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MARTIN BOUNDARY POINTS OF CONES GENERATED BY SPHERICAL JOHN REGIONS

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Dedicated to Professor Yoshihiro Mizuta on the occasion of his 60th birthday.

Abstract. We study Martin boundary points of cones generated by spherical John regions. In particular, we show that such a cone has a unique (minimal) Martin boundary point at the vertex, and also at infinity. We also study a relation between ordinary thinness and minimal thinness, and the boundary behavior of positive superharmonic functions.

1. Introduction

We work in the Euclidean space $\mathbf{R}^n$, where $n \geq 3$. Let Ω be a subdomain of $\mathbf{R}^n$ and G_Ω stand for the Green function for Ω. Let $x_0 \in \Omega$ be fixed, and let ξ be a boundary point of Ω. Suppose now that $\{y_j\}$ is a sequence in Ω converging to ξ. Then, for each bounded open set ω such that $x_0 \in \omega$ and $\overline{\omega} \subset \Omega$, there is j_0 such that $\{G_\Omega(\cdot, y_j)/G_\Omega(x_0, y_j)\}_{j=j_0}^\infty$ is a uniformly bounded sequence of positive harmonic functions in ω. Therefore some subsequence of $\{G_\Omega(\cdot, y_j)/G_\Omega(x_0, y_j)\}_j$ converges to a positive harmonic function in Ω. All limit functions obtained in this way are called *Martin kernels at ξ* or *Martin boundary points at ξ*. Note that the number of Martin boundary points at ξ depends on geometry of Ω near ξ, so it is not necessarily unique. We say that a positive harmonic function h is *minimal* if every positive harmonic function less than or equal to h coincides with a constant multiple of h. If a Martin kernel is minimal, then we call it a minimal Martin kernel or a minimal Martin boundary point. There have been many investigations for minimal Martin boundary points of several types of domains. For instance, every Euclidean boundary point of Lipschitz domains [11], NTA domains [12] or uniform domains [2], has a unique Martin boundary point and it is minimal. See also [4] and [3] for other domains. For Denjoy domains [7, 10, 16], Lipschitz-Denjoy domains [5, 8], sectorial domains [9] and quasi-sectorial domains [15], there are criteria for the number of minimal Martin boundary points at a fixed Euclidean boundary point. In [3], Aikawa, Lundh and the author investigated the number of minimal Martin boundary points at each Euclidean boundary point of a John domain. An open

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subset Ω of $\mathbf{R}^n$ is said to be a John domain with John constant C_J and John center X_0 if each point x in Ω can be connected to X_0 by a rectifiable curve γ in Ω such that

$$\operatorname{dist}(z, \partial\Omega) \geq C_J \ell(\gamma(x, z)) \quad \text{for all } z \in \gamma, \tag{1.1}$$

where $\ell(\gamma(x, z))$ denotes the length of the subarc $\gamma(x, z)$ of γ connecting x to z, and $\operatorname{dist}(z, \partial\Omega)$ stands for the distance from z to the Euclidean boundary $\partial\Omega$ of Ω. John domains include domains stated above and domains with fractal boundaries. Each Euclidean boundary point of a John domain may have many minimal Martin boundary points, but its number is finite.

Theorem A. *Let Ω be a John domain with John constant C_J. The following statements hold:*

(i) *The number of minimal Martin boundary points at every point of $\partial\Omega$ is bounded by a constant depending only on C_J.*
(ii) *If $C_J > \sqrt{3}/2$, then there are at most two minimal Martin boundary points at every point of $\partial\Omega$.*

The bound $C_J > \sqrt{3}/2$ in (ii) is sharp (cf. [3, Remark 1.1]). However, the number of minimal Martin boundary points at a given Euclidean boundary point can not be determined in terms of the John constant C_J.

In this note, we will consider a cone generated by a (relatively) open subset of the unit sphere with a John property, and will study Martin boundary points at the vertex and at infinity. For $x \in \mathbf{R}^n$ and $r > 0$, let $B(x, r)$ and $S(x, r)$ denote the open ball and the sphere of center x and radius r, respectively. When $x = 0$, we write $B(r)$ and $S(r)$ to abbreviate the notation. Let $x_0 \in S(1)$. We say that a connected (relatively) open subset V of $S(1)$ is a John region of center x_0 if there exists a positive constant c_J with the following property: for each $x \in V$ there is a rectifiable curve γ in V connecting x to x_0 such that

$$\operatorname{dist}(z, S(1) \setminus V) \geq c_J \ell(\gamma(x, z)) \quad \text{for all } z \in \gamma. \tag{1.2}$$

Throughout the note, we call Γ a cone (with vertex at the origin) generated by a John base V of center x_0 if V is a John region in $S(1)$ of center x_0 and

$$\Gamma = \left\{ x \in \mathbf{R}^n \setminus \{0\} : \frac{x}{|x|} \in V \right\}.$$

Our result is as follows.

Theorem 1.1. *Let Γ be a cone generated by a John base V of center x_0. Then there exists a unique Martin kernel $K_\Gamma(\cdot, 0)$ at the origin and it is minimal. Also, there exists a unique Martin kernel $K_\Gamma(\cdot, \infty)$ at infinity and it is minimal. Furthermore, there exist a positive continuous function f on V and $p \geq n-2$ such that for $x \in \Gamma$,*

$$K_\Gamma(x, 0) = |x|^{-p} f(x/|x|) \quad \text{and} \quad K_\Gamma(x, \infty) = |x|^{2-n+p} f(x/|x|). \tag{1.3}$$

Theorem 1.1 is an extension of Kuran's result [13, Theorem 1]. He considered an NTA cone, i.e. a cone Γ such that $\Gamma \cap B(1)$ is an NTA domain in the sense of Jerison and Kenig [12]. The boundary Harnack principle and the uniqueness theorem obtained in [12] was applied to a bounded NTA domain $\Gamma \cap B(1)$ in his arguments. It is noteworthy that cones generated by John bases do not satisfy, in general, the boundary Harnack principle at a given boundary point. For example, let $\Gamma = \{x \in \mathbf{R}^n \setminus \{0\} : x/|x| \in S(1) \setminus \gamma\}$, where γ is a closed arc in $S(1)$ with endpoints a, b. Then the boundary Harnack principle does not hold at every point in $\{x \in \mathbf{R}^n \setminus \{0\} : x/|x| \in \gamma \setminus \{a, b\}\}$. We will show the boundary Harnack principle at the origin, using ideas from our previous paper [3].

The rest of the note is organized as follows. In Section 2, we will give a proof of Theorem 1.1. In Section 3, we will show the equivalence of the ordinary thinness and the mininal thinness of a set contained in a subcone of Γ, and will show that there is no positive superharmonic function u in a domain, which contains Γ, such that $|x|^p u(x) \to +\infty$ as $x \to 0$ along a subcone of Γ, where p is the homogeneous degree of $K_\Gamma(\cdot, 0)$ in Theorem 1.1.

Throughout the note, we use the symbol C to denote an absolute positive constant whose value is unimportant and may change from line to line. If necessary, we use $C_1, C_2, \cdots$ to specify them.

2. Proof of Theorem 1.1

We start by recalling the Harnack inequality involving the quasi-hyperbolic metric. Let x and y be points in a subdomain Ω of $\mathbf{R}^n$. The quasi-hyperbolic metric on Ω is defined by

$$k_\Omega(x, y) = \inf_\gamma \int_\gamma \frac{ds(z)}{\operatorname{dist}(z, \partial\Omega)},$$

where the infimum is taken over all rectifiable curves γ in Ω connecting x to y and ds stands for the line element on γ. We say that a finite sequence of balls $\{B(x_j, 2^{-1}\operatorname{dist}(x_j, \partial\Omega))\}_{j=1}^N$ is a Harnack chain between x and y if $x_1 = x$, $x_N = y$ and $x_{j+1} \in B(x_j, 2^{-1}\operatorname{dist}(x_j, \partial\Omega))$ for $j = 1, \cdots, N-1$. The number N is called the length of the Harnack chain. We observe that the infimum of the lengths of the Harnack chains between x and y is comparable to $k_\Omega(x, y) + 1$. Therefore the Harnack inequality yields the following.

Lemma 2.1. *There exists a constant $C > 1$ depending only on the dimension n such that if $x, y \in \Omega$, then*

$$\exp(-C(k_\Omega(x, y) + 1)) \le \frac{h(x)}{h(y)} \le \exp(C(k_\Omega(x, y) + 1))$$

for every positive harmonic function h in Ω.

We next recall the notion, a system of local reference points of order N (see [3, Definition 2.1] for details). We need the case $N = 1$.

Definition 2.2. Let $0 < \eta < 1$. We say that $\xi \in \partial\Omega$ has *a system of local reference points of order* 1 *with factor* η if there exist $r_\xi > 0$ and $C_\xi > 1$ with the following property: for each positive $r < r_\xi$ there is $y_r \in \Omega \cap S(\xi, r)$ such that $\operatorname{dist}(y_r, \partial\Omega) \geq C_\xi^{-1} r$ and

$$k_{\Omega\cap B(\xi,\eta^{-3}r)}(x, y_r) \leq C_\xi \log \frac{r}{\operatorname{dist}(x, \partial\Omega)} + C_\xi \quad \text{for } x \in \Omega \cap B(\xi, \eta r). \tag{2.1}$$

We should note that this notion controls the boundary behavior of positive harmonic functions. Indeed, by Lemma 2.1 and (2.1), there exist constants $C > 1$ and $\alpha > 1$ depending only on n and C_ξ such that

$$h(x) \leq C \left(\frac{r}{\operatorname{dist}(x, \partial\Omega)} \right)^\alpha h(y_r) \quad \text{for } x \in \Omega \cap B(\xi, \eta r), \tag{2.2}$$

whenever h is a positive harmonic function in $\Omega \cap B(\xi, \eta^{-3}r)$. In view of this, we would like to show the Carleson type estimate: if h is a positive and bounded harmonic function in $\Omega \cap B(\xi, \eta^{-3}r)$ vanishing on $\partial\Omega \cap B(\xi, \eta^{-3}r)$ except for a polar set, then

$$h(x) \leq C h(y_r) \quad \text{for } x \in \Omega \cap B(\xi, \eta^2 r). \tag{2.3}$$

To do this, we need to show that each point in $\Gamma \cap B(1)$ can be connected to x_0 by a curve satisfying (1.1), and that the origin has a system of local reference points of order 1.

Lemma 2.3. *Let Γ be a cone generated by a John base V of center x_0. Then each $x \in \Gamma \cap \overline{B(1)}$ can be connected to x_0 by a rectifiable curve γ in $\Gamma \cap \overline{B(1)}$ such that*

$$\operatorname{dist}(z, \partial\Gamma) \geq C_1 \ell(\gamma(x, z)) \quad \textit{for all } z \in \gamma, \tag{2.4}$$

where C_1 is a positive constant depending only on Γ.

Proof. Let $x \in \Gamma \cap B(1)$. Then, by the definition of V, there is a rectifiable curve γ' in V connecting $x/|x|$ to x_0 and satisfying (1.2). Let γ'_x be the image of γ' under the dilation mapping $x/|x|$ to x. Then γ'_x is the curve in $\Gamma \cap S(|x|) \subset \Gamma \cap B(1)$ connecting x to $|x|x_0$ and satisfies that for $z \in \gamma'_x$,

$$\begin{aligned} \operatorname{dist}(z, \partial\Gamma) &= |x| \operatorname{dist}(z/|x|, \partial\Gamma) \\ &\geq |x| \frac{c_J}{2} \ell(\gamma'(x/|x|, z/|x|)) = \frac{c_J}{2} \ell(\gamma'_x(x, z)). \end{aligned} \tag{2.5}$$

Indeed, the above inequality can be shown as follows: If $\operatorname{dist}(z/|x|, \partial\Gamma) = \operatorname{dist}(z/|x|, \{0\}) = 1$, then we have by (1.2)

$$\begin{aligned} \operatorname{dist}(z/|x|, \partial\Gamma) &= \frac{\ell(\gamma'(x/|x|, z/|x|))}{\ell(\gamma'(x/|x|, z/|x|))} \geq \frac{\ell(\gamma'(x/|x|, z/|x|))}{c_J^{-1} \operatorname{dist}(z/|x|, S(1) \setminus V)} \\ &\geq \frac{c_J}{2} \ell(\gamma'(x/|x|, z/|x|)). \end{aligned}$$

If $\operatorname{dist}(z/|x|, \partial\Gamma) \neq \operatorname{dist}(z/|x|, \{0\})$, then there is $y \in \partial\Gamma \setminus \{0\}$ such that $\operatorname{dist}(z/|x|, \partial\Gamma) = |z/|x| - y|$. Then the angle $\angle y0z$ must be less than $\pi/2$. Therefore we have by (1.2)

$$\begin{aligned}\operatorname{dist}(z/|x|, \partial\Gamma) &= |z/|x| - y/|y|| \cos(2^{-1}\angle y0z) \geq \frac{1}{\sqrt{2}} \operatorname{dist}(z/|x|, S(1) \setminus V) \\ &\geq \frac{c_J}{\sqrt{2}} \ell(\gamma'(x/|x|, z/|x|)).\end{aligned}$$

Let $\gamma = \gamma'_x \cup [|x|x_0, x_0]$, where $[|x|x_0, x_0]$ denotes the line segment between $|x|x_0$ and x_0. To complete the lemma, it suffices to show (2.4) for $z \in [|x|x_0, x_0]$. Let $w \in [|x|x_0, x_0]$. Since $\operatorname{dist}(|x|x_0, \partial\Gamma) \leq |x| \leq |w|$, it follows from (2.5) with $z = |x|x_0$ that

$$\begin{aligned}\ell(\gamma(x, w)) &= \ell(\gamma'_x) + ||x|x_0 - w| \leq \frac{2}{c_J} \operatorname{dist}(|x|x_0, \partial\Gamma) + |w| \\ &\leq \left(\frac{2}{c_J} + 1\right)|w| = \left(\frac{2}{c_J} + 1\right)\frac{\operatorname{dist}(w, \partial\Gamma)}{\operatorname{dist}(x_0, \partial\Gamma)}.\end{aligned}$$

Hence the lemma holds with $C_1 = (2c_J^{-1} + 1)^{-1} \operatorname{dist}(x_0, \partial\Gamma)$. □

Lemma 2.4. *Let Γ be a cone generated by a John base V of center x_0. Then there exists a positive constant C_2 depending only on Γ such that*

$$k_{\Gamma\cap B(2r)}(x, rx_0) \leq C_2 \log \frac{r}{\operatorname{dist}(x, \partial\Gamma)} + C_2 \quad \text{for } x \in \Gamma \cap \overline{B(r)},$$

whenever $r > 0$. In other words, the origin has a system of local reference points of order 1.

Proof. Let $r > 0$. We note that the conclusion in Lemma 2.3 is invariant under dilation since Γ is the cone. Therefore we see that for each $x \in \Gamma \cap \overline{B(r)}$ there is a curve γ in $\Gamma \cap \overline{B(r)}$ connecting x to rx_0 such that

$$\operatorname{dist}(z, \partial(\Gamma \cap B(2r))) = \operatorname{dist}(z, \partial\Gamma) \geq C_1 \ell(\gamma(x, z)) \quad \text{for all } z \in \gamma.$$

Since $\ell(\gamma) \leq C_1^{-1} \operatorname{dist}(rx_0, \partial\Gamma) = C_1^{-1} r \operatorname{dist}(x_0, \partial\Gamma)$, we have

$$\begin{aligned}k_{\Gamma\cap B(2r)}(x, rx_0) &\leq \int_\gamma \frac{ds(z)}{\operatorname{dist}(z, \partial\Gamma)} \leq 1 + \frac{1}{C_1} \int_{2^{-1}\operatorname{dist}(x,\partial\Gamma)}^{\ell(\gamma)} \frac{dt}{t} \\ &\leq C_2 \log \frac{r}{\operatorname{dist}(x, \partial\Gamma)} + C_2,\end{aligned}$$

where a constant C_2 depends only on C_1 and $\operatorname{dist}(x_0, \partial\Gamma)$. Thus the lemma follows. □

From now on, we suppose that Γ is a cone generated by a John base of center x_0. Using Lemmas 2.3 and 2.4 and repeating similar arguments to [3, Lemmas 5.1 and 6.1], we can obtain Lemmas 2.5 and 2.7 below. We say that a property holds quasi-everywhere if it holds apart from a polar set.

Lemma 2.5. (Carleson type estimate) *Let $r > 0$. Suppose that h is a positive harmonic function in $\Gamma \cap B(2r)$ vanishing quasi-everywhere on $\partial\Gamma \cap B(2r)$. If h is bounded in $\Gamma \cap B(2r)$, then*

$$h(x) \le Ch(rx_0) \quad \text{for } x \in \Gamma \cap \overline{B(2^{-1}r)},$$

where a constant C is independent of x, h and r.

Remark 2.6. First, we could prove Lemma 2.5 for sufficiently small r, say $0 < r \le r_0$. If $r > r_0$ and h satisfies the assumptions in Lemma 2.5, then $h(\frac{r}{r_0}\cdot)$ satisfies

$$h(\tfrac{r}{r_0}x) \le Ch(\tfrac{r}{r_0}r_0x_0) = Ch(rx_0) \quad \text{for } x \in \Gamma \cap \overline{B(2^{-1}r_0)}.$$

Hence Lemma 2.5 holds for all $r > 0$.

Let $\omega(x, E, D)$ denote the harmonic measure of a Borel set E for an open set D evaluated at x.

Lemma 2.7. *Let $r > 0$. If h is a positive and bounded harmonic function in $\Gamma \cap B(2r)$ vanishing quasi-everywhere on $\partial\Gamma \cap B(2r)$, then*

$$\omega(x, \Gamma \cap S(2^{-1}r), \Gamma \cap B(2^{-1}r)) \le C\frac{h(x)}{h(rx_0)} \quad \text{for } x \in \Gamma \cap \overline{B(3^{-1}r)},$$

where a constant C is independent of x, h and r.

As a consequence of these lemmas, we can obtain the following Boundary Harnack principle at the origin. For two positive functions f_1 and f_2, we write $f_1 \approx f_2$ if there exists a constant $C > 1$ such that $C^{-1}f_1 \le f_2 \le Cf_1$. The constant C is called the constant of comparison.

Lemma 2.8. (Boundary Harnack principle) *Let $r > 0$. If h_1 and h_2 are positive and bounded harmonic functions in $\Gamma \cap B(2r)$ vanishing quasi-everywhere on $\partial\Gamma \cap B(2r)$, then*

$$\frac{h_1(y)}{h_2(y)} \approx \frac{h_1(y')}{h_2(y')} \quad \text{for } y, y' \in \Gamma \cap \overline{B(3^{-1}r)},$$

where the constant of comparison is independent of y, y', h_1, h_2 and r.

We note again that this Boundary Harnack principle holds only at the origin, that is, it does not hold at other boundary points in general. So we can not apply the arguments in [2, Lemma 4 and Proof of Theorem 3] to prove the first statement in Theorem 1.1. We need the following lemma.

Lemma 2.9. *Let Ω be a subdomain of $\mathbf{R}^n$ with $n \ge 2$, and let $\xi \in \partial\Omega$. Suppose that h is a positive harmonic function in Ω such that h vanishes quasi-everywhere on $\partial\Omega \setminus \{\xi\}$ and $\lim_{x\to\infty} h(x) = 0$ when Ω is unbounded. If h is bounded in $\Omega \setminus B(\xi, r)$ for each $r > 0$, then the measure associated with h in the Martin representation is concentrating on minimal Martin boundary points at ξ.*

Proof. Let Δ and Δ_1 denote the Martin boundary and the minimal Martin boundary of Ω, respectively, and K_Ω stand for the Martin kernel of Ω. By the

Martin representation, there is a unique measure μ_h on Δ such that $\mu_h(\Delta\setminus\Delta_1)=0$ and

$$h(x)=\int_\Delta K_\Omega(x,y)\,d\mu_h(y)\quad\text{for } x\in\Omega.$$

We now write $\Delta(\xi)$ for the set of all Martin boundary points at ξ. Let E be a compact subset of $\Delta\setminus\Delta(\xi)$, and let $\{E_j\}$ be a decreasing sequence of compact neighborhoods of E in the Martin compactification of Ω such that $\bigcap_j E_j=E$ and $(E_1\cap\Omega)\cap B(\xi,r_1)=\emptyset$ for some $r_1>0$. Then, for each $j\in\mathbf{N}$, we have by [6, Corollary 9.1.4]

$$\widehat{R}_h^{E_j\cap\Omega}(x)=\int_{\Delta_1}\widehat{R}_{K_\Omega(\cdot,y)}^{E_j\cap\Omega}(x)\,d\mu_h(y)\quad\text{for } x\in\Omega,$$

where $\widehat{R}_u^F$ denotes the regularized reduced function of a positive superharmonic function u relative to F in Ω. By assumption on h, we see that $\lim_{j\to\infty}\widehat{R}_h^{E_j\cap\Omega}$ is a bounded harmonic function in Ω vanishing quasi-everywhere on $\partial\Omega$. The maximum principle gives $\lim_{j\to\infty}\widehat{R}_h^{E_j\cap\Omega}\equiv 0$. Thus we have by the monotone convergence

$$0=\lim_{j\to\infty}\widehat{R}_h^{E_j\cap\Omega}(x_0)=\int_{\Delta_1}\lim_{j\to\infty}\widehat{R}_{K_\Omega(\cdot,y)}^{E_j\cap\Omega}(x_0)\,d\mu_h(y).\tag{2.6}$$

If $y\in E\cap\Delta_1$, then $E_j\cap\Omega$ is not minimally thin at y for each j (cf. [6, Lemma 9.1.5]). Therefore we have

$$\lim_{j\to\infty}\widehat{R}_{K_\Omega(\cdot,y)}^{E_j\cap\Omega}(x_0)=K_\Omega(x_0,y)=1\quad\text{for } y\in E\cap\Delta_1.$$

Hence this, together with (2.6), concludes $\mu_h(E)=0$, and so $\mu_h(\Delta\setminus(\Delta(\xi)\cap\Delta_1))=0$. Thus the lemma follows. □

Let us give a proof of Theorem 1.1.

Proof of Theorem 1.1. We first show that the origin has at most one minimal Martin boundary point. Let ξ and η be minimal Martin boundary points at the origin. Then, by definition, there are sequences $\{y_j\}$ and $\{y'_j\}$ in Γ converging to the origin such that $G_\Gamma(\cdot,y_j)/G_\Gamma(x_0,y_j)\to K_\Gamma(\cdot,\xi)$ and $G_\Gamma(\cdot,y'_j)/G_\Gamma(x_0,y'_j)\to K_\Gamma(\cdot,\eta)$ as $j\to\infty$. Here $K_\Gamma(\cdot,\xi)$ denotes the Martin kernel corresponding to ξ. Let $r>0$ and let $x\in\Gamma\setminus B(3r)$. We apply Lemma 2.8 to $h_1=G_\Gamma(x,\cdot)$ and $h_2=G_\Gamma(x_0,\cdot)$, and let $j\to\infty$. Then we have $K_\Gamma(x,\xi)\approx K_\Gamma(x,\eta)$. Since the constant of comparison is independent of r, it follows that $K_\Gamma(\cdot,\xi)\approx K_\Gamma(\cdot,\eta)$ on whole of Γ. By minimality and $K_\Gamma(x_0,\xi)=1=K_\Gamma(x_0,\eta)$, we obtain $K_\Gamma(\cdot,\xi)\equiv K_\Gamma(\cdot,\eta)$, and hence $\xi=\eta$. To complete the first statement of the theorem, it is enough to show that Martin boundary points at the origin are minimal. But this follows from Lemma 2.9. Indeed, if ζ is a Martin boundary point at the origin and $0<r<3^{-1}$, then Lemma 2.8 yields that

$$K_\Gamma(x,\zeta)\approx\frac{G_\Gamma(x,3^{-1}rx_0)}{G_\Gamma(x_0,3^{-1}rx_0)}\quad\text{for } x\in\Gamma\setminus\overline{B(3r)}.$$

Hence $K_\Gamma(\cdot,\zeta)$ satisfies the assumptions in Lemma 2.9, and so ζ is minimal. Also, by the Kelvin transfomation with respect to $S(1)$, we observe that there is a unique Martin boundary point at infinity and it is minimal. The last statement of the theorem can be obtained by the similar way to [13, p. 472]. □

3. Further results

Let E be a subset of $\mathbf{R}^n$ and let $\xi \in \mathbf{R}^n$ be a limit point of E. We say that E is thin at ξ (in the ordinary sense) if there exists a positive superharmonic function u in $\mathbf{R}^n$ such that $u(\xi) < +\infty$ and $u(x) \to +\infty$ as $x \to \xi$ along E. The original definition of minimal thinness by Naïm is based on the regularized reduced function of the Martin kernel. We define minimal thinness by the following equivalent condition (cf. [6, Theorem 9.2.7]). Let ξ be a minimal Martin boundary point of a domain Ω and let E be a subset of Ω, where ξ is a Martin topology limit point of E. We say that E is minimally thin at ξ with respect to Ω if there exists a Green potential $G_\Omega\mu$ in Ω such that $\int K_\Omega(x,\xi)\,d\mu(x) < +\infty$ and $G_\Omega\mu(y)/G_\Omega(x_0,y) \to +\infty$ as $y \to \xi$ along E in the Martin topology. For a subset E of $\mathbf{R}^n$, we write $\Gamma(E) = \{ry : r > 0, y \in E\}$. Note from Theorem 1.1 that a unique minimal Martin boundary point at 0 may be identified with the Euclidean boundary point 0.

Theorem 3.1. *Let Γ be a cone generated by a John base V of center x_0. Let U be a subset of $S(1)$ such that $\overline{U} \subset V$, and suppose that E is a subset of $\Gamma(U)$. Then E is thin at 0 if and only if E is minimally thin at 0 with respect to Γ.*

This was first proved in the half-space by Lelong-Ferrand [14], and was extended by Aikawa [1] to a Lipschitz domain. To prove Theorem 3.1, we need the following estimates.

Lemma 3.2. *Let Γ be a cone generated by a John base V of center x_0, and let U be a subset of $S(1)$ such that $\overline{U} \subset V$. The following statements hold:*

(i) *For $x \in \Gamma(U) \cap B(6^{-1})$,*

$$G_\Gamma(x,x_0)K_\Gamma(x,0) \approx |x|^{2-n}, \tag{3.1}$$

where the constant of comparison is independent of x.

(ii) *For $x \in \Gamma(U) \cap B(6^{-1})$ and $y \in \Gamma(U) \cap B(3|x|)$,*

$$\frac{G_\Gamma(x,x_0)G_\Gamma(x,y)}{G_\Gamma(x_0,y)} \approx |x-y|^{2-n}, \tag{3.2}$$

where the constant of comparison is independent of x and y;

(iii) *For $x \in \Gamma \cap B(6^{-1})$ and $y \in \Gamma(U) \cap (B(2^{-1}) \setminus B(3|x|))$,*

$$\frac{G_\Gamma(x,x_0)G_\Gamma(x,y)}{G_\Gamma(x_0,y)} \le C|x-y|^{2-n}, \tag{3.3}$$

where a constant C is independent of x and y.

To apply Lemma 2.1 to the Green function, we need the following: If $z \in \Omega$, then

$$k_{\Omega\setminus\{z\}}(x,y) \le 3k_\Omega(x,y) + \pi \quad \text{for } x,y \in \Omega \setminus B(z, 2^{-1}\operatorname{dist}(z,\partial\Omega)). \tag{3.4}$$

The proof of this inequality may be found in [3, Lemma 7.2].

Proof of Lemma 3.2. (i) We observe from the Harnack inequality that

$$K_\Gamma(6^{-1}x_0, \infty) \approx 1 \quad \text{and} \quad G_\Gamma(6^{-1}x_0, x_0) \approx \operatorname{dist}(x_0, \partial\Gamma)^{2-n}.$$

By Lemma 2.8 and (1.3), we have for $x \in \Gamma \cap B(6^{-1})$,

$$G_\Gamma(x,x_0) \approx \frac{G_\Gamma(x,x_0)}{G_\Gamma(6^{-1}x_0,x_0)} \approx \frac{K_\Gamma(x,\infty)}{K_\Gamma(6^{-1}x_0,\infty)} \approx K_\Gamma(x,\infty) = \frac{|x|^{2-n}}{K_\Gamma(x,0)} f(x/|x|)^2.$$

Since f is positive and continuous on $\overline{U}$, we obtain (3.1) for $x \in \Gamma(U) \cap B(6^{-1})$.

(ii) Let $x \in \Gamma(U) \cap B(6^{-1})$ and $y \in \Gamma(U) \cap B(3|x|)$. We will consider three cases.

Case 1: $|y| \le 6^{-1}|x|$. By Lemma 2.8 and (3.1), we have

$$G_\Gamma(x,x_0)\frac{G_\Gamma(x,y)}{G_\Gamma(x_0,y)} \approx G_\Gamma(x,x_0)K_\Gamma(x,0) \approx |x|^{2-n}.$$

Since $|x| \approx |x-y|$, we obtain (3.2) in this case.

Case 2: $|y| \ge 6^{-1}|x|$ and $|y-x| \ge 2^{-1}\operatorname{dist}(x,\partial\Gamma)$. Since $\operatorname{dist}(y,\partial\Gamma) \ge |y|\operatorname{dist}(\overline{U},\partial\Gamma)$ and $\operatorname{dist}(6^{-1}|x|x_0,\partial\Gamma) = 6^{-1}|x|\operatorname{dist}(x_0,\partial\Gamma) \ge 18^{-1}|y|\operatorname{dist}(x_0,\partial\Gamma)$, we have by Lemma 2.4

$$\begin{aligned} k_{\Gamma\cap B(1)}(6^{-1}|x|x_0, y) &\le k_{\Gamma\cap B(1)}(6^{-1}|x|x_0, |y|x_0) + k_{\Gamma\cap B(1)}(y, |y|x_0) \\ &\le C_2 \log\frac{|y|}{\operatorname{dist}(6^{-1}|x|x_0,\partial\Gamma)} + C_2\log\frac{|y|}{\operatorname{dist}(y,\partial\Gamma)} + 2C_2 \\ &\le C. \end{aligned}$$

Therefore Lemma 2.1, together with (3.4), gives

$$G_\Gamma(x,y) \approx G_\Gamma(x, 6^{-1}|x|x_0) \quad \text{and} \quad G_\Gamma(x_0,y) \approx G_\Gamma(x_0, 6^{-1}|x|x_0).$$

Since $|x - 6^{-1}|x|x_0| \approx |x-y|$, we obtain from Case 1 that

$$\frac{G_\Gamma(x,x_0)G_\Gamma(x,y)}{G_\Gamma(x_0,y)} \approx \frac{G_\Gamma(x,x_0)G_\Gamma(x,6^{-1}|x|x_0)}{G_\Gamma(x_0,6^{-1}|x|x_0)} \approx |x-y|^{2-n}.$$

Case 3: $|y-x| \le 2^{-1}\operatorname{dist}(x,\partial\Gamma)$. By the Harnack inequality, $G_\Gamma(x,x_0) \approx G_\Gamma(y,x_0)$. Since $G_\Gamma(x,y) \approx |x-y|^{2-n}$ in this case, we obtain (3.2).

(iii) Let $x \in \Gamma \cap B(6^{-1})$ and $y \in \Gamma(U) \cap (B(2^{-1}) \setminus B(3|x|))$. By Lemma 2.5, we have $G_\Gamma(x,x_0) \le CG_\Gamma(|y|x_0,x_0)$. It follows from Lemma 2.4 and $\operatorname{dist}(y,\partial\Gamma) \ge |y|\operatorname{dist}(\overline{U},\partial\Gamma)$ that $k_{\Gamma\cap B(1)}(y,|y|x_0) \le C$, and so Lemma 2.1 gives $G_\Gamma(|y|x_0,x_0) \approx G_\Gamma(y,x_0)$. Hence

$$\frac{G_\Gamma(x,x_0)G_\Gamma(x,y)}{G_\Gamma(x_0,y)} \le CG_\Gamma(x,y) \le C|x-y|^{2-n}.$$

The proof of the lemma is complete. □

Let $\widehat{R}_1^E$ denote the regularized reduced function of the constant function 1 relative to E in $\mathbf{R}^n$.

Proof of Theorem 3.1. We may assume, without loss of generality, that 0 is a limit point of E and $E \subset B(6^{-1})$. We first assume that E is thin at 0. By Wiener's criterion (cf. [6, Theorem 7.7.2]), there exists a sequence $\{a_j\}$ of positive numbers such that

$$\lim_{j\to\infty} a_j = +\infty \quad \text{and} \quad \sum_{j=1}^{\infty} a_j \widehat{R}_1^{E_j}(0) < +\infty,$$

where $E_j = \{x \in E : 2^{-j-1} \le |x| \le 2^{-j}\}$. Let $d\nu_j(x) = G_\Gamma(x, x_0) d\mu_j(x)$, where $\widehat{R}_1^{E_j}(x) = \int |x-y|^{2-n} d\mu_j(y)$. Since μ_j is supported on $\overline{E}_j$, it follows from (3.2) that

$$1 = \widehat{R}_1^{E_j}(y) \le C \frac{G_\Gamma \nu_j(y)}{G_\Gamma(x_0, y)} \quad \text{for quasi-every } y \in E_j.$$

Then $u(y) = \sum_{j=1}^{\infty} a_j G_\Gamma \nu_j(y)$ is a Green potential in Γ such that $u(y)/G_\Gamma(x_0, y) \to +\infty$ as $y \to 0$ along $E \setminus F$, where F is a polar set. We also have by (3.1)

$$\sum_{j=1}^{\infty} a_j \int K_\Gamma(x, 0)\, d\nu_j(x) \le C \sum_{j=1}^{\infty} a_j \widehat{R}_1^{E_j}(0) < +\infty.$$

Hence $E \setminus F$ is minimally thin at 0 with respect to Γ, and so is E.

We next assume that E is minimally thin at 0 with respect to Γ. Then there is a measure μ supported on $\overline{\Gamma(U) \cap B(6^{-1})}$ such that $\int K_\Gamma(x, 0)\, d\mu(x) < +\infty$ and $G_\Gamma \mu(y)/G_\Gamma(x_0, y) \to +\infty$ as $y \to 0$ along E. Let $d\nu(x) = G_\Gamma(x, x_0)^{-1} d\mu(x)$. Then we have by (3.2) and (3.3)

$$\frac{G_\Gamma \mu(y)}{G_\Gamma(x_0, y)} \le C \int |x-y|^{2-n}\, d\nu(x) \quad \text{for } y \in E,$$

and so $\int |x-y|^{2-n}\, d\nu(x) \to +\infty$ as $y \to 0$ along E. Also, we have by (3.1)

$$\int |x|^{2-n}\, d\nu(x) \le C \int K_\Gamma(x, 0)\, d\mu(x) < +\infty.$$

Hence E is thin at 0. The proof is complete. □

Corollary 3.3. *Let Γ be a cone generated by a John base, and suppose that E is a non-polar set such that $\overline{E} \subset \Gamma$. Then $\Gamma(E)$ is not minimally thin at 0 with respect to Γ.*

Proof. Let $r > 0$ and let $rE = \{ry : y \in E\}$. Observe that $\widehat{R}_1^E(x) = \widehat{R}_1^{rE}(rx)$ for $x \in \mathbf{R}^n$. Since E is non-polar, we have

$$\widehat{R}_1^{rE}(0) = \widehat{R}_1^E(0) > 0 \quad \text{for all } r > 0.$$

This shows that $\Gamma(E)$ is not thin at 0. Hence Theorem 3.1 concludes that $\Gamma(E)$ is not minimally thin at 0 with respect to Γ. □

Theorem 3.4. *Let Γ be a cone generated by a John base, and suppose that Ω is a domain such that $\Gamma \cap B(1) \subset \Omega$ and $0 \in \partial\Omega$. If E is a non-polar set such that $\overline{E} \subset \Gamma$, then there is no positive superharmonic function u in Ω such that*

$$\lim_{x \to 0, x \in \Gamma(E)} |x|^p u(x) = +\infty, \tag{3.5}$$

where $p > 0$ is the homogeneous degree of $K_\Gamma(\cdot, 0)$ in Theorem 1.1.

Proof. Let u be a positive superharmonic function in Ω. By [6, Theorem 9.3.3], $u/K_\Gamma(\cdot, 0)$ has a finite minimal fine limit l at 0 with respect to Γ. That is, there exists a subset F of Γ, minimally thin at 0, such that $u(x)/K_\Gamma(x, 0) \to l$ as $x \to 0$ along $\Gamma \setminus F$. By Corollary 3.3, we can find a sequence $\{x_j\}$ in $\Gamma(E) \setminus F$ converging to 0 such that $u(x_j)/K_\Gamma(x_j, 0) \to l$ as $j \to \infty$. Hence there is no positive superharmonic function in Ω satisfying (3.5). □

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