.5, we see that there exists an integer $N \ge 1$ such that $q_{m,n} > 0$ for any pair of integers m, n with $N \le m \le n$. We denote by $N(s)$ the minimum of such numbers N for $s \in (0,1)$. Lemma 3.5 implies also that $N(s) \le m$ for $s \in \big(0, (2m-2)/(2m-1)\big]$. In particular, $N(s) \le 2$ for $s \in (0, 2/3]$. If $N(s) \le 2$, then Lemma 3.6 gives us the information $q_{1,n} > q_{1,n+1}$ for $n \ge 1$. Since $q_{1,n} \to 0$ by Lemma 4.3, we now conclude that $q_{1,n} > 0$ for $n \ge 1$. Thus $N(s)$ must be 1 in this case. In particular, $N(s) = 1$ for $s \in (0, 2/3]$.

Suppose that $N(s) > 2$ for some s. Let s_0 be the infimum of the set $\{s \in (0,1) : N(s) > 2\}$. As we observed above, $N(s) = 1$ for $s < s_0$, namely, $q_{m,n}(s) > 0$ for all $1 \le m \le n$, and thus $0 < E_n(s) < +\infty$, $n \ge 1$, for $s < s_0$.

Since $N(s)$ is locally bounded in $0 < s < 1$, there exist an integer $N > 2$ and a decreasing sequence $\{s_k\}$ such that $N(s_k) = N$ and $s_k \to s_0$ as $k \to \infty$. Since $q_{N,n}(s_k) > 0$ for $n \ge N$, Lemma 3.6 yields $q_{N-1,n-1}(s_k) > q_{N-1,n}(s_k)$ for $n \ge N$. In particular, we obtain $q_{N-1,n-1}(s_k) > q_{N-1,\infty}(s_k)$ for $n \ge N$. By minimality of $N(s_k)$, we see that $-\infty \le q_{N-1,\infty}(s_k) < 0$. On the other hand, $q_{N-1,\infty}(s) \ge 0$ for $s < s_0$. By continuity, we obtain $q_{N-1,\infty}(s_0) = \infty$ or 0. Since

$$S_{N-2}(q_{N-1,\infty}) = S_{N-2}\Big(\lim_{n\to\infty} q_{N-1,n}\Big) = \lim_{n\to\infty} S_{N-2}(q_{N-1,n}) = \lim_{n\to\infty} q_{1,n} = 0,$$

we see that $q_{N-1,\infty} = T_{N-2}(0) = p_{N-2}/E_{N-2}$. So $E_{N-2}(s_0) = 0$ or $E_{N-2}(s_0) = \infty$.

Suppose first that $E_{N-2}(s_0) = 0$. Now $E_{N-2}(s)$ approaches 0 from the right as $s \to s_0^-$. On the other hand, $E_{N-1}(s) > 0$ implies

$$q_{N-1}(s) > \frac{p_{N-1}}{E_{N-2}(s)}$$

for $s < s_0$. Therefore, as $s \to s_0^-$ the right-hand side goes to ∞, which forces $q_{N-1}(s_0)$ to be infinity. Thus we have reached a contradiction.

We next suppose that $E_{N-2}(s_0) = \infty$. In this case, $E_{N-3}(s_1) = 0$ by (3.5). Then the same argument as above leads to a contradiction.

At any event, we get a contradiction. Thus the possibility that $N(s) > 2$ for some $0 < s < 1$ has been ruled out. Therefore $N(s) = 1$, namely, $q_{m,n}(s) > 0$ for $1 \le m \le n$. □

Numerical experiments suggest the following conjectures, which seem to be difficult to prove by simple induction arguments. Recall that $E_n = E_n(s)$ is defined as $(2n+1)A_n/(2n-1)A_{n-1}$ in Section 3.

Conjecture 4.4.

(i) *$E_n(s)$ is increasing in $0 < s < 1$ for each $n \ge 1$.*
(ii) *$E_n(s)$ monotonically increases to s as $n \to \infty$.*
(iii) *$E_n(s)/s$ increases from $(2n+1)/(2n+2)$ to 1 when s moves from 0 to 1.*
(iv) *Let $a \in D_\xi \cap (-\infty, 0)$. The conformal map f of the unit disk $\mathbf{D}$ onto D_ξ determined by $f(0) = a$ and $f'(0) > 0$ has positive Taylor coefficients except for the first one.*

Repeated use of (iii) would yield the inequality

$$\frac{(2n-1)!!}{(2n)!!} \cdot \frac{s^n}{n+1} \le A_n(s) \le \frac{s^n}{2n+1}$$

for each $n \ge 1$ and for each $s \in (0,1)$. Note that Stirling's formula implies

$$\frac{(2n-1)!!}{(2n)!!} = \frac{(2n)!}{2^{2n}(n!)^2} \sim \frac{1}{\sqrt{\pi n}} \quad \text{as } n \to \infty.$$

5. Applications to k-uniformly convex functions

We consider the domain

$$\Omega_k = \{u + iv \in \mathbf{C} : u^2 > k^2(u-1)^2 + k^2v^2,\ u > 0\} \tag{5.1}$$

for $k \in [0, \infty)$. Note that $1 \in \Omega_k$ for all k. Ω_0 is nothing but the right half-plane. When $0 < k < 1$, the domain Ω_k is the unbounded domain enclosed by the right half of the hyperbola

$$\left(\frac{(1-k^2)u + k^2}{k}\right)^2 - \left(\frac{(1-k^2)v}{\sqrt{1-k^2}}\right)^2 = 1$$

with foci at 1 and $-(1+k^2)/(1-k^2)$. When $k=1$, the domain Ω_1 becomes the unbounded domain enclosed by the parabola

$$v^2 = 2u - 1$$

with focus at 1. When $k > 1$, the domain Ω_k is the interior of the ellipse

$$\left(\frac{(k^2-1)u - k^2}{k}\right)^2 + \left(\frac{(k^2-1)v}{\sqrt{k^2-1}}\right)^2 = 1$$

with foci at 1 and $(k^2+1)/(k^2-1)$. For every k, the domain Ω_k is convex and symmetric in the real axis. Note also that $\Omega_{k_1} \supset \Omega_{k_2}$ if $0 \le k_1 \le k_2$ and that Ω_k converges to Ω_{k_0} in the sense of Carathéodory when $k \to k_0$.

An analytic function f in the unit disk $\mathbf{D}$ normalized by $f(0) = 0$ and $f'(0) = 1$ is called k-uniformly convex if $1 + zf''(z)/f'(z) \in \Omega_k$ for $z \in \mathbf{D}$. This concept was introduced and studied by the first author and Wiśniowska [8], [7]. Clearly, 0-uniformly convex functions are exactly same as convex functions. Moreover, uniformly convex functions introduced by Goodman are characterized as 1-uniformly convex functions (see [10] and [12]).

Let P_k be the conformal mapping of $\mathbf{D}$ onto Ω_k determined by the conditions $P_k(0) = 1$ and $P_k'(0) > 0$. They gave a concrete expression for P_k.

Theorem A (Kanas–Wiśniowska [8]). *The conformal map $P_k\colon \mathbf{D} \to \Omega_k$ with $P_k(0) = 1$ and $P_k'(0) > 0$ is given by*

$$P_k(z) = \begin{cases} (1+z)/(1-z) & \text{if } k = 0, \\ (1-k^2)^{-1}\cosh\bigl[C_k \log\bigl(1+\sqrt{z}\,\bigr)/\bigl(1-\sqrt{z}\,\bigr)\bigr] - k^2/(1-k^2) & \text{if } 0 < k < 1, \\ 1 + (2/\pi^2)\bigl[\log\bigl(1+\sqrt{z}\,\bigr)/\bigl(1-\sqrt{z}\,\bigr)\bigr]^2 & \text{if } k = 1, \\ (k^2-1)^{-1}\sin\bigl[C_k \mathbf{K}\bigl(\bigl(z/\sqrt{t}-1\bigr)/\bigl(1-\sqrt{t}z\bigr), t\bigr)\bigr] + k^2/(k^2-1) & \text{if } 1 < k, \end{cases}$$

where $C_k = (2/\pi)\arccos k$ for $0 < k < 1$ and $C_k = \pi/\bigl(2\mathbf{K}(t)\bigr)$ and $t \in (0,1)$ is chosen so that $k = \cosh\bigl(\mu(t)/2\bigr)$ for $k > 1$.

By Theorem 2.1, we can obtain another expression which is more convenient to compute the Taylor expansion about the origin.

Theorem 5.1. *For $k \le 0$, the conformal map P_k of the unit disk onto the domain Ω_k with $P_k(0) = 1$ and $P_k'(0) > 0$ is expressed as*

$$P_k(z) = 1 + \bigl(Q_k\bigl(\sqrt{z}\,\bigr)\bigr)^2, \tag{5.2}$$

where

$$Q_k(z) = \begin{cases} \sqrt{2}\,z/\sqrt{1-z^2} & \text{if } k = 0, \\ \sqrt{\dfrac{2}{1-k^2}}\,\sinh(C_k \operatorname{arctanh} z) & \text{if } 0 < k < 1, \\ \sqrt{\dfrac{1}{2\pi^2}}\,\operatorname{arctanh} z & \text{if } k = 1, \\ \sqrt{\dfrac{2}{k^2-1}}\,\sin\bigl(C_k'\mathbf{K}\bigl(z/\sqrt{s}\,, s\bigr)\bigr) & \text{if } 1 < k. \end{cases}$$

Here, $C_k = (2/\pi)\arccos k$ *when* $0 < k < 1$, *and* $C'_k = \pi/\big(2\mathbf{K}(s)\big)$ *when* $k > 1$, *where* $s \in (0,1)$ *is chosen so that* $k = \cosh \mu(s)$.

Furthermore, the function Q_k *is odd and maps the unit disk conformally onto the domain* $W_k = \{u + iv : (k-1)u^2 + (k+1)v^2 < 1\}$.

This result was presented in the first author's habilitation [6, Theorem 2.2.2] with a proof slightly different from the one here.

It is easily checked that W_k is the inside of a hyperbola when $k < 1$ and W_k is the interior of an ellipse when $k > 1$. When $k = 1$, the domain W_k becomes the parallel strip $-1/\sqrt{2} < \operatorname{Im} w < 1/\sqrt{2}$. Note that W_k is invariant under the involution $w \mapsto -w$. The relation between t and s in Theorems A and 5.1 is given by $s = 2\sqrt{t}/(1+t)$ (see [11, p. 293, (43)]).

The reader might expect that the functions P_k could be expressed in a unified way for all $0 < k < \infty$ by introducing another kind of special functions. It is, however, hopeless to do that because we discarded the left half of the "interior" of the hyperbola when k became less than 1 (see also that the forms of corresponding differential equations are different). Though we can prove Theorem 5.1 by using Theorem A, we give an independent proof so that the present article be self-contained as far as possible.

Proof of Theorem 5.1. First let $k > 1$. Choose $\xi > 0$ so that $\cosh(2\xi) = k$, namely, $k = \cosh\big(\mu(s)\big)$. Since the similarity

$$L(z) = (z + k^2)/(k^2 - 1) = 1 + (z+1)/(k^2 - 1)$$

maps -1 to 1 and 1 to $(k^2+1)/(k^2-1)$, respectively, the image $L(D_{2\xi})$ coincides with Ω_k. Thus $P_k = L \circ g_{2\xi} = 1 + (g_{2\xi} + 1)/(k^2 - 1)$. By Theorem 1.2, we obtain the relation $P_k(z) = 1 + 2\big(f_\xi(\sqrt{z}\,)\big)^2/(k^2-1)$. Hence, we conclude that $P_k(z) = 1 + \big(Q_k(\sqrt{z}\,)\big)^2$.

The case when $0 < k < 1$ can be treated as above. Indeed, take a number $\eta \in (0, \pi/4)$ so that $k = \cos 2\eta$ and let $L(z) = (z + k^2)/(k^2 - 1)$. Then the similarity L maps the left component of $\mathbf{C} \setminus H_{\pi - 2\eta}$ onto Ω_k. Noting $L'(-1) < 0$, we obtain the relation $P_k(z) = L\big(G_{\pi-2\eta}(-z)\big)$ for $z \in \mathbf{D}$. Proposition 2.4 and formula (2.4) now yield

$$\begin{aligned} P_k(z) &= L\Big(2\big(F_{\pi/2-\eta}(i\sqrt{z}\,)\big)^2 - 1\Big) = L\Big(2\sin\big((4\eta/\pi)\arctan(i\sqrt{z}\,)\big)^2 - 1\Big) \\ &= L\Big(-2\sinh\big((4\eta/\pi)\operatorname{arctanh}(\sqrt{z}\,)\big)^2 - 1\Big) = 1 + \big(Q_k(\sqrt{z}\,)\big)^2. \end{aligned}$$

When $k = 1$, the expression is obtained as the limiting case when $k \to 1$. □

As an immediate corollary, we have another characterization of k-uniformly convex functions (cf. [6, Theorem 3.8.1]).

Theorem 5.2. *For a function f locally univalent and analytic in the unit disk $\mathbf{D}$ and normalized so that $f(0)=0$, $f'(0)=1$, let g be the analytic branch of $\log f'$ determined by $g(0)=0$. The function f is k-uniformly convex if and only if $\sqrt{zg'(z)} \in W_k$ for each $z \in \mathbf{D}$.*

Proof. Set $p(z) = 1 + zf''(z)/f'(z)$. It is enough to show that $p(\mathbf{D}) \subset \Omega_k$. By assumption, there exists a point $w \in W_k$ such that $zg'(z) = w^2$ for each $z \in \mathbf{D}$. Choose a point $\zeta \in \mathbf{D}$ so that $Q_k(\zeta) = w$. Then, $p(z) = 1 + zg'(z) = 1 + \big(Q_k(\zeta)\big)^2 = P_k(\zeta^2)$ by Theorem 5.1, which implies $p(z) \in \Omega_k$. □

The next result was also claimed in [6, Theorem 2.3.3]. However, the proof for the case $k > 1$ contained a serious error. In order to correct it, the present joint research was initiated.

Theorem 5.3. *The conformal representation P_k of Ω_k with $P_k(0) = 1$, $P_k'(0) > 0$ has positive Taylor coefficients about the origin for each $k \geq 0$.*

Proof. The assertion for $0 \leq k \leq 1$ can be deduced by Theorem 5.1 because $\sinh z$ and $\operatorname{arctanh} z$ both have positive odd Taylor coefficients about the origin. The assertion for $k > 1$ follows from Theorem 1.2 since $P_k = 1 + (g_{2\xi} + 1)/(k^2 - 1)$ as observed in the proof of Theorem 5.1. □

We remark that for $0 < k < 1$, the function $w = Q_k(z)$ satisfies the linear ODE

$$(1 - z^2)^2 w'' - 2z(1 - z^2)w' - C_k^2 w = 0 \tag{5.3}$$

in $\mathbf{D}$. By using this, one can also show positivity of the Taylor coefficients of P_k about the origin. Note that this differential equation is a special case of the Legendre equation (see, for example, [1, Chapter 8]).

An analytic function p in the unit disk is called a Carathéodory function if $p(0) = 1$ and if p has positive real part, in other words, $\operatorname{Re} p(z) > 0$ holds for $|z| < 1$. The class of Carathéodory functions will be denoted by $\mathscr{P}$.

For two analytic functions f and g in the unit disk $\mathbf{D}$, we say that f is subordinate to g and denote it by $f \prec g$ if there exists an analytic map $\omega\colon \mathbf{D} \to \mathbf{D}$ such that $f = g \circ \omega$ and $\omega(0) = 0$. For each Carathéodory function q, we define the subclass

$$\mathscr{P}(q) = \{p \in \mathscr{P} : p \prec q\}$$

of $\mathscr{P}$. For instance, a normalized analytic function f in the unit disk is k-uniformly convex if and only if the function $1 + zf''(z)/f'(z)$ belongs to $\mathscr{P}(P_k)$.

If q has some nice properties, then elements of $\mathscr{P}(q)$ are dominated by q in various ways.

Proposition 5.4. *Suppose that a Carathéodory function q is convex univalent and has non-negative Taylor coefficients about the origin. Then each element p of $\mathscr{P}(q)$ satisfies*

$$q(-|z|) \leq \operatorname{Re} p(z) \leq |p(z)| \leq q(|z|)$$

for $|z| < 1$.

Proof. We write $p_r(z) = p(rz)$ for $0 < r < 1$. The Lindelöf principle says that $p \prec q$ implies $p_r \prec q_r$ for all $0 < r < 1$. Therefore, for a fixed $r \in [0, 1)$, we obtain

$$\min_{|z|=r} \operatorname{Re} q(z) \le \operatorname{Re} p(z_0) \le |p(z_0)| \le \max_{|z|=r} |q(z)|$$

for any z_0 with $|z_0| = r$. Notice now that q is symmetric, i.e., $\overline{q(z)} = q(\bar{z})$, because q has non-negative (thus real) coefficients. Since q_r is convex (see the proof of Theorem 2.1 in [4]) and symmetric and $\operatorname{Re} q_r > 0$, the relation

$$\min_{|z|=r} \operatorname{Re} q(z) = q(-r)$$

can be deduced. On the other hand, the non-negativity of the coefficients means that q can be expressed in the form

$$q(z) = 1 + \sum_{n=1}^{\infty} a_n z^n, \quad |z| < 1,$$

for some $a_n \ge 0$. Therefore,

$$|q(z)| \le 1 + \sum_{n=1}^{\infty} a_n |z|^n = q(|z|),$$

and thus,

$$\max_{|z|=r} |q(z)| = q(r)$$

follows. □

This, together with Theorem 5.3, implies the following.

Corollary 5.5. *Let* $0 \le k < \infty$ *and* $p \in \mathscr{P}(P_k)$. *Then the inequalities*

$$\operatorname{Re} p(z) \le |p(z)| \le \begin{cases} 1 + \dfrac{2}{1-k^2} \sinh^2\left(\dfrac{2 \operatorname{arctanh} \sqrt{r}}{\pi \arccos k}\right) & \text{for } k \in [0,1), \\ 1 + \dfrac{8}{\pi^2} \operatorname{arctanh}^2 \sqrt{r} & \text{for } k = 1, \\ 1 + \dfrac{2}{k^2-1} \sin^2\left(\dfrac{\pi}{2\mathbf{K}(t)} \mathbf{K}\big(\sqrt{r/t}\,, t\big)\right) & \text{for } k > 1, \end{cases}$$

and

$$|p(z)| \ge \operatorname{Re} p(z) \ge \begin{cases} 1 - \dfrac{2}{1-k^2} \sin^2\left(\dfrac{2 \arctan \sqrt{r}}{\pi \arccos k}\right) & \text{for } k \in [0,1), \\ 1 - \dfrac{8}{\pi^2} \arctan^2 \sqrt{r} & \text{for } k = 1, \\ 1 - \dfrac{2}{k^2-1} \sinh^2\left(\dfrac{\pi}{2\mathbf{K}(t)} \widetilde{\mathbf{K}}\big(\sqrt{r/t}\,, t\big)\right) & \text{for } k > 1, \end{cases}$$

hold for $z \in \mathbf{D}$ *with* $|z| = r$. *In particular, the inequalities*

$$\frac{k}{k+1} < \operatorname{Re} p(z) \le |p(z)|$$

hold for all $|z| < 1$ *and all* $k > 0$ *and* $|p(z)| < k/(k-1)$ *holds for all* $|z| < 1$ *and* $k > 1$.

In the above, we set

$$\widetilde{\mathbf{K}}(r,t) = -i\mathbf{K}(ir,t) = \int_0^r \frac{dx}{\sqrt{(1+x^2)(1+t^2x^2)}}.$$

It is expected that the positivity theorem (Theorem 5.1) has more applications to the class of k-uniformly convex functions. For instance, it gives an improvement of the bound for the lengths of the images of the unit circle under k-uniformly starlike functions (see [9, Theorem 3.5]).

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Received 29 March 2005