Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 31, 2006, 363–379

EQUIVALENCE OF WEAK AND VISCOSITY SOLUTIONS TO THE p-LAPLACE EQUATION IN THE HEISENBERG GROUP

Thomas Bieske

University of South Florida, Department of Mathematics
Tampa, FL 33620, U.S.A.; tbieske@math.usf.edu

Abstract. We prove weak and viscosity solutions to the p-Laplace equation in the Heisenberg group coincide by showing that the viscosity sub(super-) solutions coincide with the p-sub(super-)harmonic functions from potential theory. We are then able to obtain a comparison principle for viscosity solutions to the p-Laplace equation.

1. Introduction and motivation

In [JLM], Juutinen, Lindqvist, and Manfredi prove the equivalence of viscosity solutions and weak solutions to the p-Laplace equation in $\mathbf{R}^n$, given by

$$-\operatorname{div}(\|Du\|^{p-2}Du) = 0$$

for $1 < p < \infty$. Here, Du denotes the gradient of the real-valued function u. The p-Laplace equation is a well-known example of a larger class of quasi-linear equations of the form

$$-\operatorname{div}\big(\mathscr{A}_p(u, Du)\big) = 0$$

where $\mathscr{A}_p$ satisfies certain structure conditions. (See [HKM] and [HH] for complete details.) This class of equations plays a major role in non-linear potential theory and has been studied in the Euclidean environment [HKM], Carnot groups [HH], and general metric spaces [KM].

In this paper, we prove the equivalence of the potential-theoretic p-harmonic functions and viscosity solutions to the p-Laplace equation in the Heisenberg group. (See Section 3 for relevant definitions.) It should be noted that due to the geometry of the Heisenberg group, the method of proof in [JLM] cannot be used, for it relies upon the well-known $C^{1,\alpha}$ regularity of the weak solutions, which is unknown in the Heisenberg group. (Although, for p near 2, this was proved recently by Domokos and Manfredi [DM].) We therefore adopt a different strategy to obtain our results. We begin with a brief review of the Heisenberg group in Section 2, followed by the relevant definitions in Section 3. It is noted that this section highlights the nonlinear potential theory found in [HKM] and its extension to Carnot groups in [HH]. In Section 4, we prove the equivalence of viscosity solutions and p-harmonic functions and obtain a comparison principle for viscosity solutions to the p-Laplacian.

2000 Mathematics Subject Classification: Primary 31C45, 43A80; Secondary 31B05, 22E25.

2. The Heisenberg group

We begin with $\mathbf{R}^{2n+1}$ using the coordinates $(x_1, x_2, \ldots, x_{2n}, z)$ and consider the linearly independent vector fields $\{X_i, Z\}$, where the index i ranges from 1 to $2n$, defined by

$$X_i = \begin{cases} \dfrac{\partial}{\partial x_i} - \dfrac{x_{n+i}}{2}\dfrac{\partial}{\partial z}, & \text{if } 1 \le i \le n, \\ \dfrac{\partial}{\partial x_i} + \dfrac{x_{i-n}}{2}\dfrac{\partial}{\partial z}, & \text{if } n < i \le 2n, \end{cases}$$
$$Z = \frac{\partial}{\partial z}\cdot$$

For $i \le j$, these vector fields obey the relations

$$[X_i, X_j] = \begin{cases} Z, & \text{if } j = i+n, \\ 0, & \text{otherwise,} \end{cases}$$

and for all i,

$$[X_i, Z] = 0.$$

We then have a Lie algebra denoted h_n that decomposes as a direct sum

$$h_n = V_1 \oplus V_2$$

where V_1 is spanned by the X_i's and V_2 is spanned by Z. We endow h_n with an inner product $\langle \cdot, \cdot \rangle$ and related norm $\| \cdot \|$ so that this basis is orthonormal. The corresponding Lie group is called the general Heisenberg group of dimension n and is denoted by $\mathbf{H}_n$. With this choice of vector fields the exponential map can be used to identify elements of h_n and $\mathbf{H}_n$ with each other via

$$\sum_{i=1}^{2n} x_i X_i + zZ \in h_n \leftrightarrow (x_1, x_2, \ldots, x_{2n}, z) \in \mathbf{H}_n.$$

In particular, for any P, Q in $\mathbf{H}_n$, written as $P = (x_1, x_2, \ldots, x_{2n}, z_1)$ and $Q = (y_1, y_2, \ldots, y_{2n}, z_2)$ the group multiplication law is given by

$$P \cdot Q = \left(x_1 + y_1, x_2 + y_2, \ldots, x_{2n} + y_{2n}, z_1 + z_2 + \frac{1}{2}\sum_{i=1}^{n}(x_i y_{n+i} - x_{n+i} y_i) \right).$$

The natural metric on $\mathbf{H}_n$ is the Carnot–Carathéodory metric given by

$$d_C(P, Q) = \inf_{\Gamma} \int_0^1 \|\gamma'(t)\| \, dt$$

where the set Γ is the set of all curves γ such that $\gamma(0) = P$, $\gamma(1) = Q$ and $\gamma'(t) \in V_1$. By Chow's theorem (see, for example, [Be]) any two points can be connected by such a curve, which makes $d_C(P,Q)$ a left-invariant metric on $\mathbf{H}_n$. This metric induces a homogeneous norm on $\mathbf{H}_n$, denoted $|\cdot|$, by

$$|P| = d_C(0,P)$$

and we have the estimate

$$|P| \sim \sum_{i=1}^{2n} |x_i| + |z|^{1/2}.$$

This estimate leads us to define the left-invariant gauge $\mathscr{N}$ that is comparable to the Carnot–Carathéodory metric and is given by

$$\mathscr{N}(P) = \left(\left(\sum_{i=1}^{2n} x_i^2 \right)^2 + 16z^2 \right)^{1/4}.$$

We define the Carnot–Carathéodory balls $B(P,r)$ and the gauge balls $B_{\mathscr{N}}(P,r)$ in the obvious way.

Given a smooth function $u\colon \mathbf{H}_n \mapsto \mathbf{R}$, we define the horizontal gradient by

$$\nabla_0 u = (X_1 u, X_2 u, \dots, X_{2n} u),$$

the full gradient by

$$\nabla u = (X_1 u, X_2 u, \dots, X_{2n} u, Zu),$$

and the symmetrized horizontal second derivative matrix $(D^2 u)^\star$ by

$$\big((D^2 u)^\star\big)_{ij} = \tfrac{1}{2}(X_i X_j u + X_j X_i u).$$

Additionally, given a vector field $F = \sum_{i=1}^{2n} f_i X_i + f_{2n+1} Z$, we define the Heisenberg divergence of F, denoted $\operatorname{div}_{\mathscr{H}} F$, by

$$\operatorname{div}_{\mathscr{H}} F = \sum_{i=1}^{2n} X_i f_i.$$

A quick calculation shows that when $f_{2n+1} = 0$, we have

$$\operatorname{div}_{\mathscr{H}} F = \operatorname{div}_{\mathrm{eucl}} F$$

where $\operatorname{div}_{\mathrm{eucl}}$ is the standard Euclidean divergence. The main operator we are concerned with is the horizontal p-Laplacian for $1 < p < \infty$ defined by

$$\Delta_p f = \operatorname{div}_{\mathscr{H}}(\|\nabla_0 f\|^{p-2} \nabla_0 f)$$

which is a specific type of operator in an important class of operators in potential theory as detailed in [HH] and [HKM].

A function f is $\mathscr{C}^1_{\mathrm{sub}}$ if $X_i f$ is continuous for all i and f is $\mathscr{C}^2_{\mathrm{sub}}$ if f is $\mathscr{C}^1_{\mathrm{sub}}$ and $X_i X_j f$ is continuous for all i and j. Using the horizontal gradient, we may also define the Sobolev spaces $\mathrm{W}^{1,\mathrm{p}}$, $\mathrm{W}^{1,\mathrm{p}}_{\mathrm{loc}}$, etc. in the obvious way. For a more complete treatment of the Heisenberg group, the interested reader is directed to [Be], [B], [F], [FS], [G], [H], [K], [S] and the references therein.

3. Notions of solution

In this section, we highlight some results from nonlinear potential theory as detailed in [HKM]. In [HH], many of the Euclidean results were extended into the general setting of Carnot groups. A more complete treatment of viscosity solutions in the Euclidean environment can be found in [CIL] and in the Heisenberg group in [B]. All the results below can be extended into general Carnot groups.

Our main goal is to relate three different notions of solutions to the equation

$$-\Delta_p f = -\operatorname{div}_{\mathscr{H}}(\|\nabla_0 f\|^{p-2}\nabla_0 f) = 0 \tag{3.1}$$

in a bounded domain Ω.

3.1. Weak solutions. We begin by defining the concept of weak solutions to equation (3.1). We will actually do more, for we shall define weak solutions to a wider class of equations. Letting $\varepsilon \geq 0$ be a real parameter, we consider equations of the form

$$-\Delta_p f = -\operatorname{div}_{\mathscr{H}}(\|\nabla_0 f\|^{p-2}\nabla_0 f) = \varepsilon \tag{3.2}$$

in a bounded domain Ω. Note that equation (3.1) corresponds to equation (3.2) with $\varepsilon = 0$. We now give the definition of weak solutions.

Definition 1. The function $u \in \mathrm{W}^{1,\mathrm{p}}_{\mathrm{loc}}$ is an ε*-weak solution* to equation (3.2) if

$$\int_\Omega \|\nabla_0 u\|^{p-2}\langle \nabla_0 u, \nabla_0 \phi\rangle = \varepsilon \int_\Omega \phi \tag{3.3}$$

for all $\phi \in \mathrm{W}^{1,\mathrm{p}}_0(\Omega)$.

A weak solution to equation (3.1) (i.e., a 0-weak solution) is called p*-harmonic*. It is well known that a p-harmonic function u has a continuous representative that satisfies

$$\operatorname{osc}_{B_r} u \leq C\left(\frac{r}{R}\right)^{\alpha} \operatorname{osc}_{B_R}$$

when $B_R \subset \Omega$ and $r \leq R$ ([HH, Theorem 4.2]). We note that the constants $C > 0$ and $\alpha > 0$ depend only on the group $\mathbf{H}_n$. We therefore identify p-harmonic functions with their continuous representative.

In addition to weak solutions we may define weak supersolutions and weak subsolutions using the following definition (cf. [HKM]).

Definition 2. The function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is an ε*-weak supersolution* to equation (3.2) if

$$\int_\Omega \|\nabla_0 u\|^{p-2}\langle \nabla_0 u, \nabla_0 \phi\rangle \geq \varepsilon \int_\Omega \phi$$

for all non-negative $\phi \in W^{1,p}_0(\Omega)$.

The function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is an ε*-weak subsolution* to equation (3.2) if $-u$ is an ε-weak supersolution, that is, if

$$\int_\Omega \|\nabla_0 u\|^{p-2}\langle \nabla_0 u, \nabla_0 \phi\rangle \leq \varepsilon \int_\Omega \phi$$

for all non-negative $\phi \in W^{1,p}_0(\Omega)$.

Using these definitions for $\varepsilon_1 > \varepsilon_2 \geq 0$, we observe that an ε_1-weak solution is a ε_2-weak supersolution and an ε_2-weak solution is a ε_1-weak subsolution.

It is also well known that 0-weak subsolutions and supersolutions satisfy the following comparison principle

Lemma 3.1 ([HKM, Lemma 3.18]). *Let* $u \in W^{1,p}(\Omega)$ *be a weak subsolution to equation* (3.1) *and let* $v \in W^{1,p}(\Omega)$ *be a weak supersolution to equation* (3.1) *in* Ω. *If* $\gamma \equiv \min\{v-u, 0\} \in W^{1,p}_0(\Omega)$ *then* $u \leq v$ *almost everywhere in* Ω.

We are then able to formulate the existence-uniqueness of p-harmonic functions (cf. [HKM, Theorem 3.17], [HH, Section 4.10]).

Theorem 3.2. *Given a bounded domain* Ω *with boundary data* $\Theta \in W^{1,p}(\Omega)$, *there is a unique* p*-harmonic function* u *that satisfies* $u - \Theta \in W^{1,p}_0(\Omega)$.

Using standard techniques in calculus of variations, one can show that ε-weak solutions exist and Lemma 3.1 can be extended to ε-weak solutions. In addition, ε-weak solutions have a continuous representative [CDG] and therefore such solutions will be identified with that representative.

3.2. p-superharmonic functions. The next class of solutions we wish to consider are p-superharmonic functions and p-subharmonic functions defined via the following definition.

Definition 3. The function $u\colon \Omega \mapsto \mathbf{R}^N \cup \{\infty\}$ is p*-superharmonic* if the following hold:

(1) u is lower semicontinuous.
(2) u is not identically infinity in each component of Ω.
(3) For each subdomain $D \subset\subset \Omega$, a p-harmonic function g in D that is continuous in $\overline{D}$ with $g \leq u$ on ∂D implies $g \leq u$ in D.

A function u is p*-subharmonic* if $-u$ is p-superharmonic.

The key points of these definitions are that they are based on comparison with p-harmonic functions. We then are able to obtain the following comparison principle [KM, Theorem 7.2].

Lemma 3.3. *Let Ω be a bounded domain in $\mathbf{H}_n$. Let v be an p-superharmonic function and u be a p-subharmonic function in Ω so that*

$$\limsup_{Q\to P} u(Q) \le \liminf_{Q\to P} v(Q)$$

for all $P\in\partial\Omega$ with both sides not simultaneously $-\infty$ or ∞. Then $u\le v$ in Ω.

We are then able to conclude the following lemma ([HKM, Lemma 7.8]).

Lemma 3.4. *A function is p-harmonic if an only if it is both p-subharmonic and p-superharmonic.*

3.3. Viscosity solutions. In this subsection, we review the concept of viscosity solution and relate the first two notions of solution to viscosity solutions. Before we begin, we consider equation (3.2) in non-divergence form, namely,

$$-\Big(\|\nabla_0 u\|^{p-2}\operatorname{tr}\big((D^2u)^\star\big) + (p-2)\|\nabla_0 u\|^{p-4}\langle (D^2u)^\star\nabla_0 u, \nabla_0 u\rangle\Big) = \varepsilon. \tag{3.4}$$

We note that equation (3.4) is degenerate elliptic and proper in the sense of [CIL]. Given the function u, we consider the set of functions ϕ that touch from below at the point P_0. That is, the set $\mathscr{T}\mathscr{B}(u,P_0)$ given by

$$\begin{aligned}\mathscr{T}\mathscr{B}(u,P_0) = \big\{\phi\in\mathscr{C}^2_{\text{sub}}(\Omega) : u(P_0)=\phi(P_0),\\ u(P)>\phi(P) \text{ for } P\ne P_0,\ \nabla_0\phi(P_0)\ne 0\big\}.\end{aligned}$$

We are now able to define the concept of viscosity solutions to equation (3.4).

Definition 4. The function $u\colon\Omega\mapsto\mathbf{R}^N\cup\{\infty\}$ is an *ε-viscosity supersolution* to equation (3.4) if the following hold:

(1) u is lower semicontinuous.
(2) u is not identically infinity in each component of Ω.
(3) For $P_0\in\Omega$, $\phi\in\mathscr{T}\mathscr{B}(u,P_0)$ satisfies

$$-\Delta_p\phi(P_0)\ge\varepsilon.$$

A function u is an *ε-viscosity subsolution* to equation (3.4) if $-u$ is an ε-viscosity supersolution. A function u is an *ε-viscosity solution* if it is both an ε-viscosity supersolution and an ε-viscosity subsolution.

We call the collection

$$\left\{\left(\nabla\phi(P_0),(D^2\phi)^\star(P_0)\right):\phi\in\mathscr{TB}(u,P_0)\right\}$$

the subjet of u at P_0 and denote it $J^{2,-}u(P_0)$. We define the superjet of u at P_0 by $J^{2,+}u(P_0)=-J^{2,-}(-u)(P_0)$. The set-theoretic closure $\bar{J}^{2,-}u(P_0)$ is defined by all pairs $\{(\eta,X)\}$ so that there is a sequence $\{(P_n,\phi_n)\}$ with $\left(\nabla\phi_n(P_n),(D^2\phi_n)^\star(P_n)\right)\in J^{2,-}u(P_n)$ so that $P_n\to P_0, u(P_n)\to u(P_0), \nabla\phi_n(P_n)\to\eta$ and $(D^2\phi_n)^\star(P_n)\to X$. For a more complete discussion of jets, the interested reader is directed to [CIL] for the Euclidean environment and [B] for Heisenberg groups.

Given the viscosity solutions, it is natural to ask how they relate to the previous notions of solutions. It was shown via Lemma 4.1 in [B] that upper(lower) semicontinuous ε-weak sub(super-)solutions are ε-viscosity sub(super-)solutions. In addition, we have the following lemma.

Lemma 3.5. *A p-sub(super-)harmonic function is a 0-viscosity sub(super-) solution. Hence, a p-harmonic function is a 0-viscosity solution.*

Proof. We will do only the p-superharmonic case. We let u be a p-superharmonic function in Ω and choose $P_0\in\Omega$. We let $\phi\in\mathscr{C}^2_{\mathrm{sub}}(\Omega)$ be a function so that $\phi(P_0)=u(P_0)$, $\nabla_0\phi(P_0)\neq 0$, and $u(P)>\phi(P)$ for $P\neq P_0$. If $-\Delta_p\phi(P_0)<0$ then by continuity, there is a small $r>0$ so that $-\Delta_p\phi<0$ in the ball $B(P_0,r)$. Note we also have $\nabla_0\phi(P)\neq 0$ in $B(P_0,r)$. Define the strictly positive number $m\equiv\inf\{u(P)-\phi(P): d_C(P,P_0)=r\}$ and the $\mathscr{C}^2_{\mathrm{sub}}$ function $\Phi\equiv\phi+\frac{1}{2}m$. We therefore have Φ is a 0-weak subsolution in $B(P_0,r)$. Let g be the unique (continuous) p-harmonic function equal to Φ on $\partial B(P_0,r)$ whose existence is guaranteed by Theorem 3.2. Using the comparison principle (Lemma 3.1), we conclude $\Phi\le g\le u$ in $B(P_0,r)$, contrary to $\Phi(P_0)>u(P_0)$. □

We now have existence of all three notions of solutions, but a comparison principle only for the first two. In the next section, we correct this deficiency.

4. Equivalence of notions of solution

In this section, our goal is to prove the equivalence of the various notions of solution. As a consequence, we obtain a comparison principle for viscosity solutions to equation (3.4) when $\varepsilon=0$. We recall that Ω is a bounded domain. Our proof will rely heavily on the Heisenberg geometry.

We begin with a technical lemma whose Euclidean version is Lemma 3.2 in [JLM].

Lemma 4.1. *Let $v\in\mathrm{W}^{1,\mathrm{p}}_{\mathrm{loc}}$ be a continuous ε-weak solution. Let $P_0\in\Omega$ and let $\phi\in\mathscr{C}^2_{\mathrm{sub}}(\Omega)$ be a function such that $v-\phi$ has a strict local minimum at P_0. Then*

$$\limsup_{\substack{P\to P_0\\ P\neq P_0}}\left(-\Delta_p\phi(P)\right)\ge\varepsilon$$

provided that $\nabla_0\phi(P_0) \neq 0$ *or* P_0 *is an isolated critical point.*

Proof. By left translation we may assume $P_0 = 0$ and by adding constants, $\phi(0) = v(0)$. If the conclusion is false, then there is an $r_1 > 0$ so that $\nabla_0\phi(P) \neq 0$ and $-\Delta_p\phi(P) < \varepsilon$ when $0 < \mathscr{N}(P) < r_1$. Because $v - \phi$ has a strict local minimum at 0, there is an $r_2 > 0$ so that $v(P) > \phi(P)$ when $0 < \mathscr{N}(P) < r_2$. Let $r = \min\{\frac{1}{2}r_1, \frac{1}{2}r_2\}$. Then $\nabla_0\phi(P) \neq 0$, $-\Delta_p\phi(P) < \varepsilon$ and $v > \phi$ when $0 < \mathscr{N}(P) \leq r$. We next define the strictly positive constant $m = \min\{v(P) - \phi(P) : \mathscr{N}(P) = r\}$ and consider the function $\tilde{\phi} = \phi + \frac{1}{2}m$. By construction, $\tilde{\phi} \in \mathscr{C}^2_{\mathrm{sub}}(\Omega)$ and

$$\bigl(-\Delta_p\tilde{\phi}(P)\bigr) = \bigl(-\Delta_p\phi(P)\bigr).$$

Let $\psi \in C_0^\infty$ be a non-negative test function with compact support contained in the ball $B_{\mathscr{N}}(r)$. Let $0 < \varrho < r$ and consider the annulus $A \equiv B_{\mathscr{N}}(r) \setminus B_{\mathscr{N}}(\varrho)$. Using the identity

$$\operatorname{div}_{\mathscr{H}}(\psi\|\nabla_0\tilde{\phi}\|^{p-2}\nabla_0\tilde{\phi}) = \psi\Delta_p\tilde{\phi} + \|\nabla_0\tilde{\phi}\|^{p-2}\langle\nabla_0\tilde{\phi}, \nabla_0\psi\rangle$$

we obtain

$$\begin{aligned}
\int_A \|\nabla_0\tilde{\phi}\|^{p-2}\langle\nabla_0\tilde{\phi}, \nabla_0\psi\rangle\, dV &= -\int_A \psi\Delta_p\tilde{\phi}\, dV + \int_A \operatorname{div}_{\mathscr{H}}(\psi\|\nabla_0\tilde{\phi}\|^{p-2}\nabla_0\tilde{\phi})\, dV \\
&\leq \varepsilon\int_A \psi\, dV + \int_{\partial A} \psi\|\nabla_0\tilde{\phi}\|^{p-2}\nabla_0\tilde{\phi}\cdot\nu\, dS \\
&\leq \varepsilon\int_{B_{\mathscr{N}}(r)} \psi\, dV - \int_{B_{\mathscr{N}}(\varrho)} \psi\|\nabla_0\tilde{\phi}\|^{p-2}\nabla_0\tilde{\phi}\cdot\nu\, dS
\end{aligned}$$

where ν is the outward unit (Euclidean) normal. We now estimate the last integral.

$$\left|\int_{B_{\mathscr{N}}(\varrho)} \psi\|\nabla_0\tilde{\phi}\|^{p-2}\nabla_0\tilde{\phi}\cdot\nu\, dS\right| \leq \|\psi\|_\infty\|\nabla_0\tilde{\phi}\|_\infty^{p-1}\int_{B_{\mathscr{N}}(\varrho)} dS \lesssim \varrho^{2n+1}.$$

Letting $\varrho \to 0$, we have

$$\int_{B_{\mathscr{N}}(r)} \|\nabla_0\tilde{\phi}\|^{p-2}\langle\nabla_0\tilde{\phi}, \nabla_0\psi\rangle\, dV \leq \varepsilon\int_{B_{\mathscr{N}}(r)} \psi\, dV.$$

Thus, $\tilde{\phi}$ is an ε-weak subsolution.

By the comparison principle for ε-weak solutions, $\tilde{\phi} \leq v$ in $B_{\mathscr{N}}(r)$ because $\tilde{\phi} \leq v$ on $\partial B_{\mathscr{N}}(r)$ by construction. However, we have

$$\tilde{\phi}(0) = \phi(0) + \tfrac{1}{2}m = v(0) + \tfrac{1}{2}m > v(0).$$

This contradiction finishes the proof. □

Note that in the case when $p \geq 2$, by continuity we have $-\Delta_p \phi(P_0) \geq \varepsilon$ and so $\nabla_0 \phi(P) \neq 0$ near P_0.

We next consider the function φ: $\mathbf{H}_n \times \mathbf{H}_n \mapsto \mathbf{R}$ given by

$$\begin{aligned}\varphi(P,Q) &= \frac{1}{m}\sum_{i=1}^{2n}|x_i - y_i|^m + \frac{1}{m}\left|z_1 - z_2 + \frac{1}{2}\sum_{i=1}^{n}(x_{n+i}y_i - x_i y_{n+i})\right|^m \\ &\stackrel{\text{def}}{\equiv} \frac{1}{m}\sum_{i=1}^{2n}|x_i - y_i|^m + \frac{1}{m}|\zeta(P,Q)|^m\end{aligned}$$

for some large positive integer $m \geq 4$. The important properties of φ are found in the following lemma.

Lemma 4.2. *As above, let $m \geq 4$. Let the vector η be given by*

$$\eta = \begin{pmatrix} |x_1 - y_1|^{m-2}(x_1 - y_1) \\ |x_2 - y_2|^{m-2}(x_2 - y_2) \\ \vdots \\ |x_{2n} - y_{2n}|^{m-2}(x_{2n} - y_{2n}) \\ |\zeta(P,Q)|^{m-2}\zeta(P,Q) \end{pmatrix}.$$

Recall that the differential of left multiplication with respect to P, denoted DL_P, is given by

$$\begin{pmatrix} I_{2n\times 2n} & \mathscr{P} \\ 0_{1\times 2n} & 1 \end{pmatrix}$$

where the $2n \times 1$ vector $\mathscr{P}$ is given by

$$\left(-\tfrac{1}{2}x_{n+1}, -\tfrac{1}{2}x_{n+2}, \ldots, -\tfrac{1}{2}x_{2n}, \tfrac{1}{2}x_1, \tfrac{1}{2}x_2, \ldots, \tfrac{1}{2}x_n\right)^T$$

with a similar definition for DL_Q using the $2n \times 1$ vector $\mathscr{Q}$ given by

$$\left(-\tfrac{1}{2}y_{n+1}, -\tfrac{1}{2}y_{n+2}, \ldots, -\tfrac{1}{2}y_{2n}, \tfrac{1}{2}y_1, \tfrac{1}{2}y_2, \ldots, \tfrac{1}{2}y_n\right)^T.$$

Define the $(2n+1)\times(2n+1)$ matrix $\mathscr{M}$ by

$$\mathscr{M}_{ij} = \begin{cases} (m-1)|x_i - y_i|^{m-2}, & i = j,\ 1 \leq i \leq 2n, \\ (m-1)|\zeta(P,Q)|^{m-2}, & i = j = 2n+1, \\ 0, & i \neq j, \end{cases}$$

and denote Euclidean differentiation with respect to the point R by D_R. We then have the following properties:

(1) $D_P\varphi(P,Q) = DL_Q\eta$, $D_Q\varphi(P,Q) = -DL_P\eta$,

(2) $D_P\eta = \mathscr{M} DL_Q^T,\ D_Q\eta = -\mathscr{M} DL_P^T$,
(3) $DL_P DL_Q = DL_Q DL_P$,
(4) $DL_P D_P\varphi(P,Q) = -DL_Q D_Q\varphi(P,Q) \equiv \Upsilon(P,Q)$,

$$\begin{aligned}&DL_P(D_{PP}\varphi(P,Q)DL_P^T + D_{PQ}\varphi(P,Q)DL_Q^T)\\ &\quad = \frac{1}{2}|\zeta(P,Q)|^{m-2}\zeta(P,Q)\begin{pmatrix} 0_{n\times n} & -I_{n\times n} & 0_{n\times 1}\\ I_{n\times n} & 0_{n\times n} & 0_{n\times 1}\\ 0_{1\times n} & 0_{1\times n} & 0cr\end{pmatrix},\end{aligned} \tag{5}$$

$$\begin{aligned}&DL_Q(D_{QQ}\varphi(P,Q)DL_Q^T + D_{QP}\varphi(P,Q)DL_P^T)\\ &\quad = \frac{1}{2}|\zeta(P,Q)|^{m-2}\zeta(P,Q)\begin{pmatrix} 0_{n\times n} & I_{n\times n} & 0_{n\times 1}\\ -I_{n\times n} & 0_{n\times n} & 0_{n\times 1}\\ 0_{1\times n} & 0_{1\times n} & 0\end{pmatrix}.\end{aligned} \tag{6}$$

(7) *Let* $\xi \in h_n$ *be a vector. Then*

$$\begin{aligned}&\langle D_{PP}\varphi(P,Q)DL_P^T\xi, DL_P^T\xi\rangle + \langle D_{PQ}\varphi(P,Q)DL_Q^T\xi, DL_P^T\xi\rangle\\ &\quad + \langle D_{QP}\varphi(P,Q)DL_P^T\xi, DL_Q^T\xi\rangle + \langle D_{QQ}\varphi(P,Q)DL_Q^T\xi, DL_Q^T\xi\rangle = 0\end{aligned}$$

(8) *Let* $\xi \in h_n$ *be a vector. We define* $\bar{\xi}$ *to be the projection of* ξ *onto* V_1. *That is, if* $\xi = (\xi_1, \xi_2, \dots, \xi_{2n+1})$, *then* $\bar{\xi} = (\xi_1, \xi_2, \dots, \xi_{2n})$. *We then have*

$$\left\| \begin{pmatrix} D_{PP}\varphi(P,Q) & D_{PQ}\varphi(P,Q)\\ D_{QP}\varphi(P,Q) & D_{QQ}\varphi(P,Q)\end{pmatrix}\begin{pmatrix} DL_P^T\xi\\ DL_Q^T\xi\end{pmatrix}\right\|^2 = \frac{1}{2}\|\bar{\xi}\|^2|\zeta(P,Q)|^{2m-2}.$$

Proof. The first three properties are elementary calculations and left to the reader. The fourth follows from the first three. We therefore turn our attention to the last four. Let M_{PQ} be the left-hand side of Property (5). Then,

$$\begin{aligned} M_{PQ} &= DL_P\big(D_P(DL_Q\eta)DL_P^T + D_Q(DL_Q\eta)DL_Q^T\big)\\ &= DL_P\big(DL_Q D_P\eta DL_P^T + D_Q(DL_Q)\eta DL_Q^T + DL_Q D_Q\eta DL_Q^T\big)\\ &= DL_P\big(DL_Q\mathscr{M}DL_Q^T DL_P^T + D_Q(DL_Q)\eta DL_Q^T - DL_Q\mathscr{M}DL_P^T DL_Q^T\big)\\ &= DL_P\big(D_Q(DL_Q)\eta DL_Q^T\big)\end{aligned}$$

and so we are left to compute only the derivative of the matrix DL_Q. Knowing the definition of η above and the formula for DL_Q as given above, we see that when $1 \le i \le n$, we have $D_{y_i}(DL_Q)$ is a matrix with every entry 0 except for the $(i+n, 2n+1)$ entry, which is $\frac{1}{2}$. When $n < i < 2n$, we have $D_{y_i}(DL_Q)$ has all entries 0 except for the $(i-n, 2n+1)$ entry, which is $-\frac{1}{2}$. Clearly, $D_{z_2}(DL_Q)$ is the 0 matrix. We then compute

$$D_Q(DL_Q)\eta = \frac{1}{2}|\zeta(P,Q)|^{m-2}\zeta(P,Q)\begin{pmatrix} 0_{n\times n} & -I_{n\times n} & 0_{n\times 1} \\ I_{n\times n} & 0_{n\times n} & 0_{n\times 1} \\ 0_{1\times n} & 0_{1\times n} & 0 \end{pmatrix}.$$

We then have

$$\begin{aligned}
&\begin{pmatrix} I_{2n\times 2n} & \mathscr{P} \\ 0_{1\times 2n} & 1 \end{pmatrix}\begin{pmatrix} 0_{n\times n} & -I_{n\times n} & 0_{n\times 1} \\ I_{n\times n} & 0_{n\times n} & 0_{n\times 1} \\ 0_{1\times n} & 0_{1\times n} & 0 \end{pmatrix}\begin{pmatrix} I_{2n\times 2n} & 0_{2n\times 1} \\ \mathscr{Q}^T & 1 \end{pmatrix} \\
&= \begin{pmatrix} I_{2n\times 2n} & \mathscr{P} \\ 0_{1\times 2n} & 1 \end{pmatrix}\begin{pmatrix} 0_{n\times n} & -I_{n\times n} & 0_{n\times 1} \\ I_{n\times n} & 0_{n\times n} & 0_{n\times 1} \\ 0_{1\times n} & 0_{1\times n} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0_{n\times n} & -I_{n\times n} & 0_{n\times 1} \\ I_{n\times n} & 0_{n\times n} & 0_{n\times 1} \\ 0_{1\times n} & 0_{1\times n} & 0 \end{pmatrix}
\end{aligned}$$

and Property (5) follows. To prove Property (6), we let M_{PQ} be the left-hand side of the identity. Then,

$$\begin{aligned}
M_{PQ} &= DL_Q\big(-D_Q(DL_P\eta)DL_Q^T + D_P(-DL_P\eta)DL_P^T\big) \\
&= DL_Q\big(-DL_P D_Q\eta DL_Q^T - D_P(DL_P)\eta DL_P^T - DL_P D_P\eta DL_P^T\big) \\
&= DL_Q\big(DL_P\mathscr{M} DL_P^T DL_Q^T - D_P(DL_P)\eta DL_P^T - DL_P\mathscr{M} DL_Q^T DL_P^T\big) \\
&= -DL_Q\big(D_P(DL_P)\eta DL_P^T\big)
\end{aligned}$$

and we compute $D_P(DL_P)\eta$ in the same way as the above computation for $D_Q(DL_Q)\eta$ and arrive at Property (6).

To prove Property (7), we note that the right-hand side can be written as

$$\begin{aligned}
&\langle DL_P(D_{PP}\varphi(P,Q)DL_P^T + D_{PQ}\varphi(P,Q)DL_Q^T)\xi, \xi\rangle \\
&\quad + \langle DL_Q(D_{QQ}\varphi(P,Q)DL_Q^T + D_{QP}\varphi(P,Q)DL_P^T)\xi, \xi\rangle.
\end{aligned}$$

Using Properties (5) and (6) we see this is zero. Property (7) then follows.

Using the proofs of Properties (5) and (6), we have

$$\begin{aligned}
&\begin{pmatrix} D_{PP}\varphi(P,Q) & D_{PQ}\varphi(P,Q) \\ D_{QP}\varphi(P,Q) & D_{QQ}\varphi(P,Q) \end{pmatrix} \begin{pmatrix} DL_P^T\xi \\ DL_Q^T\xi \end{pmatrix} \\
&\quad = \begin{pmatrix} (D_{PP}\varphi(P,Q)DL_P^T + D_{PQ}\varphi(P,Q)DL_Q^T)\xi \\ (D_{QP}\varphi(P,Q)DL_P^T + D_{QQ}\varphi(P,Q)DL_Q^T)\xi \end{pmatrix} \\
&\quad = \frac{1}{2}|\zeta(P,Q)|^{m-2}\zeta(P,Q) \begin{bmatrix} \begin{pmatrix} 0_{n\times n} & -I_{n\times n} & 0_{n\times 1} \\ I_{n\times n} & 0_{n\times n} & 0_{n\times 1} \\ 0_{1\times n} & 0_{1\times n} & 0 \end{pmatrix}\xi \\ \begin{pmatrix} 0_{n\times n} & I_{n\times n} & 0_{n\times 1} \\ -I_{n\times n} & 0_{n\times n} & 0_{n\times 1} \\ 0_{1\times n} & 0_{1\times n} & 0 \end{pmatrix}\xi \end{bmatrix}.
\end{aligned}$$

We then see that

$$\left\| \begin{pmatrix} D_{PP}\varphi(P,Q) & D_{PQ}\varphi(P,Q) \\ D_{QP}\varphi(P,Q) & D_{QQ}\varphi(P,Q) \end{pmatrix} \begin{pmatrix} DL_P^T\xi \\ DL_Q^T\xi \end{pmatrix} \right\|^2 = \frac{1}{2}\|\bar{\xi}\|^2|\zeta(P,Q)|^{2m-2}$$

and Property (8) is proved. □

We will use the function φ above as a penalty function in our proof of a preliminary comparison principle. A key step in the proof will be the twisting of the Euclidean jets into Heisenberg jets.

Lemma 4.3 ([B, Lemma 3.4]). *Let DL_{P_0} be the differential of the left multiplication map at the point P_0, let $J^{2,+}_{\text{eucl}}u(P_0)$ be the traditional Euclidean superjet of u at the point P_0 and let $(\eta, X) \in \mathbf{R}^{2n+1} \times S^{2n+1}$. Then,*

$$(\eta, X) \in \bar{J}^{2,+}_{\text{eucl}}u(P_0)$$

gives the element

$$\big(DL_{P_0}\eta, (DL_{P_0}\, X\, (DL_{P_0})^T)_{2n}\big) \in \bar{J}^{2,+}u(P_0)$$

with the convention that for any matrix M, M_n is the $n\times n$ principal minor.

We now prove a preliminary comparison principle.

Theorem 4.4. *Fix $\varepsilon > 0$ and $1 < p < \infty$. Let v be a continuous ε-weak solution and let u be a 0-viscosity subsolution so that $u \le v$ on $\partial\Omega$. Then $u \le v$ in Ω.*

Proof. Suppose that $\sup(u-v)>0$ occurs at the interior point P_0. For each positive integer j, we consider the function $\psi_j\colon \mathbf{H}_n\times\mathbf{H}_n\mapsto\mathbf{R}$ defined by

$$\psi_j(P,Q)=u(P)-v(Q)-j\varphi(P,Q)$$

where $\varphi(P,Q)$ is the function from Lemma 4.2, with m chosen so that $m>\max\{4,p/(p-1),p\}$. Following the scheme of [B] and [CIL], we let the maximum of ψ_j occur at (P_j,Q_j) and observe for large j, these are interior points. In addition, these points tend to P_0 as $j\to\infty$. Using the Euclidean results of [CIL] and the above twisting lemma, we have

$$\big(j\Upsilon(P_j,Q_j),\mathscr{X}_j\big)\in\bar J^{2,+}u(P_j)\quad\text{and}\quad(j\Upsilon(P_j,Q_j),\mathscr{Y}_j)\in\bar J^{2,-}v(Q_j)$$

where $\Upsilon(P,Q)=DL_PD_P\varphi(P,Q)=-DL_QD_Q\varphi(P,Q)$ as detailed in Lemma 4.2.

Claim 4.5. *By passing to a subsequence if needed, we may assume $P_j\neq Q_j$.*

Proof. Fix $j>0$. By definition, we have for any P and Q,

$$u(P)-v(Q)-j\varphi(P,Q)\le u(P_j)-v(Q_j)-j\varphi(P_j,Q_j)$$

and so when $P_j=P$, we have

$$v(Q)\ge v(Q_j)+j\varphi(P_j,Q_j)-j\varphi(P_j,Q).$$

Defining the function $\beta(Q)$ by

$$\beta(Q)=v(Q_j)+j\varphi(P_j,Q_j)-j\varphi(P_j,Q)-\varphi(Q_j,Q)$$

we see that $v-\beta$ has a strict local minimum at Q_j and Q_j is an isolated critical point. Applying Lemma 4.1, we have

$$\limsup_{Q\to Q_j}\big(-\Delta_p\beta(Q)\big)\ge\varepsilon. \tag{4.1}$$

Suppose now that $P_j=Q_j$. Then $\beta(Q)=v(Q_j)-(j+1)\varphi(Q_j,Q)$. We then need to estimate $\Delta_p\beta(Q)$. Using the non-divergence form of the p-Laplacian (equation (3.4)) and the definition of $\beta(Q)$, we have

$$|\Delta_p\beta(Q)|\lesssim\|\nabla_0\varphi(Q_j,Q)\|^{p-2}\big|\operatorname{tr}(D^2\varphi)^\star(Q_j,Q)+\|(D^2\varphi)^\star(Q_j,Q)\|\,\big|.$$

Using Lemma 4.2, we have

$$\|\nabla_0\varphi(Q_j,Q)\|\sim\|\eta\|\sim\varphi(Q_j,Q)^{(m-1)/m}.$$

We note that given the standard vectors e_k with every entry 0 except for the kth entry which is equal to 1, we see that for any matrix A,

$$\operatorname{tr}(A)=\sum\langle Ae_k,e_k\rangle$$

and so

$$|\operatorname{tr}(D^2\varphi)^\star(Q_j,Q)|\lesssim\|(D^2\varphi)^\star(Q_j,Q)\|.$$

We then conclude via Lemma 4.2

$$\Bigl|\operatorname{tr}(D^2\varphi)^\star(Q_j,Q)+\|(D^2\varphi)^\star(Q_j,Q)\|\,\Bigr|\lesssim\|\mathscr{M}\|\sim\varphi(Q_j,Q)^{(m-2)/m}$$

so that

$$|\Delta_p\beta(Q)|\lesssim\bigl(\varphi(Q_j,Q)^{1/m}\bigr)^{(m-1)(p-2)+(m-2)}.$$

Since $m>p/(p-1)$, we would have

$$\lim_{\substack{Q\to Q_j\\ Q\neq Q_j}}\bigl(-\Delta_p\beta(Q)\bigr)=0.$$

This contradicts equation (4.1). □

Proceeding as in [B], u is a viscosity subsolution to equation (3.4) with $\varepsilon=0$. That is,

$$\begin{aligned}0\geq-\Bigl(&\|j\Upsilon(P_j,Q_j)\|^{p-2}\operatorname{tr}(\mathscr{X}_j)^\star\\&+(p-2)\|j\Upsilon(P_j,Q_j)\|^{p-4}\langle\mathscr{X}_j\,j\Upsilon(P_j,Q_j),j\Upsilon(P_j,Q_j)\rangle\Bigr).\end{aligned}$$

Using Lemmas 3.5 and 4.1 along with the definition of $\bar{J}^{2,-}$, we have

$$\begin{aligned}\varepsilon\leq-\Bigl(&\|j\Upsilon(P_j,Q_j)\|^{p-2}\operatorname{tr}(\mathscr{Y}_j)^\star\\&+(p-2)\|j\Upsilon(P_j,Q_j)\|^{p-4}\langle\mathscr{Y}_j\,j\Upsilon(P_j,Q_j),j\Upsilon(P_j,Q_j)\rangle\Bigr).\end{aligned}$$

Subtracting these two inequalities, we have

$$\begin{aligned}0<\varepsilon<{}&j^{p-2}\|\Upsilon(P_j,Q_j)\|^{p-2}\bigl(\operatorname{tr}(\mathscr{X}_j)-\operatorname{tr}(\mathscr{Y}_j)\bigr)\\&+(p-2)j^{p-2}\|\Upsilon(P_j,Q_j)\|^{p-4}\Bigl(\langle\mathscr{X}_j\Upsilon(P_j,Q_j),\Upsilon(P_j,Q_j)\rangle\\&\qquad-\langle\mathscr{Y}_j\Upsilon(P_j,Q_j),\Upsilon(P_j,Q_j)\rangle\Bigr).\end{aligned}\tag{4.2}$$

As in the proof of the above claim, we have

$$\|\Upsilon(P_j,Q_j)\|\sim\varphi(P_j,Q_j)^{(m-1)/m}.$$

Given a vector $\eta \in V_1$, we denote its extension to h_n by $\tilde{\eta}$. That is, $\eta = (\eta_1, \eta_2, \ldots, \eta_{2n})$ yields $\tilde{\eta} = (\eta_1, \eta_2, \ldots, \eta_{2n}, 0)$. Using the formulas for the matrices $\mathscr{X}_j$ and $\mathscr{Y}_j$ given by Lemma 4.3 and the standard estimate on the matrix ordering ([CIL, Theorem 3.2]) produces

$$\begin{aligned}
&\langle \mathscr{X}_j \Upsilon(P_j, Q_j), \Upsilon(P_j, Q_j)\rangle - \langle \mathscr{Y}_j \Upsilon(P_j, Q_j), \Upsilon(P_j, Q_j)\rangle \\
&\quad = \langle X(DL_{P_j}^T \widetilde{\Upsilon}(P_j, Q_j)), DL_{P_j}^T \widetilde{\Upsilon}(P_j, Q_j)\rangle \\
&\qquad - \langle Y(DL_{Q_j}^T \widetilde{\Upsilon}(P_j, Q_j)), DL_{Q_j}^T \widetilde{\Upsilon}(P_j, Q_j)\rangle \\
&\quad \le j\langle \mathscr{D}\xi, \xi\rangle
\end{aligned}$$

where the matrix $\mathscr{D}$ is the Euclidean second derivative of φ given by

$$\mathscr{D} = \begin{bmatrix} D_{PP}\varphi(P_j, Q_j) & D_{PQ}\varphi(P_j, Q_j) \\ D_{QP}\varphi(P_j, Q_j) & D_{QQ}\varphi(P_j, Q_j) \end{bmatrix}$$

and the vector

$$\xi = \big(DL_{P_j}^T \widetilde{\Upsilon}(P_j, Q_j) \oplus DL_{Q_j}^T \widetilde{\Upsilon}(P_j, Q_j)\big).$$

We then conclude via Properties (7) and (8) that

$$\begin{aligned}
&\langle \mathscr{X}_j \Upsilon(P_j, Q_j), \Upsilon(P_j, Q_j)\rangle - \langle \mathscr{Y}_j \Upsilon(P_j, Q_j), \Upsilon(P_j, Q_j)\rangle \\
&\quad \lesssim j\|\Upsilon(P_j, Q_j)\|^2 \varphi(P_j, Q_j)^{(2m-2)/m} \\
&\quad \lesssim j\varphi(P_j, Q_j)^{(2m-2)/m} \varphi(P_j, Q_j)^{(2m-2)/m} \\
&\quad = j\varphi(P_j, Q_j)^{(4m-4)/m}.
\end{aligned}$$

As in the proof of the claim, we write the trace difference as

$$\operatorname{tr}(\mathscr{X}_j) - \operatorname{tr}(\mathscr{Y}_j) = \sum_{k=1}^{2n} \langle \mathscr{X}_j e_k, e_k\rangle - \langle \mathscr{Y}_j e_k, e_k\rangle$$

and following the previous calculation, we obtain

$$\operatorname{tr}(\mathscr{X}_j) - \operatorname{tr}(\mathscr{Y}_j) \lesssim j\big(\varphi(P_j, Q_j)^{(2m-2)/m}\big)$$

so that with equation (4.2), we obtain

$$\begin{aligned}
0 < \varepsilon &\lesssim j^{p-1}\big(\varphi(P_j, Q_j)^{(m-1)/m}\big)^{p-2} \varphi(P_j, Q_j)^{(2m-2)/m} \\
&\quad + j^{p-1}\big(\varphi(P_j, Q_j)^{(m-1)/m}\big)^{p-4} \big(\varphi(P_j, Q_j)^{(4m-4)/m}\big) \\
&\sim j^{p-1}\big(\varphi(P_j, Q_j)^{1/m}\big)^{p(m-1)}.
\end{aligned}$$

Since $m > p$, we have $\big(p(m-1)\big)(1/m) > p-1$. We arrive at a contradiction as $j \to \infty$. □

It is here where we stray significantly from the Euclidean proof in [JLM]. That proof relies on the $C^{1,\alpha}$ regularity of the solutions. This is not known for the Heisenberg group, although a recent result [DM] has proved this regularity for p near 2. We therefore adopt a completely different approach beginning with the next lemma. The first difference is that we only have a weaker version of Lemma 3.1 in [JLM], in that our sequence converges pointwise instead of locally uniformly.

Lemma 4.6. *Let v be a p-harmonic function in Ω. For each $\varepsilon \geq 0$, let v_ε be the continuous ε-weak solution equal to v on the boundary. Then $v_\varepsilon \to v$ pointwise as $\varepsilon \to 0$.*

Proof. Arguing as in [JLM], we see that $v_\varepsilon \to v$ in L^p. We may assume that $\varepsilon \leq 1$ and observe that as noted above, if $\varepsilon_1 > \varepsilon_2$, then v_{ε_2} is a weak subsolution to equation (3.2) for ε_1. By the comparison principle (Lemma 3.1), we have that $v_{\varepsilon_2} \leq v_{\varepsilon_1}$ when $\varepsilon_1 > \varepsilon_2$. In particular for all $\varepsilon > 0$, $v \leq v_\varepsilon$. We then conclude that

$$w = \lim_{\varepsilon \to 0} v_\varepsilon = \inf_{\varepsilon > 0} \{v_\varepsilon\}$$

exists and $v \leq w$. Since $v_\varepsilon \to w$ pointwise, we have $|v_\varepsilon|^p \to |w|^p$. By the Lebesque dominated convergence theorem, (using v_1 as dominator) we have $v_\varepsilon \to w$ in L^p so that actually, $v = w$. □

Combining the previous theorem and lemma, we obtain the following consequence.

Lemma 4.7. *Let $1 < p < \infty$. 0-viscosity subsolutions are p-subharmonic. 0-viscosity supersolutions are p-superharmonic and 0-viscosity solutions are p-harmonic.*

Proof. The last statement follows from the first two and the second follows from the first by replacing u with $-u$. We let u be a 0-viscosity subsolution that is not p-subharmonic. Then there is a p-harmonic function v so that $u \leq v$ on $\partial\Omega$ but for some $P \in \Omega$, we have $u(P) > v(P)$. For $\varepsilon \leq 1$, we let v_ε be ε-weak solutions equal to v on $\partial\Omega$ so that $u \leq v_\varepsilon$ on $\partial\Omega$. By Lemma 4.6 we conclude for some ε near 0, $u(P) > v_\varepsilon(P)$, contrary to Theorem 4.4. □

Combining Lemmas 3.5 and 4.7, we have the following corollary.

Corollary 4.8. *Let $1 < p < \infty$. Then 0-viscosity sub(super-)solutions to equation (3.4) and p-sub(super-)harmonic functions coincide. In particular, for $1 < p < \infty$, a function is p-harmonic if and only if it is a 0-viscosity solution to equation (3.4).*

We are then able to conclude the following comparison principle.

Corollary 4.9. *Let $\varepsilon = 0$. Let v be a viscosity supersolution to equation* (3.4) *and let u be a viscosity subsolution of equation* (3.4) *so that $u \le v$ on $\partial\Omega$. Then $u \le v$ in Ω.*

References

[Be] Bellaïche, A.: The tangent space in sub-Riemannian geometry. - In: Sub-Riemannian Geometry, edited by A. Bellaïche and J.-J. Risler, Progress in Mathematics 144, Birkhäuser, Basel, 1996, 1–78.

[B] Bieske, T.: On infinite harmonic functions on the Heisenberg group. - Comm. Partial Differential Equations 27, 2002, 727–762.

[CDG] Capogna, L., D. Danielli, and N. Garofalo: An embedding theorem and the Harnack inequality for nonlinear subelliptic equations. - Comm. Partial Differential Equations 18, 1993, 1765–1794.

[C] Crandall, M.: Viscosity Solutions: A Primer. - Lecture Notes in Math. 1660, Springer-Verlag, Berlin, 1997.

[CIL] Crandall, M., H. Ishii, and P.-L. Lions: User's guide to viscosity solutions of second order partial differential equations. - Bull. Amer. Math. Soc. 27, 1992, 1–67.

[DM] Domokos, A., and J. Manfredi: $C^{1,\alpha}$ regularity for p-harmonic functions in the Heisenberg group for p near 2. - In: The p-harmonic Equation and Recent Advances in Analysis, Contemp. Math. 370, Amer. Math. Soc., Providence, RI, 2005, 17–23.

[F] Folland, G. B.: Subelliptic estimates and function spaces on nilpotent Lie groups. - Ark. Mat. 13, 1975, 161–207.

[FS] Folland, G. B., and E. M. Stein: Hardy Spaces on Homogeneous Groups. - Princeton Univ. Press, Princeton, NJ, 1982.

[G] Gromov, M.: Metric Structures for Riemannian and Non-Riemannian Spaces. - Birkhäuser Boston Inc., Boston, 1999.

[H] Heinonen, J.: Calculus on Carnot groups. - In: Fall School in Analysis Report No. 68, Fall School in Analysis, Jyväskylä, 1994, Univ. Jyväskylä, Jyväskylä, 1995, 1–31.

[HH] Heinonen, J., and I. Holopainen: Quasiregular maps on Carnot groups. - J. Geom. Anal. 7, 1997, 109–148.

[HKM] Heinonen, J., T. Kilpeläinen, and O. Martio: Nonlinear Potential Theory of Degenerate Elliptic Equations. - Oxford Mathematical Monographs, Oxford Univ. Press, New York, 1993.

[JLM] Juutinen, P., P. Lindqvist, and J. Manfredi: On the equivalence of viscosity solutions and weak solutions for a quasi-linear equation. - SIAM J. Math. Anal. 33, 2001, 699–717.

[K] Kaplan, A.: Lie groups of Heisenberg type. - Rend. Sem. Mat. Univ. Politec. Torino 1983, Special Issue 1984, 117–130.

[KM] Kinnunen, J., and O. Martio: Nonlinear potential theory on metric spaces. - Illinois J. Math. 46, 2002, 857–883.

[S] Stein, E. M.: Harmonic Analysis. - Princeton Univ. Press, Princeton, NJ, 1993.

Received 20 April 2005