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# HIGHER ORDER VARIATIONAL PROBLEMS ON TWO-DIMENSIONAL DOMAINS

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**Abstract.** Let $u\colon \mathbf{R}^2 \supset \Omega \to \mathbf{R}^M$ denote a local minimizer of $J[w] = \int_\Omega f(\nabla^k w)\,\mathrm{d}x$, where $k \geq 2$ and $\nabla^k w$ is the tensor of all $k^{\text{th}}$ order (weak) partial derivatives. Assuming rather general growth and ellipticity conditions for $f$, we prove that $u$ actually belongs to the class $C^{k,\alpha}(\Omega; \mathbf{R}^M)$ by the way extending the result of [BF2] to the higher order case by using different methods. A major tool is a lemma on the higher integrability of functions established in [BFZ].

## 1. Introduction

Let $\Omega$ denote a bounded domain in $\mathbf{R}^2$ and consider a function $u\colon \Omega \to \mathbf{R}^M$ which locally minimizes the variational integral

$$J[w, \Omega] = \int_\Omega f(\nabla^k w)\,\mathrm{d}x,$$

where $\nabla^k w$ represents the tensor of all $k^{\text{th}}$ order (weak) partial derivatives. Our main concern is the investigation of the smoothness properties of such local minimizers under suitable assumptions on the energy density $f$. For the first order case (i.e. $k = 1$) we have rather general results which can be found for example in the textbooks of Morrey [Mo], Ladyzhenskaya and Ural'tseva [LU], Gilbarg and Trudinger [GT] or Giaquinta [Gi], for an update of the history including recent contributions we refer to [Bi]. In order to keep our exposition simple (and only for this reason) we consider the scalar case (i.e. $M = 1$) and restrict ourselves to variational problems involving the second (generalized) derivative. Then our variational problem is related to the theory of plates: one may think of $u\colon \Omega \to \mathbf{R}$ as the displacement in vertical direction from the flat state of an elastic plate. The classical case of a potential $f$ with quadratic growth is discussed in the monographs of Ciarlet and Rabier [CR], Necăs and Hlávácek [NH], Chudinovich and Constanda [CC] or Friedman [Fr], further references are contained in Zeidler's book [Ze]. We also like to remark that plates with other hardening laws (logarithmic and power growth case) together with an additional obstacle have been studied in the papers [BF1] and [FLM] but not with optimal regularity results. The purpose of this

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note is to present a rather satisfying regularity theory for a quite large class of potentials allowing even anisotropic growth.

To be precise let $\mathbf{M}$ denote the space of all $(2\times 2)$-matrices and suppose that we are given a function $f\colon \mathbf{M}\to[0,\infty)$ of class $C^2$ which satisfies with exponents $1<p\le q<\infty$ the anisotropic ellipticity estimate

$$\lambda(1+|\xi|^2)^{(p-2)/2}|\sigma|^2 \le D^2f(\xi)(\sigma,\sigma)\le \Lambda(1+|\xi|^2)^{(q-2)/2}|\sigma|^2 \tag{1.1}$$

for all $\xi$, $\sigma\in\mathbf{M}$ with positive constants $\lambda$, $\Lambda$. Note that (1.1) implies the growth condition

$$a|\xi|^p-b\le f(\xi)\le A|\xi|^q+B \tag{1.2}$$

with suitable constants $a$, $A>0$, $b$, $B\ge 0$. Let

$$J[w,\Omega]=\int_\Omega f(\nabla^2 w)\,\mathrm{d}x,\quad \nabla^2 w=(\partial_\alpha\partial_\beta w)_{1\le\alpha,\beta\le 2}.$$

We say that a function $u\in W^2_{p,\mathrm{loc}}(\Omega)$ is a local $J$-minimizer if and only if $J[u,\Omega']<\infty$ for any subdomain $\Omega'\Subset\Omega$ and

$$J[u,\Omega']\le J[v,\Omega']$$

for all $v\in W^2_{p,\mathrm{loc}}(\Omega)$ such that $u-v\in \mathring{W}^2_p(\Omega')$ (here $W^k_{p,\mathrm{loc}}(\Omega)$ etc. denote the standard Sobolev spaces, see [Ad]). Note that (1.1) implies the strict convexity of $f$. Therefore, given a function $u_0\in W^2_q(\Omega)$, the direct method ensures the existence of a unique $J$-minimizer $u$ in the class

$$\left\{v\in W^2_p(\Omega): J[v,\Omega]<\infty,\ v-u_0\in \mathring{W}^2_p(\Omega)\right\}$$

which motivates the discussion of local $J$-minimizers. Our main result reads as follows:

**Theorem 1.1.** *Let $u$ denote a local $J$-minimizer under condition* (1.1). *Assume further that*

$$q<\min(2p,p+2) \tag{1.3}$$

*holds. Then $u$ is of class $C^{2,\alpha}(\Omega)$ for any $0<\alpha<1$.*

**Remark 1.1.** (i) Clearly the result of Theorem 1.1 extends to local minimizers of the variational integral

$$I[w,\Omega]=\int_\Omega f(\nabla^2 w)\,\mathrm{d}x+\int_\Omega g(\nabla w)\,\mathrm{d}x,$$

where $f$ is as before and where $g$ denotes a density of class $C^2$ satisfying

$$0 \le D^2 g(\xi)(\eta,\eta) \le c(1+|\xi|^2)^{(s-2)/2}|\eta|^2$$

for some suitable exponent $s$. In case $p \ge 2$ any finite number is admissible for $s$, in case $p < 2$ we require the bound $s \le 2p/(2-p)$. The details are left to the reader.

(ii) Without loss of generality we may assume that $q \ge 2$: if (1.1) holds with some exponent $q < 2$, then of course (1.1) is true with $q$ replaced by $\bar{q} := 2$ and (1.3) continues to hold for the new exponent.

(iii) If we consider the higher order variational integral $\int_\Omega f(\nabla^k w)\,\mathrm{d}x$ with $k \ge 2$ and $f$ satisfying (1.1), then (1.3) implies that local minimizers $u \in W^k_{p,\mathrm{loc}}(\Omega)$ actually belong to the space $C^{k,\alpha}(\Omega)$.

(iv) The degree of smoothness of $u$ can be improved by standard arguments provided $f$ is sufficiently regular.

(v) A typical example of an energy $J$ satisfying the assumptions of Theorem 1.1 is given by

$$J[w,\Omega] = \int_\Omega |\nabla^2 w|^2\,\mathrm{d}x + \int_\Omega (1+|\partial_1\partial_2 w|^2)^{q/2}\,\mathrm{d}x$$

with some exponent $q \in (2,4)$.

(vi) Our arguments can easily be adjusted to prove $C^{k,\alpha}$-regularity of local minimizers $u \in W^k_{p(x),\mathrm{loc}}(\Omega)$ of the energy $\int_\Omega (1+|\nabla^k w|^2)^{p(x)/2}\,\mathrm{d}x$ provided that $1 < p_* \le p(x) \le p^* < \infty$ for some numbers $p_*$, $p^*$ and if $p(x)$ is sufficiently smooth. Another possible extension concerns the logarithmic case, i.e. we now consider the variational integral $\int_\Omega |\nabla^k w| \ln(1+|\nabla^k w|)\,\mathrm{d}x$ and its local minimizers which have to be taken from the corresponding higher order Orlicz–Sobolev space.

The proof of Theorem 1.1 is organized as follows: we first introduce some suitable regularization and then prove the existence of higher order weak derivatives for this approximating sequence in Step 2. Here we also derive a Caccioppoli-type inequality using difference quotient methods. In a third step we deduce uniform higher integrability of the second generalized derivatives for any finite exponent. From this together with a lemma established in [BFZ] we finally obtain our regularity result in the last two steps.

## 2. Proof of Theorem 1.1

*Step* 1. *Approximation.* Let us fix some open domains $\Omega_1 \Subset \Omega_2 \Subset \Omega$ and denote by $\bar{u}_m$ the mollification of $u$ with radius $1/m$, in particular

$$\|\bar{u}_m - u\|_{W^2_p(\Omega_2)} \overset{m\to\infty}{\longrightarrow} 0.$$

Jensen's inequality implies

$$J[\bar{u}_m, \Omega_2] \le J[u, \Omega_2] + \tau_m,$$

where $\tau_m \to 0$ as $m \to \infty$. This, together with the lower semicontinuity of the functional $J$, shows that

$$J[\bar{u}_m, \Omega_2] \overset{m\to\infty}{\longrightarrow} J[u, \Omega_2]. \tag{2.1}$$

Next let

$$\varrho_m := \|\bar{u}_m - u\|_{W^2_p(\Omega_2)} \left[ \int_{\Omega_2} (1 + |\nabla^2 \bar{u}_m|^2)^{q/2} \, \mathrm{d}x \right]^{-1},$$

which obviously tends to $0$ as $m \to \infty$. With these preliminaries we introduce the regularized functional

$$J_m[w, \Omega_2] := \varrho_m \int_{\Omega_2} (1 + |\nabla^2 w|^2)^{q/2} \, \mathrm{d}x + J[w, \Omega_2]$$

and the corresponding regularizing sequence $\{u_m\}$ as the sequence of the unique solutions to the problems

$$J_m[\,\cdot\,, \Omega_2] \to \min \quad \text{in } \bar{u}_m + \overset{\circ}{W}{}^2_q(\Omega_2). \tag{2.2}$$

By (2.1) and (2.2) we have

$$\begin{aligned} J_m[u_m, \Omega_2] &\le J_m[\bar{u}_m, \Omega_2] \\ &= \|\bar{u}_m - u\|_{W^2_p(\Omega_2)} + J[\bar{u}_m, \Omega_2] \overset{m\to\infty}{\longrightarrow} J[u, \Omega_2], \end{aligned}$$

hence one gets

$$\limsup_{m\to\infty} J_m[u_m, \Omega_2] \le J[u, \Omega_2]. \tag{2.3}$$

On account of (2.3) and the growth of $f$ we may assume

$$u_m \overset{m\to\infty}{\rightharpoondown} : \hat{u} \quad \text{in } W^2_p(\Omega_2).$$

Moreover, lower semicontinuity gives

$$J[\hat{u}, \Omega_2] \le \liminf_{m\to\infty} J[u_m, \Omega_2],$$

which together with (2.3) and the strict convexity of $f$ implies $\hat{u} = u$ (here we also note that $\hat{u} - u \in \overset{\circ}{W}{}^2_p(\Omega_2)$). Summarizing the results it is shown up to now that (as $m \to \infty$)

$$
\begin{aligned}
u_m &\rightharpoondown u \quad \text{in } W^2_p(\Omega_2),\\
J_m[u_m, \Omega_2] &\to J[u, \Omega_2].
\end{aligned} \tag{2.4}
$$

*Step* 2. *Existence of higher order weak derivatives.* In this second step we will prove that ($f_m(\xi) := \varrho_m(1 + |\xi|^2)^{q/2} + f(\xi)$)

$$
\begin{aligned}
&\int_{\Omega_2} \eta^6 D^2 f_m(\nabla^2 u_m)(\partial_\alpha \nabla^2 u_m, \partial_\alpha \nabla^2 u_m)\, \mathrm{d}x\\
&\qquad \le c(\|\nabla\eta\|^2_\infty + \|\nabla^2\eta\|^2_\infty) \int_{\operatorname{spt}\nabla\eta} |D^2 f_m(\nabla^2 u_m)| \big[|\nabla^2 u_m|^2 + |\nabla u_m|^2\big]\, \mathrm{d}x,
\end{aligned} \tag{2.5}
$$

where $\eta \in C^\infty_0(\Omega_2)$, $0 \le \eta \le 1$, $\eta \equiv 1$ on $\Omega_1$ and where we take the sum over repeated indices. To this purpose let us recall the Euler equation

$$
\int_{\Omega_2} D f_m(\nabla^2 u_m) : \nabla^2 \varphi = 0 \quad \text{for all } \varphi \in \overset{\circ}{W}{}^2_q(\Omega_2). \tag{2.6}
$$

If $\Delta_h$ denotes the difference quotient in the coordinate direction $e_\alpha$, $\alpha = 1, 2$, then the test function $\Delta_{-h}(\eta^6 \Delta_h u_m)$ is admissible in (2.6) with the result

$$
\int_{\Omega_2} \Delta_h\{D f_m(\nabla^2 u_m)\} : \nabla^2(\eta^6 \Delta_h u_m)\, \mathrm{d}x = 0. \tag{2.7}
$$

Now denote by $\mathscr{B}_x$ the bilinear form

$$
\mathscr{B}_x = \int_0^1 D^2 f_m\big(\nabla^2 u_m(x) + th\nabla^2(\Delta_h u_m)(x)\big)\, \mathrm{d}t,
$$

and observe that

$$
\begin{aligned}
\Delta_h\{D f_m(\nabla^2 u_m)\}(x) &= \frac{1}{h}\int_0^1 \frac{d}{dt} D f_m\Big(\nabla^2 u_m(x)\\
&\qquad\qquad + t[\nabla^2 u_m(x + h e_\alpha) - \nabla^2 u_m(x)]\Big)\, \mathrm{d}t\\
&= \frac{1}{h}\int_0^1 \frac{d}{dt} D f_m\big(\nabla^2 u_m(x) + ht\nabla^2(\Delta_h u_m)(x)\big)\, \mathrm{d}t\\
&= \mathscr{B}_x\big(\nabla^2(\Delta_h u_m)(x), \cdot\,\big),
\end{aligned}
$$

hence (2.7) can be written as

$$\int_{\Omega_2} \mathscr{B}_x\big(\nabla^2(\Delta_h u_m), \nabla^2(\eta^6 \Delta_h u_m)\big)\,\mathrm{d}x = 0,$$

which means that we have

$$\begin{aligned}
&\int_{\Omega_2} \eta^6 \mathscr{B}_x\big(\nabla^2(\Delta_h u_m), \nabla^2(\Delta_h u_m)\big)\,\mathrm{d}x \\
&\quad = -\int_{\Omega_2} \mathscr{B}_x\big(\nabla^2(\Delta_h u_m), \nabla^2\eta^6 \Delta_h u_m\big)\,\mathrm{d}x \\
&\qquad - 2\int_{\Omega_2} \mathscr{B}_x\big(\nabla^2(\Delta_h u_m), \nabla\eta^6 \odot \nabla(\Delta_h u_m)\big)\,\mathrm{d}x \\
&\quad =: -T_1 - 2T_2.
\end{aligned} \tag{2.8}$$

To handle $T_1$ we just observe $\partial_\alpha\partial_\beta\eta^6 = 30\partial_\alpha\eta\partial_\beta\eta\eta^4 + 6\partial_\alpha\partial_\beta\eta\eta^5$, for $T_2$ we use $\nabla\eta^6 = 6\eta^5\nabla\eta$. The Cauchy–Schwarz inequality for the bilinear form $\mathscr{B}_x$ implies

$$\begin{aligned}
|T_2| &= 6\left|\int_{\Omega_2} \mathscr{B}_x\big(\eta^3\nabla^2(\Delta_h u_m), \eta^2\nabla\eta \odot \nabla(\Delta_h u_m)\big)\,\mathrm{d}x\right| \\
&\le 6\left[\int_{\Omega_2} \mathscr{B}_x\big(\nabla^2(\Delta_h u_m), \nabla^2(\Delta_h u_m)\big)\eta^6\,\mathrm{d}x\right]^{1/2} \\
&\quad \times \left[\int_{\Omega_2} \mathscr{B}_x\big(\nabla\eta \odot \nabla(\Delta_h u_m), \nabla\eta \odot \nabla(\Delta_h u_m)\big)\eta^4\,\mathrm{d}x\right]^{1/2},
\end{aligned}$$

an analogous estimate being valid for $T_1$. Absorbing terms, (2.8) turns into

$$\begin{aligned}
&\int_{\Omega_2} \eta^6 \mathscr{B}_x\big(\nabla^2(\Delta_h u_m), \nabla^2(\Delta_h u_m)\big)\,\mathrm{d}x \\
&\quad \le c(\|\nabla\eta\|_\infty^2 + \|\nabla^2\eta\|_\infty^2)\int_{\operatorname{spt}\nabla\eta} |\mathscr{B}_x|\big(|\nabla(\Delta_h u_m)|^2 + |\Delta_h u_m|^2\big)\,\mathrm{d}x.
\end{aligned} \tag{2.9}$$

Next we estimate (note that in the following calculations we always assume, without loss of generality, $q \ge 2$, compare Remark 1.1(ii)) for $h$ sufficiently small

$$\begin{aligned}
&\int_{\operatorname{spt}\nabla\eta} |\mathscr{B}_x|\,|\nabla(\Delta_h u_m)|^2\,\mathrm{d}x \\
&\quad \le \int_{\operatorname{spt}\nabla\eta} \big(1 + |\nabla^2 u_m|^2 + h^2|\nabla^2(\Delta_h u_m)|^2\big)^{(q-2)/2}|\nabla(\Delta_h u_m)|^2\,\mathrm{d}x \\
&\quad \le c\left[\int_{\operatorname{spt}\nabla\eta} |\nabla(\Delta_h u_m)|^{q/2}\,\mathrm{d}x\right. \\
&\qquad \left. + \int_{\operatorname{spt}\nabla\eta} (1 + |\nabla^2 u_m|^2 + h^2|\nabla^2(\Delta_h u_m)|^2)^{q/2}\,\mathrm{d}x\right] \\
&\quad \le c\int_{\operatorname{spt}\nabla\eta} (1 + |\nabla^2 u_m|^2)^{q/2}\,\mathrm{d}x.
\end{aligned}$$

In a similar way we estimate $\int_{\mathrm{spt}\nabla\eta} |\mathscr{B}_x|\,|\Delta_h u_m|^2\,\mathrm{d}x$ and end up with

$$\begin{aligned}(2.10)\qquad &\limsup_{h\to 0}\int_{\Omega_2}\eta^6\mathscr{B}_x\big(\nabla^2(\Delta_h u_m),\nabla^2(\Delta_h u_m)\big)\,\mathrm{d}x\\ &\quad\le c(\|\nabla\eta\|_\infty^2+\|\nabla^2\eta\|_\infty^2)\int_{\mathrm{spt}\nabla\eta}(1+|\nabla u_m|^2+|\nabla^2 u_m|^2)^{q/2}\,\mathrm{d}x.\end{aligned}$$

Since $q\ge 2$ is assumed, (2.10) implies that $\nabla^2 u_m\in W^1_{2,\mathrm{loc}}(\Omega_2)$ and

$$\Delta_h(\nabla^2 u_m)\overset{h\to 0}{\longrightarrow}\partial_\alpha(\nabla^2 u_m)\quad\text{in } L^2_{\mathrm{loc}}(\Omega_2)\text{ and a.e.}$$

**Remark 2.1.** With (2.10) we have

$$|\Delta_h\{Df_m(\nabla^2 u_m)\}|^{q/(q-1)}\in L^1_{\mathrm{loc}}(\Omega_2)\text{ uniformly with regard to } h,$$

and, as a consequence,

$$Df_m(\nabla^2 u_m)\in W^1_{q/(q-1),\mathrm{loc}}(\Omega_2).$$

This follows exactly as outlined in the calculations after (3.12) of [BF3].

With the above convergences and Fatou's lemma we find the lower bound

$$\int_{\Omega_2}\eta^6 D^2 f_m(\nabla^2 u_m)(\partial_\alpha\nabla^2 u_m,\partial_\alpha\nabla^2 u_m)\,\mathrm{d}x$$

for the left-hand side of (2.10) which gives using (1.1)

$$\begin{aligned}&\int_{\Omega_2}\eta^6(1+|\nabla^2 u_m|^2)^{(p-2)/2}|\nabla^3 u_m|^2\,\mathrm{d}x\\ &\quad\le c(\|\nabla\eta\|_\infty^2+\|\nabla^2\eta\|_\infty^2)\int_{\mathrm{spt}\nabla\eta}(1+|\nabla u_m|^2+|\nabla^2 u_m|^2)^{q/2}\,\mathrm{d}x<\infty,\end{aligned}$$

in particular

$$(2.11)\qquad h_m:=(1+|\nabla^2 u_m|^2)^{p/4}\in W^1_{2,\mathrm{loc}}(\Omega_2).$$

But (2.11) implies $h_m\in L^r_{\mathrm{loc}}(\Omega_2)$ for any $r<\infty$, i.e.

$$(2.12)\qquad \nabla^2 u_m\in L^t_{\mathrm{loc}}(\Omega_2)\quad\text{for any } t<\infty.$$

Using Fatou's lemma again we obtain from (2.8)

$$\begin{aligned}
&\int_{\Omega_2} \eta^6 D^2 f_m(\nabla^2 u_m)(\partial_\alpha \nabla^2 u_m, \partial_\alpha \nabla^2 u_m)\,\mathrm{d}x \\
(2.13)\quad &\le \liminf_{h\to 0} \int_{\Omega_2} \eta^6 \Delta_h\{Df_m(\nabla^2 u_m)\} : \nabla^2(\Delta_h u_m)\,\mathrm{d}x \\
&= \liminf_{h\to 0} -\int_{\Omega_2} \Delta_h\{Df_m(\nabla^2 u_m)\} : [\nabla^2\eta^6 \Delta_h u_m + 2\nabla\eta^6 \odot \nabla(\Delta_h u_m)]\,\mathrm{d}x.
\end{aligned}$$

On account of (2.12), Remark 2.1 and Vitali's convergence theorem we may pass to the limit $h \to 0$ on the right-hand side of (2.13) and obtain

$$\begin{aligned}
&\int_{\Omega_2} \eta^6 D^2 f_m(\nabla^2 u_m)(\partial_\alpha \nabla^2 u_m, \partial_\alpha \nabla^2 u_m)\,\mathrm{d}x \\
&\qquad \le -\int_{\Omega_2} D^2 f_m(\nabla^2 u_m)(\partial_\alpha \nabla^2 u_m, \nabla^2\eta^6 \partial_\alpha u_m + 2\nabla\eta^6 \odot \nabla\partial_\alpha u_m)\,\mathrm{d}x.
\end{aligned}$$

This immediately gives (2.5) by repeating the calculations leading from (2.8) to (2.9).

*Step* 3. *Uniform higher integrability of* $\nabla^2 u_m$. Let $\chi$ denote any real number satisfying $\chi > p/(2p-q)$, moreover we set $\alpha = \chi p/2$. For all discs $B_r \Subset B_R \Subset \Omega_2$ any $\eta \in C_0^\infty(B_R)$, $\eta \equiv 1$ on $B_r$, $|\nabla^k \eta| \le c/(R-r)^k$, $k = 1, 2$, we have by Sobolev's inequality

$$\int_{B_r} (1 + |\nabla^2 u_m|^2)^\alpha\,\mathrm{d}x \le \int_{B_R} (\eta^3 h_m)^{2\chi}\,\mathrm{d}x \le c\Big[\int_{B_R} |\nabla(\eta^3 h_m)|^t\,\mathrm{d}x\Big]^{2\chi/t},$$

where $t \in (1,2)$ satisfies $2\chi = 2t/(2-t)$. Hölder's inequality implies

$$\begin{aligned}
\int_{B_r} (1 + |\nabla^2 u_m|^2)^\alpha\,\mathrm{d}x &\le c(r,R)\Big[\int_{B_R} |\nabla(\eta^3 h_m)|^2\,\mathrm{d}x\Big]^\chi \\
&\le c(r,R)\Big[\int_{B_R} \eta^6 |\nabla h_m|^2\,\mathrm{d}x + \int_{\operatorname{spt}\nabla\eta} |\nabla\eta^3|^2 h_m^2\,\mathrm{d}x\Big]^\chi.
\end{aligned}$$

Observing that obviously

$$\int_{\operatorname{spt}\nabla\eta} |\nabla\eta^3|^2 h_m^2\,\mathrm{d}x \le c(r,R) \int_{\operatorname{spt}\nabla\eta} (1 + |\nabla^2 u_m|^2)^{p/2}\,\mathrm{d}x$$

and that by (2.5)

$$\begin{aligned}
\int_{B_R} \eta^6 |\nabla h_m|^2\,\mathrm{d}x &\le c(r,R) \int_{\operatorname{spt}\nabla\eta} (1 + |\nabla^2 u_m|^2)^{(q-2)/2} \big[|\nabla^2 u_m|^2 + |\nabla u_m|^2\big]\,\mathrm{d}x \\
&\le c(r,R)\Big[\int_{\operatorname{spt}\nabla\eta} (1 + |\nabla^2 u_m|^2)^{q/2}\,\mathrm{d}x + \int_{\operatorname{spt}\nabla\eta} |\nabla u_m|^q\,\mathrm{d}x\Big],
\end{aligned}$$

we deduce

$$\begin{aligned}\int_{B_r}(1+|\nabla^2 u_m|^2)^\alpha\,\mathrm{d}x \le c(r,R)\bigg[\int_{\operatorname{spt}\nabla\eta}&(1+|\nabla^2 u_m|^2)^{q/2}\,\mathrm{d}x\\ &+\int_{\operatorname{spt}\nabla\eta}|\nabla u_m|^q\,\mathrm{d}x\bigg]^\chi,\end{aligned}\tag{2.14}$$

where $c(r,R) = c(R-r)^{-\beta}$ for some suitable $\beta > 0$. For discussing (2.14) we first note that the term $\int_{\operatorname{spt}\nabla\eta}|\nabla u_m|^q\,\mathrm{d}x$ causes no problems. In fact, since $\|u_m\|_{W^2_p(\Omega_2)} \le c < \infty$ we know that $\nabla u_m \in L^t_{\mathrm{loc}}(\Omega_2)$ for any $t < \infty$ in case $p \ge 2$. If $p < 2$, then we have local $L^t$-integrability of $\nabla u_m$ provided that $t < 2p/(2-p)$, but $q < 2p < 2p/(2-p)$ on account of (1.3). As a consequence, we may argue exactly as in [ELM] or [Bi, p. 60], to derive from (2.14) by interpolation and hole-filling (here $q < 2p$ enters in an essential way)

$$\nabla^2 u_m \in L^t_{\mathrm{loc}}(\Omega_2) \quad \text{for any } t<\infty \text{ and uniformly with regard to } m. \tag{2.15}$$

Note that (2.15) implies with Step 2 the uniform bound

$$\int_{\Omega_2}\eta^6 D^2 f_m(\nabla^2 u_m)(\partial_\alpha\nabla^2 u_m, \partial_\alpha\nabla^2 u_m)\,\mathrm{d}x \le c(\eta) < \infty, \tag{2.16}$$

in particular (2.16) shows

$$h_m \in W^1_{2,\mathrm{loc}}(\Omega_2) \quad \text{uniformly with regard to } m. \tag{2.17}$$

**Remark 2.2.** (i) If $u$ is a local $J$-minimizer subject to an additional constraint of the form $u \ge \psi$ a.e. on $\Omega$ for a sufficiently regular function $\psi\colon \Omega \to \mathbf{R}$, then it is an easy exercise to adjust the technique used in [BF1] to the present situation which means that we still have (2.15) so that (recall (2.4)) $u \in W^2_{t,\mathrm{loc}}(\Omega)$ for any $t < \infty$, hence $u \in C^{1,\alpha}(\Omega)$ for all $0 < \alpha < 1$. In [Fr, Theorem 10.6, p. 98], it is shown for the special case $f(w) = |\Delta w|^2$ that actually $u \in C^2(\Omega)$ is true, and it would be interesting to see if this result also holds for the energy densities discussed here.

(ii) We remark that the proof of (2.15) just needs the inequality $q < 2p$, whereas the additional assumption $q < p+2$ enters in the next step.

*Step* 4. *$C^2$-regularity.* Now we consider an arbitrary disc $B_{2R} \Subset \Omega_1$ and $\eta \in C_0^\infty(B_{2R})$ satisfying $\eta \equiv 1$ on $B_R$ and $|\nabla\eta| \le c/R$, $|\nabla^2\eta| \le c/R^2$. Moreover we denote by $T_{2R}$ the annulus $T_{2R} := B_{2R} - B_R$ and by $P_m$ a polynomial function of degree less than or equal to 2. Exactly as in Step 2 (replacing $u_m$ by $u_m - P_m$) we obtain

$$\int_{B_{2R}} \eta^6 D^2 f_m(\nabla^2 u_m)(\partial_\alpha \nabla^2 u_m, \partial_\alpha \nabla^2 u_m)\,\mathrm{d}x \\ \leq -\int_{T_{2R}} D^2 f_m(\nabla^2 u_m)\Big(\partial_\alpha \nabla^2 u_m, \nabla^2 \eta^6 \partial_\alpha [u_m - P_m] \\ + 2\nabla \eta^6 \odot \nabla \partial_\alpha (u_m - P_m)\Big)\,\mathrm{d}x.$$

With the notation

$$H_m := \Big[D^2 f_m(\nabla^2 u_m)(\partial_\alpha \nabla^2 u_m, \partial_\alpha \nabla^2 u_m)\Big]^{1/2}, \quad \sigma_m := D f_m(\nabla^2 u_m)$$

we therefore have

$$\int_{B_{2R}} \eta^6 H_m^2\,\mathrm{d}x \leq c\int_{T_{2R}} |\nabla \sigma_m|\big[|\nabla^2 \eta^6|\,|\nabla u_m - \nabla P_m| + |\nabla \eta^6|\,|\nabla^2 u_m - \nabla^2 P_m|\big]\,\mathrm{d}x.$$

Moreover, by the Cauchy–Schwarz inequality and (1.1)

$$|\nabla \sigma_m|^2 \leq H_m \big[D^2 f_m(\nabla^2 u_m)(\partial_\alpha \sigma_m, \partial_\alpha \sigma_m)\big]^{1/2} \leq H_m |\nabla \sigma_m| \Gamma_m^{(q-2)/4},$$

where $\Gamma_m := 1 + |\nabla^2 u_m|^2$. Finally we let

$$\tilde{h}_m := \max\big[\Gamma_m^{(q-2)/4}, \Gamma_m^{(2-p)/4}\big]$$

and obtain

$$|\nabla \sigma_m| \leq c H_m \Gamma_m^{(q-2)/4} \leq c H_m \tilde{h}_m,$$

hence

$$\int_{B_{2R}} \eta^6 H_m^2\,\mathrm{d}x \leq c\int_{T_{2R}} H_m \tilde{h}_m \big[|\nabla^2 \eta^6|\,|\nabla u_m - \nabla P_m| \\ + |\nabla \eta^6|\,|\nabla^2 u_m - \nabla^2 P_m|\big]\,\mathrm{d}x. \tag{2.18}$$

Letting $\gamma = 4/3$ we discuss the right-hand side of (2.18):

$$\int_{T_{2R}} H_m \tilde{h}_m |\nabla \eta^6|\,|\nabla^2 u_m - \nabla^2 P_m|\,\mathrm{d}x \\ \leq \frac{c}{R}\bigg[\int_{B_{2R}} (H_m \tilde{h}_m)^\gamma\,\mathrm{d}x\bigg]^{1/\gamma} \bigg[\int_{B_{2R}} |\nabla^2 u_m - \nabla^2 P_m|^4\,\mathrm{d}x\bigg]^{1/4}.$$

Next the choice of $P_m$ is made more precise by the requirement

$$\nabla^2 P_m = \fint_{B_{2R}} \nabla^2 u_m \,\mathrm{d}x. \tag{2.19}$$

Then Sobolev–Poincaré's inequality together with the definition of $\tilde{h}_m$ gives

$$\begin{aligned}\left[\fint_{B_{2R}} |\nabla^2 u_m - \nabla^2 P_m|^4 \,\mathrm{d}x\right]^{1/4} &\le c\left[\fint_{B_{2R}} |\nabla^3 u_m|^\gamma \,\mathrm{d}x\right]^{1/\gamma}\\ &\le c\left[\fint_{B_{2R}} (H_m\tilde{h}_m)^\gamma \,\mathrm{d}x\right]^{1/\gamma},\end{aligned}$$

hence

$$\int_{T_{2R}} H_m\tilde{h}_m|\nabla\eta^6|\,|\nabla^2 u_m - \nabla^2 P_m| \,\mathrm{d}x \le \frac{c}{R}\left[\fint_{B_{2R}} (H_m\tilde{h}_m)^\gamma \,\mathrm{d}x\right]^{2/\gamma}. \tag{2.20}$$

To handle the remaining term on the right-hand side of (2.18) we need in addition to (2.19)

$$\fint_{B_{2R}} (\nabla u_m - \nabla P_m) \,\mathrm{d}x = 0,$$

which can be achieved by adjusting the linear part of $P_m$. Then we have by Poincaré's inequality

$$\begin{aligned}&\int_{B_{2R}} H_m\tilde{h}_m|\nabla^2\eta^6|\,|\nabla u_m - \nabla P_m| \,\mathrm{d}x\\ &\quad\le \frac{c}{R^2}\left[\fint_{B_{2R}} (H_m\tilde{h}_m)^\gamma \,\mathrm{d}x\right]^{1/\gamma}\left[\fint_{B_{2R}} |\nabla u_m - \nabla P_m|^4 \,\mathrm{d}x\right]^{1/4}\\ &\quad\le \frac{c}{R}\left[\fint_{B_{2R}} (H_m\tilde{h}_m)^\gamma \,\mathrm{d}x\right]^{1/\gamma}\left[\fint_{B_{2R}} |\nabla^2 u_m - \nabla^2 P_m|^4 \,\mathrm{d}x\right]^{1/4},\end{aligned}$$

and the right-hand side is bounded by the right-hand side of (2.20). Hence, recalling (2.18) and (2.20), we have established the inequality

$$\left[\fint_{B_R} H_m^2 \,\mathrm{d}x\right]^{\gamma/2} \le c\fint_{B_{2R}} (H_m\tilde{h}_m)^\gamma \,\mathrm{d}x. \tag{2.21}$$

Given this starting inequality we like to apply the following lemma which is proved in [BFZ].

**Lemma 2.1.** *Let $d > 1$, $\beta > 0$ be two constants. With a slight abuse of notation let $f$, $g$, $h$ now denote any non-negative functions on $\Omega \subset \mathbf{R}^n$ satisfying*

$$f \in L^d_{\mathrm{loc}}(\Omega), \quad \exp(\beta g^d) \in L^1_{\mathrm{loc}}(\Omega), \quad h \in L^d_{\mathrm{loc}}(\Omega).$$

*Suppose that there is a constant $C > 0$ such that*

$$\left[ \fint_B f^d \,\mathrm{d}x \right]^{1/d} \le C \fint_{2B} f g \,\mathrm{d}x + C \left[ \fint_{2B} h^d \,\mathrm{d}x \right]^{1/d}$$

*holds for all balls $B = B_r(x)$ with $2B = B_{2r}(x) \Subset \Omega$. Then there is a real number $c_0 = c_0(n, d, C)$ such that if $h^d \log^{c_0\beta}(e+h) \in L^1_{\mathrm{loc}}(\Omega)$, then the same is true for $f$. Moreover, for all balls $B$ as above we have*

$$\fint_B f^d \log^{c_0\beta}\left[ e + \frac{f}{\|f\|_{d,2B}} \right] \mathrm{d}x \le c \left[ \fint_{2B} \exp(\beta g^d) \,\mathrm{d}x \right] \left[ \fint_{2B} f^d \,\mathrm{d}x \right] + c \fint_{2B} h^d \log^{c_0\beta}\left[ e + \frac{h}{\|f\|_{d,2B}} \right] \mathrm{d}x,$$

*where $c = c(n, d, \beta, C) > 0$ and $\|f\|_{d,2B} = (\fint_{2B} f^d \,\mathrm{d}x)^{1/d}$.*

The appropriate choices in the setting at hand are $d = 2/\gamma = 3/2$, $f = H_m^\gamma$, $g = \tilde{h}_m^\gamma$, $h \equiv 0$. We claim that

$$\fint_{B_{2R}} \exp(\tilde{h}_m^2 \beta) \,\mathrm{d}x \le c \quad \text{and} \quad \fint_{B_{2R}} H_m^2 \,\mathrm{d}x \le c$$

for a constant being uniform in $m$. The uniform bound of the second integral follows from (2.16); thus let us discuss the first one. By (2.17) and Trudinger's inequality (see e.g. Theorem 7.15 of [GT]) we know that for any disc $B_\varrho \Subset \Omega_1$

$$\int_{B_\varrho} \exp(\beta_0 h_m^2) \,\mathrm{d}x \le c(\varrho) < \infty,$$

where $\beta_0$ just depends on the uniformly bounded quantities $\|h_m\|_{W^1_2(\Omega_1)}$. This implies for any $\beta > 0$ and $\kappa \in (0,1)$

$$\int_{B_\varrho} \exp(\beta h_m^{2-\kappa}) \,\mathrm{d}x \le c(\varrho, \beta, \kappa) < \infty.$$

Moreover, on account of $q < p + 2$ we have

$$\Gamma_m^{(q-2)/2} \le h_m^{2-\kappa} \quad \text{and clearly} \quad \Gamma_m^{(2-p)/2} \le h_m^{2-\kappa}$$

for $\kappa$ sufficiently small, which gives our claim and we may indeed apply the lemma with the result

$$\fint_{B_\varrho} H_m^2 \log^{c_0\beta}(e + H_m)\, \mathrm{d}x \le c(\beta, \varrho) < \infty$$

for all discs $B_\varrho \subset \Omega_1$ and all $\beta > 0$. Thus we have established the counterparts of (2.7) and (2.10) in [BFZ], and exactly the same arguments as given there lead to (2.11) from [BFZ]. Thus we deduce the uniform continuity of the sequence $\{\sigma_m\}$ (see again [BFZ], end of Section 2), hence we have uniform convergence $\sigma_m \to: \sigma$ for some continuous tensor $\sigma$. In order to identify $\sigma$ with $Df(\nabla^2 u)$, we recall the weak convergence stated in (2.4) and also observe that $\nabla^2 u_m \to \nabla^2 u$ a.e. which can be deduced along the same lines as in Lemma 4.5c) of [BF3], we also refer to Proposition 3.29 iii) of [Bi]. Therefore $Df(\nabla^2 u)$ is a continuous function, i.e. $\nabla^2 u$ is of class $C^0$, and finally $u \in C^2(\Omega)$ follows.

*Step* 5. $C^{2,\alpha}$*-regularity of* $u$. To finish the proof of Theorem 1.1 we observe that with Step 4 we get from (2.5) the estimate

$$\int_{\Omega_1} |\nabla^3 u_m|^2\, \mathrm{d}x \le c(\Omega_1) < \infty,$$

in particular one has for $\alpha = 1, 2$

$$U := \partial_\alpha u \in W^2_{2,\mathrm{loc}}(\Omega).$$

Moreover we have

$$\int_\Omega D^2 f_m(\nabla^2 u_m)(\nabla^2 \partial_\alpha u_m, \nabla^2 \varphi)\, \mathrm{d}x = 0 \quad \text{for any } \varphi \in C_0^\infty(\Omega).$$

Together with the convergences (as $m \to \infty$)

$$D^2 f_m(\nabla^2 u_m) \to D^2 f(\nabla^2 u) \quad \text{in } L^\infty_{\mathrm{loc}}(\Omega),$$
$$\nabla^2 \partial_\alpha u_m \rightharpoondown \nabla^2 U \quad \text{in } L^2_{\mathrm{loc}}(\Omega)$$

we therefore arrive at the limit equation

$$\int_\Omega D^2 f(\nabla^2 u)(\nabla^2 U, \nabla^2 \varphi)\, \mathrm{d}x = 0.$$

Hence $U$ is a weak solution of an equation with continous coefficients and $u \in C^{2,\alpha}(\Omega)$ for any $0 < \alpha < 1$ follows from [GM, Theorem 4.1]. ▫

**References**


[Ad] Adams, R. A.: Sobolev spaces. - Academic Press, New York–San Francisco–London, 1975.

[Bi] Bildhauer, M.: Convex Variational Problems: Linear, Nearly Linear and Anisotropic Growth Conditions. - Lecture Notes in Math. 1818, Springer, Berlin–Heidelberg–New York, 2003.

[BF1] Bildhauer, M., and M. Fuchs: Higher order variational inequalities with non-standard growth conditions in dimension two: plates with obstacles. - Ann. Acad. Sci. Fenn. Math. 26, 2001, 509–518.

[BF2] Bildhauer, M., and M. Fuchs: Two-dimensional anisotropic variational problems. - Calc. Var. 16, 2003, 177–186.

[BF3] Bildhauer, M., and M. Fuchs: Variants of the Stokes problem: the case of anisotropic potentials. - J. Math. Fluid Mech. 5, 2003, 364–402.

[BFZ] Bildhauer, M., M. Fuchs, and X. Zhong: A lemma on the higher integrability of functions with applications to the regularity theory of two-dimensional generalized Newtonian fluids. - Manuscripta Math. 116, 2005, 135–156.

[CC] Chudinovich, I., and C. Constanda: Variational and Potential Methods in the Theory of Bending of Plates with Transverse Shear Deformation. - Chapman and Hall, 2000.

[CR] Ciarlet, P., and P. Rabier: Les équations de von Kármán. - Lecture Notes in Math. 826, Springer, Berlin–Heidelberg–New York, 1980.

[ELM] Esposito, L., F. Leonetti, and G. Mingione: Regularity results for minimizers of irregular integrals with $(p,q)$-growth. - Forum Math. 14, 2002, 245–272.

[Fr] Friedman, A.: Variational Principles and Free Boundary Problems. - Wiley-Interscience, 1982.

[FLM] Fuchs, M., G. Li, and O. Martio: Second order obstacle problems for vectorial functions and integrands with subquadratic growth. - Ann. Acad. Sci. Fenn. Math. 23, 1998, 549–558.

[Gi] Giaquinta, M.: Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems. - Ann. of Math. Stud. 105, Princeton University Press, Princeton 1983.

[GM] Giaquinta, M., and G. Modica: Regularity results for some classes of higher order non linear elliptic systems. - J. Reine Angew. Math. 311/312, 1979, 145–169.

[GT] Gilbarg, D., and N. Trudinger: Elliptic Partial Differential Equations of the Second Order. - Second Edition, Springer-Verlag, Berlin–Heidelberg, 1983.

[LU] Ladyzhenskaya, O. A., and N. N. Ural'tseva: Linear and Quasilinear Elliptic Equations. - Nauka, Moskow, 1964. English transl:: Academic Press, New York 1968.

[Mo] Morrey, C. B.: Multiple Integrals in the Calculus of Variations. - Grundlehren Math. Wiss. 130, Springer, Berlin–Heidelberg–New York, 1966.

[NH] Nečăs, J., and I. Hlávácek: Mathematical Theory of Elastic and Elasto-Plastic Bodies. - Elsevier, New York, 1981.

[Ze] Zeidler, E.: Nonlinear Functional Analysis and Its Applications IV. Applications to Mathematical Physics. - Springer, Berlin–Heidelberg–New York, 1987.


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