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# SOBOLEV EMBEDDINGS FOR VARIABLE EXPONENT RIESZ POTENTIALS ON METRIC SPACES

**Toshihide Futamura, Yoshihiro Mizuta and Tetsu Shimomura**

Daido Institute of Technology, Department of Mathematics
Nagoya 457-8530, Japan; futamura@daido-it.ac.jp

Hiroshima University, The Division of Mathematical and Information Sciences
Faculty of Integrated Arts and Sciences, Higashi-Hiroshima 739-8521, Japan;
mizuta@mis.hiroshima-u.ac.jp

Hiroshima University, Graduate School of Education
Department of Mathematics, Higashi-Hiroshima 739-8524, Japan; tshimo@hiroshima-u.ac.jp

**Abstract.** In the metric space setting, our aim in this paper is to deal with the boundedness of Hardy–Littlewood maximal functions in generalized Lebesgue spaces $L^{p(\cdot)}$ when $p(\cdot)$ satisfies a log-Hölder condition. As an application of the boundedness of maximal functions, we study Sobolev's embedding theorem for variable exponent Riesz potentials on metric space.

## 1. Introduction

Let $X$ be a metric space with a metric $d$. Write $d(x,y)=|x-y|$ for simplicity. We denote by $B(x,r)$ the open ball centered at $x\in X$ of radius $r>0$. Let $\mu$ be a Borel measure on $X$. Assume that $0<\mu(B)<\infty$ and there exist constants $C>0$ and $s\geq 1$ such that

$$\frac{\mu(B')}{\mu(B)}\geq C\left(\frac{r'}{r}\right)^{s} \tag{1.1}$$

for all balls $B=B(x,r)$ and $B'=B(x',r')$ with $x'\in B$ and $0<r'\leq r$. Note that $\mu$ is a doubling measure on $X$, that is, there exists a constant $C'>0$ such that

$$\mu\bigl(B(x,2r)\bigr)\leq C'\mu\bigl(B(x,r)\bigr) \tag{1.2}$$

for all $x\in X$ and $r>0$.

We define the Riesz potential of order $\alpha$ for a locally integrable function $f$ on $X$ defined by

$$U_\alpha f(x)=\int_X \frac{|x-y|^\alpha f(y)}{\mu\bigl(B(x,|x-y|)\bigr)}\,d\mu(y).$$

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Here $\alpha > 0$. Following Orlicz [26] and Kováčik and Rákosník [19], we consider a positive continuous function $p(\cdot)$ on $X$ and a function $f$ satisfying

$$\int_X |f(y)|^{p(y)}\, d\mu(y) < \infty.$$

In this paper we treat $p(\cdot)$ such that $p > 1$ on $X$ and $p$ satisfies a log-Hölder condition:

$$|p(x) - p(y)| \le \frac{a_1 \log\bigl(\log(1/|x-y|)\bigr)}{\log(1/|x-y|)} + \frac{a_2}{\log(1/|x-y|)}$$

whenever $|x-y| < \frac{1}{4}$, where $a_1$ and $a_2$ are nonnegative constants. If $a_1 > 0$, then we can not expect the usual boundedness of maximal functions in $L^{p(\cdot)}$, according to the recent works by Diening [3], [4], Pick and Růžička [27] and Cruz-Uribe, Fiorenza and Neugebauer [2]. Our typical example is a variable exponent $p(\cdot)$ on $X$ such that

$$p(x) = p_0 + \frac{a_1 \log\bigl(\log\bigl(1/\varrho_K(x)\bigr)\bigr)}{\log\bigl(1/\varrho_K(x)\bigr)} + \frac{a_2}{\log\bigl(1/\varrho_K(x)\bigr)}$$

when $\varrho_K(x)$ is small, where $p_0 > 1$, $a_1 \ge 0$, $a_2 \ge 0$ and $\varrho_K(x)$ denotes the distance of $x$ from a compact subset $K$ of $X$.

Our first task is then to establish the boundedness of Hardy–Littlewood maximal functions from $L^{p(\cdot)}$ to some Orlicz classes, as an extension of Harjulehto–Hästö–Pere [13] with $a_1 = 0$ in metric setting and the authors' [11, Theorem 2.4] in Euclidean setting. As an application of the boundedness of maximal functions, we establish Sobolev's embedding theorem for variable exponent Riesz potentials on metric space; in the case $a_1 = 0$, see also Diening [4] and Kokilashvili–Samko [17], [18],

In the borderline case of Sobolev's theorem, we are concerned with exponential integrabilities of Trudinger type, which extend the results by Edmunds–Gurka–Opic [5], [6], Edmunds–Krbec [7] and the authors' [9], [22]. We also discuss the pointwise continuity of Riesz potentials defined in the $n$-dimensional Euclidean space, as an extension of the authors [9], [23], [24].

For related results, see Adams–Hedberg [1], Heinonen [16], Musielak [25] and Růžička [28].

## 2. Variable exponents

Throughout this paper, let $C$ denote various constants independent of the variables in question.

Let $G$ be a bounded open set in $X$. In this section let us assume that $p(\,\cdot\,)$ is a positive continuous function on $G$ satisfying:

(p1) $1 < p_-(G) = \inf_G p(x) \le \sup_G p(x) = p_+(G) < \infty$;

(p2) $|p(x) - p(y)| \le \dfrac{a_1 \log\bigl(\log(1/|x-y|)\bigr)}{\log(1/|x-y|)} + \dfrac{a_2}{\log(1/|x-y|)}$,
whenever $|x-y| < \frac{1}{4}$, $x \in G$ and $y \in G$, for some constants $a_1 \ge 0$ and $a_2 \ge 0$.

**Example 2.1.** Let $F$ be a closed subset of $G$. For $a \ge 0$ and $b \ge 0$, consider

$$p(x) = p_0 + \omega_{a,b}\bigl(\varrho_F(x)\bigr),$$

where $1 < p_0 < \infty$, $\varrho_F(x)$ denotes the distance of $x$ from $F$ and

$$\omega_{a,b}(t) = \frac{a \log\bigl(\log(1/t)\bigr)}{\log(1/t)} + \frac{b}{\log(1/t)}$$

for $0 < t \le r_0$ $(< \frac{1}{4})$; set $\omega_{a,b}(t) = \omega_{a,b}(r_0)$ when $t > r_0$ and $\omega_{a,b}(0) = 0$. Then we can find $r_0 > 0$ sufficiently small that $p$ satisfies (p1) and (p2).

For a proof, we prepare the following result.

**Lemma 2.2.** *Let $\omega$ be a nonnegative continuous function on the interval $[0, r_0]$ such that*

(i) $\omega(0) = 0$;

(ii) $\omega'(t) \ge 0$ *for* $0 < t \le r_0$;

(iii) $\omega''(t) \le 0$ *for* $0 < t \le r_0$.

*Then*

$$\omega(s+t) \le \omega(s) + \omega(t) \quad \text{for } s,t \ge 0 \text{ and } s+t \le r_0. \tag{2.1}$$

It is easy to find $r_0 \in \bigl(0, \frac{1}{4}\bigr)$ such that $\omega_{a,b}$ satisfies (i)–(iii) on $[0, r_0]$.
Let

$$1/p'(x) = 1 - 1/p(x).$$

Then, noting that

$$\begin{aligned} p'(y) - p'(x) &= \frac{p(x) - p(y)}{\bigl(p(x)-1\bigr)\bigl(p(y)-1\bigr)} \\ &= \frac{p(x) - p(y)}{\bigl(p(x)-1\bigr)^2} + \frac{\{p(x)-p(y)\}^2}{\bigl(p(x)-1\bigr)^2\bigl(p(y)-1\bigr)}, \end{aligned} \tag{2.2}$$

we have the following result.

**Lemma 2.3.** *There exists a positive constant $c$ such that*

$$|p'(x) - p'(y)| \le \omega_{a,c}(|x-y|) \quad \text{whenever } x \in G \text{ and } y \in G,$$

*where* $a = a(x) = a_1\bigl(p(x)-1\bigr)^{-2}$.

## 3. Boundedness of maximal functions

Define the $L^{p(\cdot)}(G)$ norm by

$$\|f\|_{p(\cdot)} = \|f\|_{p(\cdot),G} = \inf\left\{\lambda > 0 : \int_G \left|\frac{f(y)}{\lambda}\right|^{p(y)} d\mu(y) \le 1\right\}$$

and denote by $L^{p(\cdot)}(G)$ the space of all measurable functions $f$ on $G$ with $\|f\|_{p(\cdot)} < \infty$.

By the decay condition (1.1), we have

$$\mu\big(B(x,r)\big) \ge Cr^s \tag{3.1}$$

for all $x \in G$ and $0 < r < d_G$, where $d_G$ denotes the diameter of $G$. For $f \in L^{p(\cdot)}(G)$, define the maximal function

$$\begin{aligned} Mf(x) &= \sup_{r>0} \frac{1}{\mu\big(B(x,r)\big)} \int_{G\cap B(x,r)} |f(y)|\, d\mu(y) \\ &= \sup_{0<r<d_G} \frac{1}{\mu\big(B(x,r)\big)} \int_{G\cap B(x,r)} |f(y)|\, d\mu(y). \end{aligned}$$

Our first aim is to discuss the boundedness of the maximal functions.

**Theorem 3.1.** *Let $a > a_1$ when $a_1 > 0$ and $a = 0$ when $a_1 = 0$. Set $A(x) = as/p(x)^2$. If $\|f\|_{p(\cdot)} \le 1$, then*

$$\int_G \left\{Mf(x)\big(\log\big(e + Mf(x)\big)\big)^{-A(x)}\right\}^{p(x)} d\mu(x) \le C.$$

When $a_1 = 0$, Theorem 3.1 was proved by Harjulehto–Hästö–Pere [13], which is an extension of Diening [3]. For the boundedness of maximal functions in general domains, see Cruz-Uribe, Fiorenza and Neugebauer [2].

**Remark 3.2.** Set $\Phi(r,x) = \left\{r\big(\log(e+r)\big)^{-A(x)}\right\}^{p(x)}$ for $r > 0$ and $x \in G$. Then Theorem 3.1 assures the existence of $C > 0$ such that

$$\int_G \Phi\big(Mf(x)/C, x\big)\, d\mu(x) \le 1 \quad \text{whenever } \|f\|_{p(\cdot)} \le 1.$$

As in Edmunds and Rákosník [8], we define

$$\|f\|_\Phi = \|f\|_{\Phi,G} = \inf\left\{\lambda > 0 : \int_G \Phi\big(|f(x)|/\lambda, x\big)\, d\mu(x) \le 1\right\};$$

then it follows that

$$\|Mf\|_\Phi \le C\|f\|_{p(\cdot)} \quad \text{for } f \in L^{p(\cdot)}(G).$$

Theorem 3.1 is proved along the same lines as in the authors' [11, Theorem 2.4], but we give a proof of Theorem 3.1 for the readers' convenience.

To complete the proof, we prepare the following lemma.

**Lemma 3.3.** *Let $f$ be a nonnegative measurable function on $G$ with $\|f\|_{p(\,\cdot\,)} \le 1$. Then*

$$Mf(x) \le C\bigl\{Mg(x)^{1/p(x)}\bigl(\log\bigl(e+Mg(x)\bigr)\bigr)^{A_1(x)}+1\bigr\},$$

*where $g(y)=f(y)^{p(y)}$ and $A_1(x)=a_1s/p(x)^2$.*

*Proof.* Let $f$ be a nonnegative measurable function on $G$ with $\|f\|_{p(\,\cdot\,)} \le 1$. First note that

$$\int_G f(y)^{p(y)}\,d\mu(y) \le 1. \tag{3.2}$$

Then, if $r \ge r_0$, then

$$\frac{1}{\mu\bigl(B(x,r)\bigr)}\int_{B(x,r)} f(y)\,d\mu(y) \le \frac{1}{\mu\bigl(B(x,r)\bigr)}\int_{B(x,r)}\{1+f(y)^{p(y)}\}\,d\mu(y) \le C \tag{3.3}$$

by our assumption.

For $0<k\le 1$ and $r>0$, we have by Lemma 2.3

$$\begin{aligned}
&\frac{1}{\mu\bigl(B(x,r)\bigr)}\int_{B(x,r)} f(y)\,d\mu(y)\\
&\quad\le k\biggl\{\frac{1}{\mu\bigl(B(x,r)\bigr)}\int_{B(x,r)}(1/k)^{p'(y)}\,d\mu(y)+\frac{1}{\mu\bigl(B(x,r)\bigr)}\int_{B(x,r)} f(y)^{p(y)}\,d\mu(y)\biggr\}\\
&\quad\le k\{(1/k)^{p'(x)+\omega(r)}+F\},
\end{aligned}$$

where $F=\bigl(\mu\bigl(B(x,r)\bigr)\bigr)^{-1}\int_{B(x,r)} f(y)^{p(y)}\,d\mu(y)$ and $\omega(r)=\omega_{a,c}(r)$ as in Example 2.1. Here, considering

$$k = F^{-1/\{p'(x)+\omega(r)\}} = F^{-1/p'(x)+\beta(x)}$$

with $\beta(x)=\omega(r)/\bigl\{p'(x)\bigl(p'(x)+\omega(r)\bigr)\bigr\}$ when $F\ge 1$, we have

$$\frac{1}{\mu\bigl(B(x,r)\bigr)}\int_{B(x,r)} f(y)\,d\mu(y) \le 2F^{1/p(x)}F^{\omega(r)/p'(x)^2};$$

if $F<1$, then we can take $k=1$ to obtain

$$\frac{1}{\mu\bigl(B(x,r)\bigr)}\int_{B(x,r)} f(y)\,d\mu(y) \le 2.$$

Hence it follows that

$$\frac{1}{\mu\big(B(x,r)\big)} \int_{B(x,r)} f(y)\, d\mu(y) \le 2\big\{F^{1/p(x)} F^{\omega(r)/p'(x)^2} + 1\big\}. \tag{3.4}$$

If $r \le F^{-1}$, then we see from (3.4) that

$$\frac{1}{\mu\big(B(x,r)\big)} \int_{B(x,r)} f(y)\, d\mu(y) \le C\big\{F^{1/p(x)} \big(\log(e+F)\big)^{A_1(x)} + 1\big\}.$$

If $r_0 > r > F^{-1}$, then we have by the lower bound (3.1)

$$\begin{aligned} F^{1/p(x)+\omega(r)/p'(x)^2} &\le C\mu\big(B(x,r)\big)^{-1/p(x)} r^{-s\omega(r)/p'(x)^2} \\ &\quad \times \left\{\int_{B(x,r)} f(y)^{p(y)}\, d\mu(y)\right\}^{1/p(x)+\omega(r)/p'(x)^2}. \end{aligned}$$

In view of (3.2), we find

$$\begin{aligned} &F^{1/p(x)+\omega(r)/p'(x)^2} \\ &\le C\mu\big(B(x,r)\big)^{-1/p(x)} \big(\log(1/r)\big)^{A_1(x)} \left\{\int_{B(x,r)} f(y)^{p(y)}\, d\mu(y)\right\}^{1/p(x)+\omega(r)/p'(x)^2} \\ &\le C\mu\big(B(x,r)\big)^{-1/p(x)} \big(\log(1/r)\big)^{A_1(x)} \left\{\int_{B(x,r)} f(y)^{p(y)}\, d\mu(y)\right\}^{1/p(x)} \\ &\le C\mu\big(B(x,r)\big)^{-1/p(x)} (\log F)^{A_1(x)} \left\{\int_{B(x,r)} f(y)^{p(y)}\, d\mu(y)\right\}^{1/p(x)} \\ &= CF^{1/p(x)} (\log F)^{A_1(x)}. \end{aligned}$$

Now we have established

$$\frac{1}{\mu\big(B(x,r)\big)} \int_{B(x,r)} f(y)\, d\mu(y) \le C\big\{F^{1/p(x)} \big(\log(e+F)\big)^{A_1(x)} + 1\big\} \tag{3.5}$$

for all $r > 0$ and $x \in G$, which completes the proof.

*Proof of Theorem* 3.1. Let $p_1(x) = p(x)/p_1$ for $1 < p_1 < p_-(G)$. Then Lemma 3.3 yields

$$\{Mf(x)\}^{p_1(x)} \le C\big\{Mg(x)\big(\log\big(e + Mg(x)\big)\big)^{\tilde{a}_1 s/p_1(x)} + 1\big\}$$

for $x \in G$, where $g(y) = f(y)^{p_1(y)}$ and $\tilde{a}_1 = a_1/p_1$. Letting $a > a_1$ when $a_1 > 0$ and $a = 0$ when $a_1 = 0$, we set $A(x) = as/p(x)^2$. Then we can choose $p_1$ so that $a_1 p_1 \le a$ and

$$\{Mf(x)\}^{p(x)} \le C\bigl\{Mg(x)\bigl(\log\bigl(e + Mg(x)\bigr)\bigr)^{A(x)p(x)/p_1} + 1\bigr\}^{p_1},$$

which yields

$$\bigl\{Mf(x)\bigl(\log\bigl(e + Mf(x)\bigr)\bigr)^{-A(x)}\bigr\}^{p(x)} \le C\{Mg(x) + 1\}^{p_1}.$$

Now Theorem 3.1 follows from the boundedness of maximal functions in $L^{p_1}$ (in the case of constant exponent).

**Remark 3.4.** Let $p(\,\cdot\,)$ be a positive continuous function on $G$ such that $1 \le p(x) \le p_+(G) < \infty$. Then, as in Harjulehto–Hästö–Pere [13], we can prove the following weak type result for maximal functions:

$$|E_f(t)| \le C \int_G \left|\frac{f(y)}{t}\right|^{p(y)} d\mu(y)$$

whenever $t > 0$ and $f \in L^{p(\,\cdot\,)}(G)$, where $E_f(t) = \{x \in G : Mf(x) \ge t\}$; see also Cruz-Uribe, Fiorenza and Neugebauer [2, Theorem 1.8].

To prove this, we may assume that $t = 1$. We have for $k > 1$

$$\begin{aligned}
&\frac{1}{\mu\bigl(B(x,r)\bigr)} \int_{B(x,r)} |f(y)|\, d\mu(y) \\
&\quad \le k\biggl\{\frac{1}{\mu\bigl(B(x,r)\bigr)} \int_{B(x,r)} (1/k)^{p'(y)}\, d\mu(y) + \frac{1}{\mu\bigl(B(x,r)\bigr)} \int_{B(x,r)} |f(y)|^{p(y)}\, d\mu(y)\biggr\} \\
&\quad \le k\bigl\{(1/k)^{(p_+)'} + F\bigr\},
\end{aligned}$$

where $F = \bigl(\mu\bigl(B(x,r)\bigr)\bigr)^{-1} \int_{B(x,r)} |f(y)|^{p(y)}\, d\mu(y)$. Here, considering $k = F^{-1/(p_+)'}$ when $F < 1$, we find

$$1 \le 2F^{1/p_+},$$

so that

$$\left(\tfrac{1}{2}\right)^{p_+} \le M\bigl(|f|^{p(\,\cdot\,)}\bigr)(x) \quad \text{for } x \in E_f(1),$$

which proves the required assertion.

**Remark 3.5.** For $0 < r < \frac{1}{2}$, let

$$G = \{x = (x_1, x_2) : 0 < x_1 < 1,\ -1 < x_2 < 1\}.$$

Define

$$p(x_1, x_2) = \begin{cases} p_0 + a_1\big(\log\big(\log(1/x_2)\big)\big)/\log(1/x_2) & \text{when } 0 < x_2 \le r_0, \\ p_0 & \text{when } x_2 \le 0; \end{cases}$$

and $p(x_1, x_2) = p(x_1, r_0)$ when $x_2 > r_0$. Setting

$$G(r) = \{x = (x_1, x_2) : 0 < x_1 < r,\ -r < x_2 < 0\},$$

we consider

$$f_r(y) = \chi_{G(r)}(y)$$

and set $g_r = f_r/\|f_r\|_{p(\,\cdot\,),G}$, where $\chi_E$ denotes the characteristic function of a measurable set $E$. Then we claim for $0 < r < \frac{1}{2}r_0$:

(i) $\|f_r\|_{p(\,\cdot\,),G} = r^{2/p_0}$;

(ii) $M(g_r)(x) \ge C_1 r^{-2/p_0}$ for $0 < x_1 < r$ and $r < x_2 < 2r$;

(iii) $\displaystyle\int_G \big\{M(g_r)(x)\big(\log\big(e + M(g_r)(x)\big)\big)^{-A(x)}\big\}^{p(x)}\,dx \ge C_2\big(\log(1/r)\big)^{2(a_1-a)/p_0}$ for $A(x) = 2a/p(x)^2$,

so that the conclusion of Theorem 3.1 does not hold for $0 < a < a_1$.

## 4. Sobolev's inequality

For $0 < \alpha < s$, we consider the Riesz potential $U_\alpha f$ of $f \in L^{p(\,\cdot\,)}(G)$ defined by

$$U_\alpha f(x) = \int_G \frac{|x-y|^\alpha f(y)}{\mu\big(B(x, |x-y|)\big)}\,d\mu(y);$$

recall that $s$ is the decay constant in (1.1). In this section, suppose $p(\,\cdot\,)$ satisfies (p1), (p2) and

$$p_+(G) < s/\alpha. \tag{p3}$$

Let

$$1/p^\sharp(x) = 1/p(x) - \alpha/s.$$

In what follows we establish Sobolev's inequality for $\alpha$-potentials on $G$, as an extension of the case of constant exponent which was discussed by Hajłasz and Koskela [12] and Heinonen [16]; for the Euclidean case, see the books by Adams and Hedberg [1] and the second author [21]. In the next two sections, we are concerned with the exponential integrability, which extends the results by Edmunds, Gurka and Opic [5], [6], and the authors [22].

Now we show our result, which gives an extension of Diening [4]; for further investigations we also refer the reader to the results by Kokilashvili–Samko [17], [18], where the index $\alpha$ is a variable exponent.

**Theorem 4.1.** *Letting $a > a_1$ when $a_1 > 0$ and $a = 0$ when $a_1 = 0$, we set $A(x) = as/p(x)^2$. Suppose $p_+(G) < s/\alpha$. Let $f$ be a nonnegative measurable function on $G$ with $\|f\|_{p(\,\cdot\,)} \le 1$. Then*

$$\int_G \left\{ U_\alpha f(x) \big(\log\big(e + U_\alpha f(x)\big)\big)^{-A(x)} \right\}^{p^\sharp(x)} d\mu(x) \le C.$$

In spite of the fact that the proof of Theorem 4.1 is quite similar to that of Theorem 3.4 in [11], we give a proof of Theorem 4.1 for the readers' convenience.

For this purpose, we prepare the following two lemmas.

**Lemma 4.2.** *Let $f$ be a nonnegative measurable function on $G$ with $\|f\|_{p(\,\cdot\,)} \le 1$. Then*

$$\int_{G \setminus B(x,\delta)} \frac{|x-y|^\alpha f(y)}{\mu\big(B(x,|x-y|)\big)}\, d\mu(y) \le C \delta^{-s/p^\sharp(x)} \log(1/\delta)^{A_1(x)}$$

*for $x \in G$ and $0 < \delta < \frac{1}{4}$, where $A_1(x) = a_1 s/p(x)^2$ as before.*

*Proof.* Let $f$ be a nonnegative measurable function on $G$ with $\|f\|_{p(\,\cdot\,)} \le 1$. Then, for $k > 1$, we have

$$\begin{aligned}
&\int_{G \setminus B(x,\delta)} \frac{|x-y|^\alpha f(y)}{\mu\big(B(x,|x-y|)\big)}\, d\mu(y) \\
&\quad \le k \left\{ \int_{G \setminus B(x,\delta)} \left( \frac{|x-y|^\alpha}{k\mu\big(B(x,|x-y|)\big)} \right)^{p'(y)} d\mu(y) + \int_{G \setminus B(x,\delta)} f(y)^{p(y)}\, d\mu(y) \right\} \\
&\quad \le k \left\{ \int_{G \setminus B(x,\delta)} \left( \frac{|x-y|^\alpha}{k\mu\big(B(x,|x-y|)\big)} \right)^{p'(y)} d\mu(y) + 1 \right\}.
\end{aligned}$$

Consider the set

$$E = \left\{ y \in G : |x-y|^\alpha / \big(k\mu\big(B(x,|x-y|)\big)\big) > 1 \right\}.$$

Note here that by (1.2), (3.1) and Lemma 2.3

$$
\begin{aligned}
&\int_{E\setminus B(x,\delta)} \left( \frac{|x-y|^{\alpha}}{k\mu\big(B(x,|x-y|)\big)} \right)^{p'(y)} d\mu(y) \\
&\le \int_{E\setminus B(x,\delta)} \left( \frac{|x-y|^{\alpha}}{k\mu\big(B(x,|x-y|)\big)} \right)^{p'(x)+\omega(|x-y|)} d\mu(y) \\
&\le C \sum_j \int_{B(x,2^j\delta)\setminus B(x,2^{j-1}\delta)} \left( \frac{(2^j\delta)^{\alpha}}{k\mu\big(B(x,2^j\delta)\big)} \right)^{p'(x)+\omega(2^j\delta)} d\mu(y) \\
&\le Ck^{-p'(x)-\omega(\delta)} \sum_j (2^j\delta)^{\alpha(p'(x)+\omega(2^j\delta))} \big(\mu(B(x,2^j\delta)\big)^{-(p'(x)+\omega(2^j\delta))+1} \\
&\le Ck^{-p'(x)-\omega(\delta)} \sum_j (2^j\delta)^{(\alpha-s)(p'(x)+\omega(2^j\delta))+s} \\
&\le Ck^{-p'(x)-\omega(\delta)} \int_{\delta}^{\infty} t^{(\alpha-s)(p'(x)+\omega(t))+s} t^{-1}\, dt \\
&\le Ck^{-p'(x)-\omega(\delta)} \delta^{(\alpha-s)(p'(x)+\omega(\delta))+s} \\
&\le Ck^{-p'(x)-\omega(\delta)} \delta^{p'(x)(\alpha-s/p(x))} \big(\log(1/\delta)\big)^{(s-\alpha)a_1/(p(x)-1)^2} \\
&= Ck^{-p'(x)-\omega(\delta)} \delta^{-p'(x)s/p^{\sharp}(x)} \big(\log(1/\delta)\big)^{(s-\alpha)a_1/(p(x)-1)^2},
\end{aligned}
$$

where $\omega(r) = \omega_{a,c}(r)$. Now, letting $k = \delta^{-s/p^{\sharp}(x)}\big(\log(1/\delta)\big)^{A_1(x)}$, we see that

$$
\int_{E\setminus B(x,\delta)} \left( \frac{|x-y|^{\alpha}}{k\mu\big(B(x,|x-y|)\big)} \right)^{p'(y)} d\mu(y) \le C.
$$

Further we find

$$
\int_{G\setminus E} \left( \frac{|x-y|^{\alpha}}{k\mu\big(B(x,|x-y|)\big)} \right)^{p'(y)} d\mu(y) \le C.
$$

Consequently it follows that

$$
\int_{G\setminus B(x,\delta)} \frac{|x-y|^{\alpha} f(y)}{\mu\big(B(x,|x-y|)\big)}\, d\mu(y) \le C\delta^{-s/p^{\sharp}(x)}\big(\log(1/\delta)\big)^{A_1(x)}
$$

for $x \in G$ and $0 < \delta < \frac{1}{4}$, as required.

**Lemma 4.3.** *Let $f$ be a nonnegative measurable function on $G$ with $\|f\|_{p(\cdot)} \le 1$. Then*

$$U_\alpha f(x) \le C\{Mf(x)^{p(x)/p^\sharp(x)}\big(\log\big(e+Mf(x)\big)\big)^{a_1\alpha/p(x)}+1\}.$$

*Proof.* To give the required estimate, we borrow the idea of Hedberg [15]. In fact, for $x \in G$ and $0<\delta<\frac{1}{4}$ we have by Lemma 4.1

$$\begin{aligned} U_\alpha f(x) &= \int_{G\cap B(x,\delta)} \frac{|x-y|^\alpha f(y)}{\mu\big(B(x,|x-y|)\big)}\,d\mu(y) + \int_{G\setminus B(x,\delta)} \frac{|x-y|^\alpha f(y)}{\mu\big(B(x,|x-y|)\big)}\,d\mu(y) \\ &\le C\delta^\alpha Mf(x) + C\delta^{-s/p^\sharp(x)}\big(\log(1/\delta)\big)^{A_1(x)}. \end{aligned}$$

Considering $\delta = Mf(x)^{-p(x)/s}\big(\log\big(e+Mf(x)\big)\big)^{a_1/p(x)}$ when $Mf(x)$ is large enough, we obtain the required inequality.

*Proof of Theorem* 4.1. Let $a>a_1>0$ or $a=a_1=0$, and set $A(x) = as/p(x)^2$. Then Lemma 4.3 yields

$$\Big\{U_\alpha f(x)\big(\log\big(e+U_\alpha f(x)\big)\big)^{-A(x)}\Big\}^{p^\sharp(x)} \le C\Big[\big\{Mf(x)\big(\log\big(e+Mf(x)\big)\big)^{-A(x)}\big\}^{p(x)}+1\Big],$$

which together with Theorem 3.1 completes the proof.

**Remark 4.4.** In Remark 3.5, we see that

$$U_\alpha g_r(x) \ge C_3 r^{-2/p^\sharp(x)}\big(\log(1/r)\big)^{A_1(x)}$$

for $0<x_1<r$ and $r<x_2<2r$, where $A_1(x) = 2a_1/p(x)^2$. Hence we have

$$\int_G \big\{U_\alpha g_r(x)\big(\log\big(e+U_\alpha g_r(x)\big)\big)^{-A(x)}\big\}^{p^\sharp(x)}\,dx \ge C_4\big(\log(1/r)\big)^{2(a_1-a)/p_0},$$

where $A(x) = 2a/p(x)^2$. This implies that the conclusion of Theorem 4.1 does not hold when $a<a_1$.

**Remark 4.5.** By Theorem 4.1 we see that $U_\alpha f \in L^{p(\cdot)}(G)$ whenever $f \in L^{p(\cdot)}(G)$. Then, as was pointed out by Lerner [20], the inequality

$$\int_G |U_\alpha f(x)|^{p(x)}\,d\mu(x) \le C\int_G |f(y)|^{p(y)}\,d\mu(y)$$

holds whenever $f \in L^{p(\cdot)}(G)$ if and only if $p$ is constant, under the additional assumption that

$$\mu(E) = \sup\big\{\mu(K) \mid K\subset E,\ K:\text{compact}\big\}$$

for every measurable set $E \subset X$. In fact, the if part is clear. We here assume that $p$ is not constant. Then we can find numbers $p_1$ and $p_2$ such that $1 \le p_1 < p_2 < \infty$, and both $E_1 = \{x \in G : p(x) \le p_1\}$ and $E_2 = \{x \in G : p(x) \ge p_2\}$ have positive $\mu$ measure. Further by our assumption, there exist compact sets $K_i$, $i = 1, 2$, such that $K_i \subset E_i$. If $f = k\chi_{K_1}$ with $k > 1$, then

$$U_\alpha f(x) \ge Ck\mu(K_1) \quad \text{for } x \in K_2,$$

so that

$$\int_G |U_\alpha f(x)|^{p(x)}\, d\mu(x) \ge Ck^{p_2}\mu(K_2).$$

On the other hand,

$$\int_G |f(x)|^{p(x)}\, d\mu(x) \le k^{p_1}\mu(K_1).$$

If the inequality holds, then we should have

$$k^{p_2} \le Ck^{p_1},$$

which gives a contradiction by letting $k \to \infty$.

In the same manner, we see that the inequality

$$\int_G \{Mf(x)\}^{p(x)}\, d\mu(x) \le C \int_G |f(y)|^{p(y)}\, d\mu(y)$$

holds whenever $f \in L^{p(\cdot)}(G)$ if and only if $p$ is constant.

**Remark 4.6.** Let $\omega(r)$ be a continuous function on $(0, \infty)$ such that

$$\omega(r) = \frac{a_1 \log\bigl(\log(1/r)\bigr)}{\log(1/r)} + \frac{a_2}{\log(1/r)}$$

for $0 < r \le r_0 < \frac{1}{4}$, with $a_1 > 0$ and $a_2 > 0$; set $\omega(r) = \omega(r_0)$ for $r > r_0$. Consider a variable exponent $p(\cdot)$ on the unit ball $B$ in $\mathbf{R}^n$ defined by

$$p(x) = p_0 + \omega\bigl(\varrho(x)\bigr),$$

where $1 < p_0 < n/\alpha$ and $\varrho(x) = 1 - |x|$. Take $r_0$ so small that $p(x) < n/\alpha$ for all $x \in B$. In view of Theorem 4.1, we see that if $a > a_1$ and $A(x) = an/p(x)^2$, then

$$\int_B \Bigl\{U_\alpha f(x)\bigl(\log\bigl(e + U_\alpha f(x)\bigr)\bigr)^{-A(x)}\Bigr\}^{p^\sharp(x)}\, dx \le C$$

whenever $f$ is a nonnegative measurable function on $B$ with $\|f\|_{p(\cdot)} \le 1$.

## 5. Exponential integrability

For fixed $x_0 \in G$, let us assume that an exponent $p(x)$ is a continuous function on $G$ satisfying

$$p(x) > p_0 \quad \text{when } x \neq x_0 \tag{5.1}$$

and

$$\left| p(x) - \left\{ p_0 + \frac{a \log\bigl(\log(1/|x_0 - x|)\bigr)}{\log(1/|x_0 - x|)} \right\} \right| \le \frac{b}{\log(1/|x_0 - x|)} \tag{5.2}$$

for $x \in B(x_0, r_0)$, where $0 < r_0 < \frac{1}{4}$, $p_0 = s/\alpha$, $0 < a \le (s-\alpha)/\alpha^2$ and $b > 0$.

Our aim in this section is to give an exponential integrability of Trudinger type. Before doing so, we prepare several lemmas. In view of (2.2) and (5.2), we have the following result.

**Lemma 5.1.** *There exist $C > 0$ and $0 < r_0 < \frac{1}{4}$ such that*

$$p'(y) \le p_0' - \omega(|x_0 - y|) \tag{5.3}$$

*for all $y \in B_0 = B(x_0, r_0)$, where $p_0' = p_0/(p_0 - 1) = s/(s-\alpha)$ and $\omega$ is a nonnegative nondecreasing function on $(0, \infty)$ such that*

$$\omega(r) = \frac{a\alpha^2}{(s-\alpha)^2} \frac{\log\bigl(\log(1/r)\bigr)}{\log(1/r)} - \frac{C}{\log(1/r)}$$

*when $0 < r \le r_0$; set $\omega(r) = \omega(r_0)$ when $r > r_0$ as before.*

**Lemma 5.2.** *If $0 < a < (s-\alpha)/\alpha^2$, then*

$$\int_{B_0 \setminus B(x,\delta)} \left( \frac{|x-y|^\alpha}{\mu\bigl(B(x, |x-y|)\bigr)} \right)^{p'(y)} d\mu(y) \le C\bigl(\log(1/\delta)\bigr)^{1 - a\alpha^2/(s-\alpha)}$$

*and if $a = (s-\alpha)/\alpha^2$, then*

$$\int_{B_0 \setminus B(x,\delta)} \left( \frac{|x-y|^\alpha}{\mu\bigl(B(x, |x-y|)\bigr)} \right)^{p'(y)} d\mu(y) \le C \log\bigl(\log(1/\delta)\bigr)$$

*for $x \in B_0$ and $0 < \delta < \delta_0$, where $0 < \delta_0 < \frac{1}{4}$.*

*Proof.* First consider the case $0 < a < (s-\alpha)/\alpha^2$. Let $E = \{y \in B_0 : |x-y|^\alpha/\mu(B(x,|x-y|)) > 1\}$ for fixed $x \in B_0$. Let $j_0$ be the smallest integer such that $2^{j_0}\delta > 2r_0$. Since $|x-y| \le 3|x_0-y|$ for $y \in B_0 \setminus B(x_0, |x_0-x|/2)$, we have by (1.2), (3.1) and (5.3)

$$\begin{aligned}
I_1 &= \int_{E\setminus\{B(x_0,|x_0-x|/2)\cup B(x,\delta)\}} \left(\frac{|x-y|^\alpha}{\mu(B(x,|x-y|))}\right)^{p'(y)} d\mu(y) \\
&\le C\sum_{j=1}^{j_0}\int_{B(x,2^j\delta)\setminus B(x,2^{j-1}\delta)} \left(\frac{(2^j\delta)^\alpha}{\mu(B(x,2^j\delta))}\right)^{p_0'-\omega(2^{j-1}\delta/3)} d\mu(y) \\
&\le C\sum_{j=1}^{j_0}(2^j\delta)^{\alpha(p_0'-\omega(2^{j-1}\delta/3))}\big(\mu(B(x,2^j\delta))\big)^{-(p_0'-\omega(2^{j-1}\delta/3))+1} \\
&\le C\sum_{j=1}^{j_0}(2^j\delta)^{(\alpha-s)(p_0'-\omega(2^{j-1}\delta/3))+s} \\
&\le C\sum_{j=1}^{j_0}\big(\log 1/(2^j\delta)\big)^{-a\alpha^2/(s-\alpha)} \\
&\le C\int_\delta^{3r_0}\big(\log(1/t)\big)^{-a\alpha^2/(s-\alpha)}t^{-1}\,dt \le C\big(\log(1/\delta)\big)^{1-a\alpha^2/(s-\alpha)}
\end{aligned}$$

for $0 < \delta < \delta_0$, since $1 - a\alpha^2/(s-\alpha) > 0$.

Next we give an estimate for

$$I_2 = \int_{B(x_0,|x-x_0|/2)\setminus B(x,\delta)} \left(\frac{|x-y|^\alpha}{\mu(B(x,|x-y|))}\right)^{p'(y)} d\mu(y).$$

We may assume that $2|x-x_0| > \delta$. Then we see from Lemma 5.1 that if $y \in B(x_0, |x-x_0|/2)$, then $p'(y) \le p_0' + \eta$, where $\eta = C/\log(1/|x_0-x|)$. Hence we obtain by (1.1) and (3.1)

$$\begin{aligned}
I_2 &\le C\int_{B(x_0,|x-x_0|/2)} \left(\frac{|x-x_0|^\alpha}{\mu(B(x,|x_0-x|/2))}\right)^{p'(y)} d\mu(y) \\
&\le C\int_{B(x_0,|x-x_0|/2)} \left\{\left(\frac{|x-x_0|^\alpha}{\mu(B(x,|x_0-x|/2))}\right)^{p_0'+\eta} + 1\right\} d\mu(y) \\
&\le C\mu(B(x_0,|x_0-x|/2))\left\{\left(\frac{|x-x_0|^\alpha}{\mu(B(x,|x_0-x|/2))}\right)^{p_0'+\eta} + 1\right\} \\
&\le C\big\{|x-x_0|^{\alpha(p_0'+\eta)}\mu(B(x_0,|x_0-x|/2))^{-(p_0'+\eta)+1} + 1\big\} \le C.
\end{aligned}$$

Thus it follows that

$$\int_{B_0\setminus B(x,\delta)} \left(\frac{|x-y|^\alpha}{\mu\big(B(x,|x-y|)\big)}\right)^{p'(y)} d\mu(y) \le C\big(\log(1/\delta)\big)^{1-a\alpha^2/(s-\alpha)}$$

for $0<\delta<\frac{1}{4}$, which proves the first case.

The second case $a=(s-\alpha)/\alpha^2$ is similarly proved.

**Lemma 5.3.** *Let $f$ be a nonnegative measurable function on $B_0$ with $\|f\|_{p(\,\cdot\,)}\le 1$. If $\beta_1>\beta=\big(1-a\alpha^2/(s-\alpha)\big)/p'_0=(s-\alpha-a\alpha^2)/s>0$, then*

$$\int_{B_0\setminus B(x,\delta)} \frac{|x-y|^\alpha f(y)}{\mu\big(B(x,|x-y|)\big)}\, d\mu(y) \le C\big(\log(1/\delta)\big)^{\beta_1} \tag{5.4}$$

*for $x\in B_0$ and $0<\delta<\delta_0$, where $0<\delta_0<\frac{1}{4}$.*

*Proof.* Take $p_1$ such that $1<p_1<p'_0$ and $\beta_1>\gamma=\big(1-a\alpha^2/(s-\alpha)\big)/p_1>\beta$. We may assume that $p'(y)>p_1$ for $y\in B_0$.

Let $f$ be a nonnegative measurable function on $B_0$ with $\|f\|_{p(\,\cdot\,)}\le 1$. For $k>1$ and $0<\delta<\frac{1}{4}$, we have by Lemma 5.2

$$\begin{aligned}
&\int_{B_0\setminus B(x,\delta)} \frac{|x-y|^\alpha f(y)}{\mu\big(B(x,|x-y|)\big)}\, d\mu(y)\\
&\quad\le k\left\{\int_{B_0\setminus B(x,\delta)} \left(\frac{|x-y|^\alpha}{k\mu\big(B(x,|x-y|)\big)}\right)^{p'(y)} d\mu(y) + \int_{B_0\setminus B(x,\delta)} f(y)^{p(y)}\, d\mu(y)\right\}\\
&\quad\le k\big\{Ck^{-p_1}\big(\log(1/\delta)\big)^{1-a\alpha^2/(s-\alpha)}+1\big\}.
\end{aligned}$$

Now, considering $k$ such that $k^{-p_1}\big(\log(1/\delta)\big)^{1-a\alpha^2/(s-\alpha)}=1$, we have

$$\int_{B_0\setminus B(x,\delta)} \frac{|x-y|^\alpha f(y)}{\mu\big(B(x,|x-y|)\big)}\, d\mu(y) \le C\big(\log(1/\delta)\big)^{\gamma} \le C\big(\log(1/\delta)\big)^{\beta_1},$$

as required.

In what follows we show that (5.4) remains true with $\beta_1$ replaced by $\beta=(s-\alpha-a\alpha^2)/s$.

**Lemma 5.4.** *Let $f$ be a nonnegative measurable function on $B_0$ with $\|f\|_{p(\,\cdot\,)}\le 1$. If $\beta=(s-\alpha-a\alpha^2)/s>0$, then*

$$\int_{B_0\setminus B(x,\delta)} \frac{|x-y|^\alpha f(y)}{\mu\big(B(x,|x-y|)\big)}\, d\mu(y) \le C\big(\log(1/\delta)\big)^{\beta}$$

*for $x\in B_0$ and $0<\delta<\delta_0$, where $0<\delta_0<\frac{1}{4}$.*

*Proof.* Let $f$ be a nonnegative measurable function on $B_0$ with $\|f\|_{p(\cdot)} \le 1$. Let $\eta = \big(\log(1/\delta)\big)^{-\log\log(1/\delta)}$ for small $\delta$, say $0 < \delta < \delta_0 < \frac{1}{4}$. Then note from Lemma 5.3 that

$$\int_{B_0\setminus B(x,\eta)} \frac{|x-y|^{\alpha} f(y)}{\mu\big(B(x,|x-y|)\big)}\, d\mu(y) \le C\big(\log(1/\eta)\big)^{\beta_1} \le C\big(\log(1/\delta)\big)^{\beta}. \tag{5.5}$$

Letting $k = \big(\log(1/\delta)\big)^{\beta}$ and $B(x) = B(x_0, |x_0 - x|/2)$, we find

$$k^{\omega(\eta/3)} \le C,$$

so that we obtain from Lemmas 5.1 and 5.2 that

$$\begin{aligned}
&\int_{B(x,\eta)\setminus\{B(x,\delta)\cup B(x)\}} \left(\frac{|x-y|^{\alpha}}{k\mu\big(B(x,|x-y|)\big)}\right)^{p'(y)} d\mu(y)\\
&\quad\le \int_{B(x,\eta)\setminus\{B(x,\delta)\cup B(x)\}} \left\{\left(\frac{|x-y|^{\alpha}}{k\mu\big(B(x,|x-y|)\big)}\right)^{p_0'-\omega(|x-y|/3)} + 1\right\} d\mu(y)\\
&\quad\le Ck^{-p_0'} \int_{B_0\setminus B(x,\delta)} \left(\frac{|x-y|^{\alpha}}{\mu\big(B(x,|x-y|)\big)}\right)^{p_0'-\omega(|x-y|/3)} d\mu(y) + C\\
&\quad\le Ck^{-p_0'}\big(\log(1/\delta)\big)^{1-a\alpha^2/(s-\alpha)} + C \le C.
\end{aligned}$$

Hence it follows from the proof of Lemma 5.3 that

$$\int_{B(x,\eta)\setminus\{B(x,\delta)\cup B(x)\}} \frac{|x-y|^{\alpha} f(y)}{\mu\big(B(x,|x-y|)\big)}\, d\mu(y) \le C(\log(1/\delta))^{\beta}. \tag{5.6}$$

Next we show that

$$\int_{B(x)\setminus B(x,\delta)} \frac{|x-y|^{\alpha} f(y)}{\mu\big(B(x,|x-y|)\big)}\, d\mu(y) \le C\big(\log(1/\delta)\big)^{\beta}. \tag{5.7}$$

Since $a > 0$, we have by the latter half of the proof of Lemma 5.2

$$\begin{aligned}
&\int_{B(x)\setminus B(x,\delta)} \frac{|x-y|^{\alpha} f(y)}{\mu\big(B(x,|x-y|)\big)}\, d\mu(y)\\
&\quad\le \int_{B(x)\setminus B(x,\delta)} \left(\frac{|x-y|^{\alpha}}{\mu\big(B(x,|x-y|)\big)}\right)^{p'(y)} d\mu(y) + \int_{B(x)\setminus B(x,\delta)} f(y)^{p(y)}\, d\mu(y)\\
&\quad\le C.
\end{aligned}$$

Now we claim from (5.5), (5.6) and (5.7) that

$$\int_{B_0\setminus B(x,\delta)} \frac{|x-y|^{\alpha} f(y)}{\mu\big(B(x,|x-y|)\big)}\, d\mu(y) \le C\big(\log(1/\delta)\big)^{\beta}.$$

Thus the proof is completed.

**Lemma 5.5.** *Let $f$ be a nonnegative measurable function on $B_0$ with $\|f\|_{p(\,\cdot\,)} \le 1$. If $\beta = (s-\alpha-a\alpha^2)/s > 0$, then*

$$U_\alpha f(x) \le C\big(\log\big(e+Mf(x)\big)\big)^{\beta} \quad \text{for } x \in B_0.$$

*Proof.* We see from Lemma 5.4 that

$$\begin{aligned} U_\alpha f(x) &= \int_{B(x,\delta)} \frac{|x-y|^\alpha f(y)}{\mu\big(B(x,|x-y|)\big)}\,d\mu(y) + \int_{B_0\setminus B(x,\delta)} \frac{|x-y|^\alpha f(y)}{\mu\big(B(x,|x-y|)\big)}\,d\mu(y) \\ &\le C\delta^\alpha Mf(x) + C\big(\log(1/\delta)\big)^{\beta}. \end{aligned}$$

Here, letting

$$\delta = \big(Mf(x)\big)^{-1/\alpha}\big(\log\big(e+Mf(x)\big)\big)^{\beta/\alpha}$$

when $Mf(x)$ is large enough, we have

$$U_\alpha f(x) \le C\big(\log\big(e+Mf(x)\big)\big)^{\beta},$$

as required.

It follows from Lemma 5.5 that

$$\exp\big(C^{-1}\big(U_\alpha f(x)\big)^{1/\beta}\big) \le e + Mf(x)$$

whenever $f$ is a nonnegative measurable function on $B_0$ with $\|f\|_{p(\,\cdot\,)} \le 1$. By the classical fact that $Mf \in L^{p_-}(B_0)$, we establish the following exponential inequality of Trudinger type.

**Theorem 5.6.** *Let $0 < a < (s-\alpha)/\alpha^2$. If $\beta = (s-\alpha-a\alpha^2)/s$, then there exist positive constants $c_1$ and $c_2$ such that*

$$\int_{B_0} \exp\big(c_1\big(U_\alpha f(x)\big)^{1/\beta}\big)\,d\mu(x) \le c_2$$

*for all nonnegative measurable functions $f$ on $B_0$ with $\|f\|_{p(\,\cdot\,)} \le 1$.*

**Remark 5.7.** Let $B_0$ be a ball in the $n$-dimensional space $\mathbf{R}^n$. If $f$ is a nonnegative measurable function on $B_0$ such that

$$\int_{B_0} f(y)^{p(y)}\,dy < \infty,$$

then we claim by applying an idea by Hästö [14] that

$$\int_{B_0} f(y)^{n/\alpha}\big(\log\big(e+f(y)\big)\big)^{a\alpha}\,dy < \infty. \tag{5.8}$$

In fact, if $y \in E = \{x \in B_0 : f(x) \geq |x_0 - x|^{-\alpha}(\log(e + |x_0 - x|^{-1}))^{-1}\}$, then

$$f(y)^{p(y)} \geq Cf(y)^{n/\alpha}(\log(e + f(y)))^{a\alpha}$$

so that

$$\int_E f(y)^{n/\alpha}(\log(e + f(y)))^{a\alpha}\, dy < \infty,$$

which proves (5.8), since $0 < a < (n-\alpha)/\alpha^2$. With the aid of Edmunds–Krbec [7] and the authors [22] we also obtain Theorem 5.6 in the Euclidean case.

Finally we are concerned with the case $a = (s-\alpha)/\alpha^2$.

**Lemma 5.8.** *Let $f$ be a nonnegative measurable function on $B_0$ with $\|f\|_{p(\cdot)} \leq 1$. If $a = (s-\alpha)/\alpha^2$, then*

$$U_\alpha f(x) \leq C(\log(e + \log(e + Mf(x))))^{p_0'} \quad \text{for } x \in B_0.$$

*Proof.* Let $f$ be a nonnegative measurable function on $B_0$ with $\|f\|_{p(\cdot)} \leq 1$. For $k > 1$ and $0 < \delta < \delta_0 < \frac{1}{4}$, we have by applications of the arguments in the proof of Lemma 5.4

$$\int_{B_0 \setminus B(x,\delta)} \frac{|x-y|^\alpha f(y)}{\mu(B(x,|x-y|))}\, d\mu(y) \leq C(\log(\log(1/\delta)))^{1/p_0'}.$$

Consequently it follows that

$$\begin{aligned} U_\alpha f(x) &= \int_{B(x,\delta)} \frac{|x-y|^\alpha f(y)}{\mu(B(x,|x-y|))}\, d\mu(y) + \int_{B_0 \setminus B(x,\delta)} \frac{|x-y|^\alpha f(y)}{\mu(B(x,|x-y|))}\, d\mu(y) \\ &\leq C\delta^\alpha Mf(x) + C(\log(\log(1/\delta)))^{1/p_0'}. \end{aligned}$$

Here let

$$\delta = Mf(x)^{-1/\alpha}(\log(e + \log(e + Mf(x))))^{1/\{\alpha p_0'\}}$$

when $Mf(x)$ is large enough. Then we have

$$U_\alpha f(x) \leq C(\log(e + \log(e + Mf(x))))^{1/p_0'},$$

as required.

By Lemma 5.8 and the fact that $Mf \in L^{p_0}(B_0)$, we establish the following double exponential inequality for $f \in L^{p(\cdot)}(B_0)$.

**Theorem 5.9.** *If $a = (s-\alpha)/\alpha^2$, then there exist positive constants $c_1$ and $c_2$ such that*

$$\int_{B_0} \exp(\exp(c_1(U_\alpha f(x))^{s/(s-\alpha)}))\, d\mu(x) \leq c_2$$

*for all nonnegative measurable functions $f$ on $B_0$ with $\|f\|_{p(\cdot)} \leq 1$.*

**Remark 5.10.** In case $f$ belongs to more general variable exponent Lebesgue spaces, we will be expected to discuss the corresponding exponential integrability as in Edmunds, Gurka and Opic [5], [6]. But we do not go into details any more.

## 6. Exponential integrability, II

In this section, let $B = B(0,1)$ be the unit ball in $\mathbf{R}^n$. We consider a variable exponent $p(\cdot)$ on $B$ which is a continuous function on $B$ satisfying

$$p(x) > p_0 \quad \text{on } B \tag{6.1}$$

and

$$\left| p(x) - \left\{ p_0 + \frac{a \log\bigl(\log\bigl(1/\varrho(x)\bigr)\bigr)}{\log\bigl(1/\varrho(x)\bigr)} \right\} \right| \le \frac{b}{\log\bigl(1/\varrho(x)\bigr)} \tag{6.2}$$

when $\varrho(x) < r_0$, where $0 < r_0 < \frac{1}{4}$, $a > 0$, $b > 0$, $p_0 = n/\alpha$ and $\varrho(x) = 1 - |x|$ denotes the distance of $x$ from the boundary $\partial B$.

For $f \in L^{p(\cdot)}(B)$, the Riesz potential of order $\alpha$, $0 < \alpha < n$, is defined by

$$U_\alpha f(x) = \int_B |x-y|^{\alpha-n} f(y)\, dy.$$

If $a > (n-\alpha)/\alpha^2$, then we see from Theorem 7.7 below that

$$|U_\alpha f(x) - U_\alpha f(z)| \le C\bigl(\log(1/|x-z|)\bigr)^{(n-\alpha-a\alpha^2)/n}$$

whenever $x, z \in B$ and $|x-z| < \frac{1}{2}$; for this fact, see also [10, Theorem 4.3].

In what follows, when $0 < a \le (n-\alpha)/\alpha^2$, we discuss exponential inequalities of $U_\alpha f$ as in Section 5.

As in Lemma 5.1, we have the following result.

**Lemma 6.1.** *There exist positive constants $t_0 < \frac{1}{4}$ and $C$ such that*

$$p'(x) \le p'_0 - \omega\bigl(\varrho(x)\bigr)$$

*for $x \in B$, where $\omega(t) = \bigl(a\alpha^2/(n-\alpha)^2\bigr)\log\bigl(\log(1/t)\bigr)/\log(1/t) - C/\log(1/t)$ for $0 < t \le t_0$ and $\omega(t) = \omega(t_0)$ for $t > t_0$.*

**Lemma 6.2.** *If $0 < a < (n-\alpha)/\alpha^2$, then*

$$I \equiv \int_{B\setminus B(x,r)} |x-y|^{(\alpha-n)p'(y)}\, dy \le C\bigl(\log(1/r)\bigr)^{\gamma}$$

*for all $x \in B$ and $0 < r < \frac{1}{2}$, where $\gamma = 1 - a\alpha^2/(n-\alpha)$.*

*Proof.* First consider the case $\frac{1}{2}\varrho(x) \le r < \frac{1}{2}$. Letting $E_1 = \{y \in B\setminus B(x,r) : |\varrho(x) - \varrho(y)| > 2r\}$, we find by polar coordinates,

$$\begin{aligned} I_1 &\equiv \int_{E_1} |x-y|^{(\alpha-n)p'(y)}\,dy \\ &\le C\int_{\{t:|t-\varrho(x)|>2r\}} |t-\varrho(x)|^{(\alpha-n)(p'_0-\omega(t))+n-1}\,dt \\ &\le C\int_{\{t:t>2r\}} t^{(n-\alpha)\omega(t)-1}\,dt \\ &\le C\int_{2r}^{1/2} \bigl(\log(1/t)\bigr)^{-a\alpha^2/(n-\alpha)} t^{-1}\,dt + C \le C\bigl(\log(1/r)\bigr)^{\gamma}. \end{aligned}$$

Letting $E_2 = \{y \in B \setminus B(x,r) : |\varrho(x) - \varrho(y)| \le 2r\}$, we find by polar coordinates,

$$\begin{aligned} I_2 &\equiv \int_{E_2} |x-y|^{(\alpha-n)p'(y)}\,dy \\ &\le C\int_{\{t:|t-\varrho(x)|\le 2r\}} r^{(\alpha-n)p'_0+n-1}\,dt \le Cr^{-1}\int_0^{4r} dt \le C. \end{aligned}$$

Hence it follows that

$$\int_{B\setminus B(x,r)} |x-y|^{(\alpha-n)p'(y)}\,dy \le C\bigl(\log(1/r)\bigr)^{\gamma}$$

when $\frac{1}{2}\varrho(x) \le r < \frac{1}{2}$. In particular, we obtain

$$\int_{B\setminus B(x,\varrho(x)/2)} |x-y|^{(\alpha-n)p'(y)}\,dy \le C\bigl(\log\bigl(1/\varrho(x)\bigr)\bigr)^{\gamma}. \tag{6.3}$$

Next consider the case $0 < r < \frac{1}{2}\varrho(x)$. Let $E_3 = B\bigl(x, \frac{1}{2}\varrho(x)\bigr) \setminus B(x,r)$. In view of Lemma 6.1, we find

$$p'(y) \le p'_0 - \omega\bigl(\varrho(x)\bigr) + C/\log\bigl(1/\varrho(x)\bigr) \le p'_0 - \omega_1(|x-y|)$$

for $y \in E_3$, where $\omega_1(t) = \omega(t) - C/\log(1/t)$ for small $t > 0$. Hence, we see that

$$\begin{aligned} I_3 &\equiv \int_{E_3} |x-y|^{(\alpha-n)p'(y)}\,dy \\ &\le \int_{E_3} |x-y|^{(\alpha-n)\{p'_0-\omega_1(|x-y|)\}}\,dy \\ &\le C\int_r^{\varrho(x)/2} \bigl(\log(1/t)\bigr)^{-a\alpha^2/(n-\alpha)} t^{-1}\,dt \le C\bigl(\log(1/r)\bigr)^{\gamma}. \end{aligned}$$

In view of (6.3), we establish

$$\int_{B\setminus B(x,r)} |x-y|^{(\alpha-n)p'(y)}\,dy \le C\bigl(\log(1/r)\bigr)^{\gamma}$$

when $0 < r < \frac{1}{2}\varrho(x)$. Thus the required result is proved.

As in the proof of Lemma 5.4, we can prove the following result.

**Lemma 6.3.** *Let $f$ be a nonnegative function on $B$ such that $\|f\|_{p(\cdot)} \le 1$. If $0 < a < (n-\alpha)/\alpha^2$ and $\beta = \gamma/p_0' = (n-\alpha-a\alpha^2)/n$, then*

$$\int_{B\setminus B(x,r)} |x-y|^{\alpha-n} f(y)\,dy \le C\bigl(\log(1/r)\bigr)^{\beta}$$

*whenever $0 < r < \frac{1}{2}$.*

We see from Lemma 6.3 that

$$\begin{aligned} U_\alpha f(x) &= \int_{B(x,\delta)} |x-y|^{\alpha-n} f(y)\,dy + \int_{B\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)\,dy \\ &\le C\delta^{\alpha} Mf(x) + C\bigl(\log(1/\delta)\bigr)^{\beta}. \end{aligned}$$

Here, letting

$$\delta = \bigl(Mf(x)\bigr)^{-1/\alpha}\bigl(\log\bigl(e+Mf(x)\bigr)\bigr)^{\beta/\alpha}$$

when $Mf(x)$ is large enough, we have

$$U_\alpha f(x) \le C\bigl(\log\bigl(e+Mf(x)\bigr)\bigr)^{\beta},$$

so that

$$\exp\bigl(C^{-1}\bigl(U_\alpha f(x)\bigr)^{1/\beta}\bigr) \le e + Mf(x)$$

whenever $f$ is a nonnegative measurable function on $B_0$ with $\|f\|_{p(\cdot)} \le 1$. By the classical fact that $Mf \in L^{p_0}(B)$, we establish the following exponential inequality of Trudinger type.

**Theorem 6.4.** *Let $0 < a < (n-\alpha)/\alpha^2$. If $\beta = (n-\alpha-a\alpha^2)/n$, then there exist positive constants $c_1$ and $c_2$ such that*

$$\int_B \exp\bigl(c_1\bigl(U_\alpha f(x)\bigr)^{1/\beta}\bigr)\,dx \le c_2$$

*for all nonnegative measurable functions $f$ on $B$ with $\|f\|_{p(\cdot)} \le 1$.*

Finally we are concerned with the case $a = (n-\alpha)/\alpha^2$. The following can be proved in the same way as Lemma 5.4.

**Lemma 6.5.** *Let $f$ be a nonnegative function on $B$ such that $\|f\|_{p(\cdot)} \le 1$. If $a = (n-\alpha)/\alpha^2$, then*

$$\int_{B\setminus B(x,r)} |x-y|^{(\alpha-n)p'(y)} f(y)\,dy \le C\bigl(\log\bigl(\log(1/r)\bigr)\bigr)^{(n-\alpha)/n}$$

*for small $r > 0$.*

As in the proof of Theorem 6.4, we establish the following double exponential inequality for $f \in L^{p(\cdot)}(B)$.

**Theorem 6.6.** *If $a = (n-\alpha)/\alpha^2$, then there exist positive constants $c_1$ and $c_2$ such that*

$$\int_B \exp\big(\exp\big(c_1\big(U_\alpha f(x)\big)^{n/(n-\alpha)}\big)\big)\,dx \le c_2$$

*for all nonnegative measurable functions $f$ on $B$ with $\|f\|_{p(\cdot)} \le 1$.*

## 7. Continuity

Let $G$ be a bounded open set in the $n$-dimensional space $\mathbf{R}^n$, and fix $x_0 \in G$. In this section, we deduce the continuity at $x_0$ of Riesz potentials $U_\alpha f$ when $f \in L^{p(\cdot)}(G)$ with $p(\cdot)$ satisfying

$$\left| p(x) - \left\{ \frac{n}{\alpha} + \frac{a \log\big(\log(1/|x_0 - x|)\big)}{\log(1/|x_0-x|)} \right\} \right| \le \frac{b}{\log(1/|x_0-x|)},$$

where $a > (n-\alpha)/\alpha^2$, $b > 0$ and $x$ runs over the small ball $B_0 = B(x_0, r_0)$.

Consider a positive continuous nonincreasing function $\varphi$ on the interval $(0,\infty)$ such that

($\varphi$) $\big(\log(1/t)\big)^{-\varepsilon_0}\varphi(t)$ is nondecreasing on $(0, r_0]$ for some $\varepsilon_0 > 0$ and $r_0 > 0$;

set $\varphi(r) = \varphi(r_0)$ for $r > r_0$. We see from condition ($\varphi$) that $\varphi$ satisfies the doubling condition.

Set

$$\Phi(r) = \left( \int_0^r \varphi(t)^{-\alpha^2/(n-\alpha)} t^{-1}\,dt \right)^{(n-\alpha)/n}.$$

Our final goal is to establish the following result, which deals with the continuity of $\alpha$-potentials in $\mathbf{R}^n$.

**Theorem 7.1.** *Let $p(\cdot)$ satisfy*

$$p(x) = \frac{n}{\alpha} + \frac{\log\varphi(|x_0-x|)}{\log(1/|x_0-x|)} \quad \textit{for } x \in B_0 = B(x_0, r_0)$$

*and $f \in L^{p(\cdot)}(B_0)$. If $\Phi(1) < \infty$, then $U_\alpha f$ is continuous at $x_0$; in this case,*

$$|U_\alpha f(x) - U_\alpha f(z)| \le C\Phi(|x-z|)$$

*whenever $x, z \in B\big(x_0, \tfrac{1}{2}r_0\big)$.*

**Remark 7.2.** Let $\varphi(r) = \big(\log(e + 1/r)\big)^a$. Then $\Phi(1) < \infty$ if and only if $a > (n-\alpha)/\alpha^2$, so that Theorem 7.1 gives an extension of the authors' [9].

For a proof of Theorem 7.1, we may assume that $x_0 = 0$ without loss of generality. Before the proof we prepare the following two results.

**Lemma 7.3.** *For $x \in B\big(0, \frac{1}{2}r_0\big)$ and small $\delta > 0$,*

$$\int_{B(x,\delta)} |x-y|^{p'(y)(\alpha-n)}\,dy \le C\int_0^\delta \varphi(r)^{-\alpha^2/(n-\alpha)} r^{-1}\,dr.$$

*Proof.* First note from (2.2) and $(\varphi)$ that

$$p'(y) \le p'_0 - \omega(|y|) \quad \text{for } y \in B_0,$$

where $p'_0 = n/(n-\alpha)$ and $\omega(r) = \big(\alpha^2/(n-\alpha)^2\big)\big(\log\varphi(r)\big)/\log(1/r) - C/\log(1/r)$ for $0 < r \le r_0$; set $\omega(r) = \omega(r_0)$ for $r > r_0$. If $0 < \delta \le \frac{1}{2}|x|$, then we have

$$\begin{aligned}\int_{B(x,\delta)} |x-y|^{p'(y)(\alpha-n)}\,dy &\le \sum_j \int_{B(x,2^{-j+1}\delta)\setminus B(x,2^{-j}\delta)} |x-y|^{p'(y)(\alpha-n)}\,dy\\ &\le \sum_j (2^{-j}\delta)^{(\alpha-n)(p'_0-\omega(2^{-j}\delta))}\sigma_n(2^{-j+1}\delta)^n\\ &\le C\sum_j \varphi(2^{-j}\delta)^{-\alpha^2/(n-\alpha)}\\ &\le C\int_0^\delta \varphi(r)^{-\alpha^2/(n-\alpha)} r^{-1}\,dr,\end{aligned}$$

where $\sigma_n$ denotes the volume of the unit ball. Similarly, if $\frac{1}{2}|x| < \delta < \frac{1}{3}r_0$, we have

$$\begin{aligned}\int_{B(x,\delta)\setminus B(x,|x|/2)} |x-y|^{p'(y)(\alpha-n)}\,dy &\le C\int_{B(0,3\delta)} |y|^{p'(y)(\alpha-n)}\,dy\\ &\le C\int_0^{3\delta} \varphi(r)^{-\alpha^2/(n-\alpha)} r^{-1}\,dr.\end{aligned}$$

Therefore it follows from the doubling property that

$$\int_{B(x,\delta)} |x-y|^{p'(y)(\alpha-n)}\,dy \le C\int_0^\delta \varphi(r)^{-\alpha^2/(n-\alpha)} r^{-1}\,dr$$

when $0 < \delta < \frac{1}{3}r_0$. Now the proof is completed.

**Lemma 7.4.** *Let $f$ be a nonnegative measurable function on $B_0$ with $\|f\|_{p(\cdot)} \le 1$. Then*

$$\int_{B_0\setminus\{B(0,\delta)\cup B(x,\delta)\}} |x-y|^{\alpha-n-1} f(y)\,dy \le C\delta^{-1}\varphi(\delta)^{-\alpha^2/n}$$

*for $x \in B\big(0, \frac{1}{2}r_0\big)$ and small $\delta > 0$.*

*Proof.* Let $f$ be a nonnegative measurable function on $B_0$ with $\|f\|_{p(\cdot)} \le 1$. For $k > 1$ we have

$$\begin{aligned}
&\int_{B_0 \setminus \{B(0,\delta) \cup B(x,\delta)\}} |x-y|^{\alpha-n-1} f(y)\, dy \\
&\quad \le k \Big\{ \int_{B_0 \setminus \{B(x,\delta) \cup B(0,\delta)\}} (|x-y|^{\alpha-n-1}/k)^{p'(y)}\, dy \\
&\qquad\qquad + \int_{B_0 \setminus \{B(x,\delta) \cup B(0,\delta)\}} f(y)^{p(y)}\, dy \Big\} \\
&\quad \le k \Big\{ \int_{B_0 \setminus \{B(x,\delta) \cup B(0,\delta)\}} (|x-y|^{\alpha-n-1}/k)^{p'(y)}\, dy + 1 \Big\}.
\end{aligned}$$

In view of the assumption of $\varphi$, we obtain

$$\begin{aligned}
&\int_{B_0 \setminus \{B(x,\delta) \cup B(0,\delta)\}} (|x-y|^{\alpha-n-1}/k)^{p'(y)}\, dy \\
&\quad \le C \Big\{ \int_{B_0 \setminus \{B(x,\delta) \cup B(0,\delta)\}} (|x-y|^{\alpha-n-1}/k)^{p'_0-\omega(\delta)}\, dy + 1 \Big\} \\
&\quad \le C \Big\{ k^{-p'_0+\omega(\delta)} \int_\delta^\infty t^{(\alpha-n-1)(p'_0-\omega(\delta))+n} t^{-1}\, dt + 1 \Big\} \\
&\quad \le C k^{-p'_0+\omega(\delta)} \delta^{(\alpha-n-1)(p'_0-\omega(\delta))+n}.
\end{aligned}$$

Considering $k$ such that $k^{-p'_0+\omega(\delta)} \delta^{(\alpha-n-1)(p'_0-\omega(\delta))+n} = 1$, we see that

$$\int_{B_0 \setminus \{B(0,\delta) \cup B(x,\delta)\}} |x-y|^{\alpha-n-1} f(y)\, dy \le C \delta^{-1} \varphi(\delta)^{-\alpha^2/n},$$

as required.

*Proof of Theorem* 7.1. Let $f$ be a nonnegative measurable function on $B_0$ with $\|f\|_{p(\cdot)} \le 1$. For $0 < k < 1$, we have by Lemma 7.3

$$\begin{aligned}
&\int_{B(x,\delta)} |x-y|^{\alpha-n} f(y) dy \\
&\quad \le k \int_{B(x,\delta)} \big\{ (|x-y|^{\alpha-n}/k)^{p'(y)} + f(y)^{p(y)} \big\}\, dy \\
&\quad \le k \Big\{ k^{-n/(n-\alpha)} \int_{B(x,\delta)} |x-y|^{(\alpha-n)p'(y)}\, dy + 1 \Big\} \\
&\quad \le k \big\{ C k^{-n/(n-\alpha)} \Phi(\delta)^{n/(n-\alpha)} + 1 \big\}
\end{aligned}$$

whenever $x \in B\big(0, \frac{1}{2}r_0\big)$ and $0 < \delta < \frac{1}{2}r_0$. Now, considering $k = \Phi(\delta)$, we find

$$\int_{B_0 \cap B(x,\delta)} |x-y|^{\alpha-n} f(y)\,dy \le C\Phi(\delta). \tag{7.1}$$

Hence, if $x, z \in B\big(0, \frac{1}{2}r_0\big)$ and $|x-z| < \frac{1}{4}r_0$, then we have

$$\int_{B(x,2|x-z|)} |x-y|^{\alpha-n} f(y)\,dy \le C\Phi(|x-z|). \tag{7.2}$$

On the other hand we have

$$\begin{aligned}
&\int_{B_0 \setminus B(x,2|x-z|)} \big||x-y|^{\alpha-n} - |z-y|^{\alpha-n}\big| f(y)\,dy \\
&\quad \le C|x-z| \int_{B_0 \setminus B(x,2|x-z|)} |x-y|^{\alpha-n-1} f(y)\,dy \\
&\quad = C|x-z| \bigg\{ \int_{B_0 \setminus \{B(x,2|x-z|) \cup B(0,2|x-z|)\}} |x-y|^{\alpha-n-1} f(y)\,dy \\
&\qquad + \int_{\{B_0 \cap B(0,2|x-z|)\} \setminus B(x,2|x-z|)} |x-y|^{\alpha-n-1} f(y)\,dy \bigg\}.
\end{aligned}$$

It follows from Lemma 7.4 that

$$\int_{B_0 \setminus \{B(x,2|x-z|) \cup B(0,2|x-z|)\}} |x-y|^{\alpha-n-1} f(y)\,dy \le C|x-z|^{-1} \varphi(|x-z|)^{-\alpha^2/n}.$$

Moreover we see from (7.1) that

$$\begin{aligned}
&\int_{\{B_0 \cap B(0,2|x-z|)\} \setminus B(x,2|x-z|)} |x-y|^{\alpha-n-1} f(y)\,dy \\
&\quad \le C|x-z|^{-1} \int_{B_0 \cap B(0,2|x-z|)} |y|^{\alpha-n} f(y)\,dy \le C|x-z|^{-1}\Phi(|x-z|).
\end{aligned}$$

Since $\varphi(r)^{-\alpha^2/n} \le C\Phi(r)$ by the doubling property of $\varphi$, we obtain

$$\int_{B_0 \setminus B(x,2|x-z|)} \big||x-y|^{\alpha-n} - |z-y|^{\alpha-n}\big| f(y)\,dy \le C\Phi(|x-z|).$$

Further we obtain by (7.2)

$$\int_{B(x,2|x-z|)} |z-y|^{\alpha-n} f(y)\,dy \le C\Phi(|x-z|).$$

Now, we establish

$$|U_\alpha f(x) - U_\alpha f(z)| \le C\Phi(|x-z|),$$

as required.

**Remark 7.5.** If $\Phi(1) = \infty$, then we can find $f \in L^{p(\cdot)}(B_0)$ such that $U_\alpha f(0) = \infty$, which means that $U_\alpha f$ is not continuous at $0$.

For this purpose set

$$\psi(r) = \int_r^1 \varphi(t)^{-\alpha^2/(n-\alpha)} t^{-1}\, dt$$

and

$$f(y) = |y|^{-(n-\alpha)/(p(y)-1)} \psi(|y|)^{-1}.$$

Take $r_0$ so small that $\psi(r) > e$ when $0 < r < r_0$. Note that

$$r^{-(n-\alpha)p/(p-1)+n} = r^{-(n-\alpha p)/(p-1)} = \varphi(r)^{-\alpha/(p-1)}$$

for $r = |y|$ and $p = p(y)$. By $(\varphi)$ we have

$$\varphi(r)^{-\alpha/(p-1)} \geq C\varphi(r)^{-\alpha^2/(n-\alpha)},$$

so that

$$\begin{aligned} U_\alpha f(0) &= \int_{B_0} |y|^{\alpha-n-(n-\alpha)/(p(y)-1)} \psi(|y|)^{-1}\, dy \\ &\geq C \int_0^{r_0} \varphi(t)^{-\alpha^2/(n-\alpha)} \psi(t)^{-1}\, dt/t = \infty \end{aligned}$$

since $\psi(0) = \infty$ by our assumption.

On the other hand, taking a number $\delta$ such that $1 < \delta < n/\alpha$ and noting by $(\varphi)$ that

$$\varphi(r)^{-\alpha/(p-1)} \leq C\varphi(r)^{-\alpha^2/(n-\alpha)},$$

we have

$$\begin{aligned} \int_{B_0} f(y)^{p(y)}\, dy &= \int_{B_0} |y|^{-(n-\alpha)p(y)/(p(y)-1)} \psi(|y|)^{-p(y)}\, dy \\ &\leq C \int_0^{r_0} \varphi(t)^{-\alpha^2/(n-\alpha)} \psi(t)^{-\delta}\, dt/t < \infty \end{aligned}$$

since $1 < \delta < n/\alpha \leq p(y)$ and $\psi(0) = \infty$, as required.

Finally, we consider a variable exponent $p(\cdot)$ on the unit ball $B$ such that

$$p(x) = p_0 + \frac{\log \varphi\big(\varrho(x)\big)}{\log\big(e/\varrho(x)\big)} \tag{7.3}$$

for $x \in B$, where $p_0 = n/\alpha$; assume as above that

$$p(x) > p_0 \quad \text{on } B.$$

**Theorem 7.6.** *If $\Phi(1) < \infty$ and $f \in L^{p(\cdot)}(B)$, then*

$$|U_\alpha f(x) - U_\alpha f(z)| \leq C\Phi(|x-z|)$$

*whenever $x, z \in B$.*

For a proof of Theorem 7.6, it suffices to show that

$$\int_{B(x,\delta)} |x-y|^{p'(y)(\alpha-n)}\,dy \le C \int_0^\delta \varphi(r)^{-\alpha^2/(n-\alpha)} r^{-1}\,dr$$

for $x \in B$ and small $\delta > 0$, as in Lemma 7.3. We obtain, in fact, this inequality in the same way as in Lemmas 6.2 and 7.3.

**Remark 7.7.** We do not know the best condition which assures the continuity of Riesz potentials in the metric space setting.

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