Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 32, 2007, 125–139

THURSTON UNBOUNDED EARTHQUAKE MAPS

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Abstract. In this paper, we obtain an analogue in earthquake theory to the David theorem on the solution to the Beltrami differential equation, that is, we introduce a sufficient condition for a type of Thurston unbounded earthquake measures to be induced by earthquake maps.

1. Introduction

Let $\mathbf{D}$ be the unit open disk centered at the origin of the complex plane $\mathbf{C}$. An orientation-preserving homeomorphism $F\colon \mathbf{D} \to F(\mathbf{D}) \subset \mathbf{C}$ is said to be *quasiconformal* if there exists a constant $K > 0$ such that for every point $z \in \mathbf{D}$,

$$K_F(z) = \limsup_{r\to 0} \frac{\max\limits_{0\le\theta\le 1} |f(z+re^{i2\pi\theta})|}{\min\limits_{0\le\theta\le 1} |f(z+re^{i2\pi\theta})|} \le K.$$

The *maximal complex dilatation* $K(F)$ of a quasiconformal map F is defined to be

$$K(F) = \inf_{\mathrm{supp}(K_F)} \sup_{z\in\mathrm{supp}(K_F)} K_F(z).$$

See Section IV.4 of [12] for the equivalence of such a definition to others on quasiconformal mappings. A quasiconformal map F has the following two properties:

(i) F is absolutely continuous on almost all horizontal or vertical lines. This implies that the partial derivatives and then the Beltrami coefficient

$$\mu_F(z) = \frac{\overline{\partial} F(z)}{\partial F(z)}$$

of F exists for almost all $z \in \mathbf{D}$ with respect to the Lebesgue measure; and

(ii) μ_F is a Borel measurable function on $\mathbf{D}$ satisfying

$$\|\mu_F\|_\infty = \inf_{\mathrm{supp}(\mu_F)} \sup_{z\in\mathrm{supp}(\mu_F)} |\mu_F(z)| \le k$$

for a constant $0 < k < 1$.

In fact, the above two conditions are also sufficient for F to be quasiconformal (see [12] for a proof). More remarkably, it is also true, due to Morrey, that for every

2000 Mathematics Subject Classification: Primary 37E10; Secondary 30F60, 37E20.
Key words: Circle homeomorphism, earthquake map, cross ratio.
The first author wishes to thank the PSC-CUNY grants for partial supports.

Borel measurable function μ on $\mathbf{D}$ satisfying condition (ii), there exists a unique quasiconformal mapping F (up to post-composition by a conformal map) such that $\mu_F = \mu$ almost everywhere with respect to the Lebesgue measure on $\mathbf{D}$. Futhermore, Ahlfors and Bers showed in [1] that F holomorphically depends on μ.

Theorem A. (Measurable Riemann Mapping Theorem) *If μ is a Borel measurable function defined on $\mathbf{D}$ with $\|\mu\|_\infty < 1$, then there exists a unique quasiconformal mapping, up to post-composition by a conformal map, satisfying the Beltrami differential equation*

$$\overline{\partial} F = \mu \cdot \partial F$$

for almost all z with respect to the Lebesgue measure on $\mathbf{D}$.

Later in [2], David extended the existence and uniqueness of the solution to the Beltrami differential equation to the situation in which $\|\mu\|_\infty = 1$ but the growth of $|\mu|$ to 1 is under some asymptotic control.

Theorem B. (David Theorem) *Let m denote the Lebesgue measure on $\mathbf{D}$. If a measurable function μ defined on $\mathbf{D}$ satisfies*

$$m\{z \in \mathbf{D} : |\mu(z)| > 1 - \varepsilon\} \leq C \exp\Big(-\frac{\alpha}{\varepsilon}\Big)$$

for all $0 < \varepsilon < \varepsilon_0$ and for some positive constants C, α, ε_0, then the Beltrami differential equation has a homeomorphism solution $F \in W^{1,1}_{\text{loc}}(\mathbf{D})$, which is unique up to post-composition by a conformal map.

In this paper, we obtain an analogue of the David Theorem to earthquake maps. We will leave the long definition of an earthquake map to the next section. Roughly speaking, an earthquake map E on the hyperbolic plane $\mathbf{D}$ is a piecewise Möbius transformation defined on domains divided by nonintersecting geodesics on $\mathbf{D}$ such that the comparisons of the Möbius transformations on different domains are hyperbolic Möbius transformations with axes separating their domains and translating in the same direction, where the collection $\mathscr{L}$ of the nonintersecting geodesics forms a lamination on $\mathbf{D}$ (that is, it foliates a closed subset of $\mathbf{D}$). For an earthquake map $(E, \mathscr{L})$, the amount of shearing or twisting along the geodesics in $\mathscr{L}$ naturally induces a transversal measure σ supported on $\mathscr{L}$, called an earthquake measure induced by $(E, \mathscr{L})$. In general, by *an earthquake measure* $(\sigma, \mathscr{L})$ we mean a transversal measure σ supported on a lamination $\mathscr{L}$ on $\mathbf{D}$. The *Thurston norm* of an earthquake measure $(\sigma, \mathscr{L})$ is defined to be

$$\|\sigma\|_{Th} = \sup_{l(\beta)\leq 1} \sigma(\beta) = \sup_{l(\beta)=1} \sigma(\beta),$$

where β is a closed geodesic segment transversal to $\mathscr{L}$ and $l(\beta)$ denotes the hyperbolic length of β. We say that an earthquake measure is *Thurston bounded* if it has a finite Thurston norm, and an earthquake map is *Thurston bounded* if its induced earthquake measure is Thurston bounded.

Thurston showed in [14] that if an earthquake map is a quasi-isometry with respect to the hyperbolic metric on $\mathbf{D}$, then its induced earthquake measure is

Thurston bounded. In the same paper, he also pointed out that the converse is also true (see [5] for a detailed proof).

Theorem C. (Thurston) *If an earthquake measure $(\sigma, \mathscr{L})$ is Thurston bounded, then there exists an earthquake map $(E, \mathscr{L})$ such that σ is the induced earthquake measure by E. Moreover, up to post-composition by a Möbius transformation, σ determines the isometries of E on all gaps, and for any leaf $l \in \mathscr{L}$, two possibly different isometries on l only differ by a hyperbolic isometry with axis l and translation length between 0 and the measure $\sigma(l)$ of l.*

Here we present the following analogue of the David Thereom to Thurston unbounded earthquake measures. Given a lamination $\mathscr{L}$, we use β to denote an arbitrary geodesic arc transversal to $\mathscr{L}$ of hyperbolic length ≤ 1 and $\delta(\beta)$ to denote the Euclidean distance from the arc β to the boundary of $\mathbf{D}$.

Main Theorem. *If an earthquake measure $(\sigma, \mathscr{L})$ satisfies*

$$\sigma(\beta) \leq \frac{2}{3} \ln \ln \frac{1}{\delta(\beta)} + C \tag{1}$$

for a constant $C > 0$ and any geodesic arc β transversal to $\mathscr{L}$ of hyperbolic length ≤ 1 and sufficiently close to the boundary in the Euclidean metric, then there exists an earthquake map $(E, \mathscr{L})$ such that σ is the earthquake measure induced by E. Moreover, up to post-composition by a Möbius transformation σ determines the isometries of E on all gaps, and for any leaf $l \in \mathscr{L}$, two possibly different isometries on l only differ by a hyperbolic isometry with axis l and translation length between 0 and the measure $\sigma(l)$ of l.

In the next section, we first give the precise definitions of earthquake maps and earthquake measures, then we provide some details to see how an earthquake map induces an earthquake measure, and finally we summarize what is known about constructing an earthquake map given the measure. In the third section, we prove our main theorem.

Acknowledgement. The authors wish to thank Profs. Frederick P. Gardiner and Linda Keen for their interests on this paper. They are grateful to the referee for providing them many useful suggestions to improve the presentation of the paper.

2. Earthquake maps and earthquake measures

Consider $\mathbf{D}$ as a hyperbolic plane. A *geodesic lamination* $\mathscr{L}$ in $\mathbf{D}$ is a collection of geodesics which foliate a closed subset L of $\mathbf{D}$. The set L is called *the locus* of $\mathscr{L}$, the geodesics are called the *leaves* of $\mathscr{L}$, the connected components of $\mathbf{D} \setminus L$ are called the *gaps*, and the gaps and the leaves of $\mathscr{L}$ are called the *strata* of the lamination.

Let $\mathscr{L}$ be a geodesic lamination in $\mathbf{D}$. By an *$\mathscr{L}$-left earthquake map E* we mean that E is an injective and surjective (and often discontinuous) map from $\mathbf{D}$ to $\mathbf{D}$ satisfying:

(i) for any stratum A, the restriction of E on A is the restriction of a Möbius transformation, which maps $\mathbf{D}$ onto $\mathbf{D}$, on A; and

(ii) for any two strata A and B, the *comparison map*

$$cmp(A,B) = (E|_A)^{-1} \circ E|_B : \mathbf{D} \to \mathbf{D}$$

is a hyperbolic transformation whose axis weakly separates A and B and which translates to the left as viewed from A. Here $E|_A$ and $E|_B$ denote the Möbius transformations representing E on A and B, and we say that a line l *weakly separates* two sets A and B if any path connecting a point $a \in A$ to a point $b \in B$ intersects l.

Similarly one can define a right earthquake map, and by an earthquake map we mean it is either a left or right earthquake. In general, there are parallel results between left and right earthquakes. In this paper, we assume earthquakes are left earthquakes.

As showed by Thurston in [14], it is not hard to see that each left earthquake map $(E, \mathscr{L})$ extends to an orientation-preserving homeomorphism h on the boundary circle $\mathbf{S}^1$. The greatness of the invention of earthquake maps is that each orientation-preserving circle homeomorphism h can be realized in such a way. Furthermore, the quasi-symmetry of h on $\mathbf{S}^1$ corresponds to the quasi-isometry of E on $\mathbf{D}$ in the hyperbolic metric. For details and some new developments, see [14], [3], [6], [5], [13], [7], [8], [9], [10], [4] and [11]. In this paper, we focus on the connection between earthquake maps and their infinitesimal expressions, namely, earthquake measures. As a parallel to the connection between quasiconformal mappings and Beltrami differential coefficients, we find an analogue of the David Theorem between earthqake maps and earthquake measures.

Now let us see how an earthquake map induces an earthquake measure. Roughly speaking, the earthquake measure induced by an earthquake map E quantifies the amount of shearing along the geodesics in the lamination of E. More precisely, it can be introduced as follows. Given an earthquake map $(E, \mathscr{L})$ and two geodesic lines l_* and l^* in $\mathscr{L}$, let β be a closed geodesic segment which is transversal to both l_* and l^* and intersects them at its endpoints. The amount $\nu(\beta)$ of *relative transversal shearing* of the earthquake map $(E, \mathscr{L})$ along β is defined as follows. Let $P = \{I_i\}_{i=1}^n$ be a partition of β into small geodesic segments, and let T_i be the comparison map between the strata containing the endpoints of the segment I_i. The *translation length*, denoted by $\tau(T_i)$, of each hyperbolic Möbius transformation $T_i\colon \mathbf{D} \to \mathbf{D}$ can be defined as the logarithm of the derivative of T_i at its expanding fixed point. Let $\nu(P) = \sum_{i=1}^n \tau(T_i)$. Now we define

$$\nu(\beta) = \inf_P \nu(P). \tag{2}$$

The *earthquake measure* $\sigma(\beta)$ of β, induced by the earthquake map $(E, \mathscr{L})$, is defined to be

$$\sigma(\beta) = \inf_{\beta'} \nu(\beta'), \tag{3}$$

where β' is a closed geodesic segment containing β in its interior. Let $\mathbf{X}$ denote the space $\mathbf{S}^1 \times \mathbf{S}^1 \setminus \{\text{the diagonal}\}$ factorized by the equivalence relation $(a, b) \sim (b, a)$. Then σ naturally extends to a Borel measure on $\mathbf{X}$ with its support consisting of all pairs of the endpoints of leaves in $\mathscr{L}$. That is, a Borel measure on $\mathbf{X}$ supported on $\mathscr{L}$. (See [7] for the details to show σ is indeed a Borel measure on $\mathbf{X}$.) More generally, by *an earthquake measure* we mean any Borel measure on $\mathbf{X}$ supported on a lamination $\mathscr{L}$.

Each earthquake map induces an earthquake measure, but the converse is not always true. (See [5] for examples of earthquake measures which can not be induced by earthquake maps, and one example will be briefly reviewed later in this section.) On the other hand, if an earthquake measure is Thurston bounded, then the converse is true. That is the Theorem C in the introduction.

In order to briefly summarize the known results on the construction of an earthquake map given an earthquake measure, we define a *generalized earthquake map* as a map that satisfies all the conditions of an earthquake map E except that E is not required to be onto. In the same way as for an earthquake map, a generalized earthquake map induces an earthquake measure. There are examples of earthquake measures that can be only induced by generalized earthquake maps but not earthquake maps. Thurston introduced the following example in [14]. For simplicity, we work with the upper half plane $\mathbf{H}$. Let l_n be the geodesic line connecting $-n$ to ∞, where $n = 0, 1, 2, \cdots$, and let $\mathscr{L}$ be the collection of all l_n's. Suppose that the weight on each geodesic l_n is 1. Define a generalized earthquake map E to be the identity map on the right half hyperbolic plane, and $E|_{l_{n-1}}^{-1} \circ E|_{l_n}$ is defined to be the hyperbolic map with l_n as its axis, $-n$ as its attracting fixed point and 1 as its translation length. Take $E|_{l_0}(z) = \frac{1}{e}z$, and then $E|_{l_{n-1}}^{-1} \circ E|_{l_n}(z) = \frac{1}{e}(z + n) - n$ for each $n > 0$. Since

$$E|_{l_n} = E|_{l_0} \circ (E|_{l_0}^{-1} \circ E|_{l_1}) \circ \cdots \circ (E|_{l_{n-2}}^{-1} \circ E|_{l_{n-1}}) \circ (E|_{l_{n-1}}^{-1} \circ E|_{l_n}),$$

E maps the geodesic l_n to the geodesic connecting $E(-n) = -(\frac{1}{e} + \frac{1}{e^2} + \cdots + \frac{1}{e^n})$ to ∞. Clearly, $E(-n)$ converges to $-\frac{1}{e-1}$ as n goes to ∞, and hence E maps the full hyperbolic plane $\mathbf{H}$ into the hyperbolic half plane to the right of the geodesic connecting $-\frac{1}{e-1}$ to ∞. Therefore E is not onto and hence it is not an earthquake map.

A more interesting example was constructed in [5]. In that example, an earthquake measure $(\sigma, \mathscr{L})$ was inductively constructed such that the generalized earthquake map E_σ corresponding to σ is not an earthquake map but the generalized earthquake map $E_{2\sigma}$ corresponding to $(2\sigma, \mathscr{L})$ is acturally an earthquake map.

In [5], the work in sections 2.1 and 2.2 was devoted to the construction of an earthquake map for a given earthquake measure with finite Thurston norm. It was also pointed out there that the procedure applies to the construction of a map similar to an earthquake map, but not satisfying the "onto" condition, for any Borel measure supported on a lamination. In our terminology, the work of sections 2.1, 2.2 and 2.3 in [5] shows the following theorem.

Theorem D. *Given any earthquake measure $(\sigma, \mathscr{L})$, there exists a generalized earthquake map $(E, \mathscr{L})$ such that σ is the earthquake measure induced by E. Moreover, up to post-composition by a Möbius transformation, σ determines the isometries of E on all gaps, and for any leaf $l \in \mathscr{L}$, two possibly different isometries on l only differ by a hyperbolic isometry with axis l and translation length between 0 and the measure $\sigma(l)$ of l.*

3. Proof of Main Theorem

Given an earthquake measure σ, by Theorem D, there exists a generalized earthquake map E_σ that induces the measure σ. The proof of the Main Theorem of this paper therefore reduces to showing the following:

Theorem 1. *If σ satisfies condition* (1) *in our Main Theorem then any generalized earthquake E_σ inducing σ is actually an earthquake map.*

In order to prove a generalized earthquake to be an earthquake, we only need to show that for any infinite sequence $\{l_n\}_{n=0}^{\infty}$ of geodesics in the lamination $\mathscr{L}$ that shrink to a point on $\mathbf{S}^1$ as $n \to \infty$, the images $E(l_n)$ also shrink to a point on $\mathbf{S}^1$. Note that because the geodesics in $\mathscr{L}$ cannot intersect, there are only two different ways for $\{l_n\}_{n=0}^{\infty}$ to shrink to a point. One way is that l_n's eventually share one endpoint and the other is that there exists an infinite subsequence $\{l_{n_k}\}_{k=0}^{\infty}$ of $\{l_n\}_{n=0}^{\infty}$ such that any two of them have different endpoints on both sides of the limit point. Under condition (1), we will show first that in the first case $E(l_n)$ converges to a point; then we will show that in a special situation of the second case they converge to a point; and finally we will show that they converge to a pont in the general situation of the second case.

Let us first develop some background. Given a quadruple $Q = \{a, b, c, d\}$ consisting of four points a, b, c, d on the unit circle $\mathbf{S}^1$ arranged in the counterclockwise order, we denote one cross ratio of Q by

$$cr(Q) = \frac{(b-a)(d-c)}{(c-b)(d-a)}. \tag{4}$$

It is easy to prove the following two lemmas (see [7] and [9]).

Lemma 1. *A quadruple Q has $cr(Q) = 1$ if and only if the geodesic $\overline{ac}$ from a to c is perpendicular to the geodesic $\overline{bd}$ from b to d, and if and only if the hyperbolic distance from $\overline{ab}$ to $\overline{cd}$ (or $\overline{bc}$ to $\overline{da}$) is equal to $\ln(3 + 2\sqrt{2})$.*

Lemma 2. *Let Q be a quadruple of four points a, b, c, d on $\mathbf{S}^1$ arranged in the counterclockwise order. Suppose y denotes the cross ratio $cr(Q)$ and x denotes the hyperbolic distance between the geodesics connecting a to d and b to c respectively. Then $y = \left(\frac{e^{x/2} - e^{-x/2}}{2}\right)^2$. Furthermore, there exists a universal constant $\delta > 0$ such that if $y \leq \delta$, then $x \geq \sqrt{y}$, and indeed $\lim_{y \to 0^+} \frac{x}{\sqrt{y}} = 2$.*

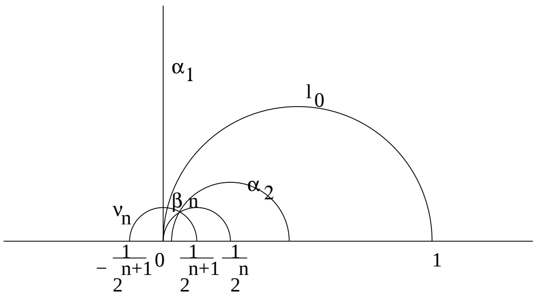

Figure 1.

Proposition 1. *Let* $(\sigma, \mathscr{L})$ *be an earthquake measure satisfying condition* (1) *and let* E *be a generalized earthquake map that induces the measure* σ*. Suppose that at a point* $a \in \mathbf{S}^1$*, there are infinitely many geodesics in* $\mathscr{L}$ *emanating from* a *and shrinking to* a (*in the Euclidean metric*)*. Let* l_0 *be one of those. Then* E *maps the hyperbolic half plane bounded by* l_0 *onto a hyperbolic half plane bounded by* $E(l_0)$*.*

Proof. For simplicity, let us work with the upper half plane $\mathbf{H}$. Without loss of generality, we assume that $a = 0$, l_0 connects 0 to 1 and $E|_{l_0}$ is the identity map. For each $n \geq 1$, let $\mathscr{L}_n$ be the collection of geodesics in $\mathscr{L}$ connecting 0 to points on $(1/2^n, 1]$, let $\sigma_n = \sigma(\mathscr{L}_n)$, and let β_n be the segment on the geodesic γ_n connecting $-1/2^{n+1}$ to $1/2^{n+1}$ and between the geodesics connecting 0 to $1/2^n$ and 1 respectively (see Figure 1). Clearly,

$$\delta(\beta_n) = \frac{\sqrt{3}}{2}\frac{1}{2^{n+1}} > \frac{1}{2^{n+2}}.$$

Now we show that the hyperbolic length of β_n is less than 1. Let α_1 be the geodesic passing through 0 and perpendicular to γ_n, that is the goedesic connecting 0 to ∞. Let P be the intersection point between γ_n and the geodesic connecting 0 to $1/2^n$ and α_2 be the geodesic passing through P and perpendicular to γ_n. Through elementary calculation, one can work out that α_2 is a semicircle centered at $1/2^n$ and with radius $\sqrt{3}/2^{n+1}$. Hence α_2 connects $\frac{1}{2^n} - \frac{\sqrt{3}}{2^{n+1}}$ to $\frac{1}{2^n} + \frac{\sqrt{3}}{2^{n+1}}$. Clearly, the hyperbolic length of β_n is less than the length of the segment β'_n on γ_n between α_1 and α_2. Let x denote the hyperbolic length of β'_n and let y denote the cross ratio of the quadruple $\{0, \frac{1}{2^n} - \frac{\sqrt{3}}{2^{n+1}}, \frac{1}{2^n} + \frac{\sqrt{3}}{2^{n+1}}, \infty\}$. By Lemma 2,

$$e^{x/2} - e^{-x/2} = 2\sqrt{y}.$$

Clearly,

$$y = \frac{\frac{1}{2^n} - \frac{\sqrt{3}}{2^{n+1}}}{2\frac{\sqrt{3}}{2^{n+1}}} = \frac{2-\sqrt{3}}{2\sqrt{3}} = \frac{1}{2\sqrt{3}(2+\sqrt{3})} < \left(\frac{1}{2\sqrt{3}}\right)^2 < \left(\frac{1}{3.4}\right)^2.$$

Let $\phi(t) = e^{t/2} - e^{-t/2}$. Since $\phi(t)$ is a strictly increasing function of t and

$$\phi(1) = e^{1/2} - e^{-1/2} = \frac{e-1}{e^{1/2}} > 1 > \frac{2}{3.4} > 2\sqrt{y} = \phi(x),$$

$1 > x$, that is, the hyperbolic length $l(\beta_n')$ of β_n' is less than 1. Therefore,

$$l(\beta_n) < l(\beta_n') < 1.$$

By condition (1),

$$\sigma_n = \sigma(\mathscr{L}_n) \le \frac{2}{3} \ln \ln \frac{1}{\delta(\beta_n)} + C < \frac{2}{3} \ln \ln 2^{n+2} + C.$$

Hence

$$e^{\sigma_n} = O((\ln 2^{n+2})^{2/3}) = O((n+2)^{2/3}).$$

Let B be the Möbius transformation from $\mathbf{H}$ onto itself that maps 1 to -1, 0 to ∞ and $1/2^n$ to -2^n for each $n \ge 1$. Then $\tilde{E} = B \circ E \circ B^{-1}$ is also a generalized earthquake map with its induced earthquake measure equal to the pushforward $(\tilde{\sigma}, \tilde{\mathscr{L}})$ of $(\sigma, \mathscr{L})$ by B. In order to show that E maps the half hyperbolic plane bounded by l_0 onto the half plane bounded by $E(l_0)$, it is equivalent to show that $\tilde{E}$ maps the left half plane bounded by the geodesic connecting -1 to ∞ onto itself. It is sufficient to prove

$$\lim_{n\to\infty} \tilde{E}(-2^n) = -\infty.$$

Clearly, $\tilde{E}$ moves each point -2^n to the right on the real line.

Let $\tilde{\mathscr{L}_n}$ be the collection of the geodesics in $\tilde{\mathscr{L}}$ connecting points on $(-2^n, -1]$ to ∞ and $\tilde{\sigma}_n = \tilde{\sigma}(\tilde{\mathscr{L}_n})$, where $n \ge 1$. Clearly, for each $n \ge 1$,

$$\tilde{\sigma}_n = \sigma_n.$$

By moving all the measure on $\tilde{\mathscr{L}_n}$ to the geodesic connecting -1 to ∞, the resulting new earthquake map $\tilde{E}_n$ (assumed to be the identity map on the geodesic connecting -1 to ∞) will possibly move -2^n further to the right than the map $\tilde{E}$, that is,

$$\tilde{E}(-2^n) \le \tilde{E}_n(-2^n) = -\frac{2^n}{e^{\tilde{\sigma}_n}} = -\frac{2^n}{O((n+2)^{2/3})}.$$

Hence

$$\lim_{n\to\infty} \tilde{E}(-2^n) = -\infty,$$

and then the proof is complete. □

Remark. In the proof of the previous proposition, one can compute $l(\beta_n')$ directly through an integration. That is

$$l(\beta_n') = \int_{\pi/6}^{\pi/2} \frac{d\theta}{\sin\theta} = -\ln(\cot\theta + \csc\theta)|_{\pi/6}^{\pi/2} = \ln\sqrt{3} < 1.$$

The method used in our proof focuses on the close relationship between the cross ratio of a quadruple and the hyperbolic distance between the corresponding two geodesics as indicated in Lemma 2. It doesn't require to work out explicitly the measures of the angles used in the above integration, and therefore it makes easier to estimate $l(\beta'_n)$ when the measures of the angles are quite arbitrary. We will apply this method two more times in the proof of Proposition 2, which is the main step for the proof of our main theorem.

The following two lemmas are useful in estimating the cross ratio of an image quadruple under an earthquake map. They have been applied to prove several results in [5] and [7]. See Corollaries 1 and 2 in [5] for their proofs.

Lemma 3. ([5]) *Let $Q = \{a, b, c, d\}$ be a quadruple on the real line with $-\infty \le a < b < c < d$, $s \in [c, d]$ and $t \in [d, \infty]$. Suppose that $A_{(s,t)}$ is the hyperbolic Möbius transformation with repelling fixed point at s and attracting fixed point at t and derivative at the repelling fixed point equal to $\lambda > 1$. Suppose $f_{(s,t)} \colon \mathbf{R} \to \mathbf{R}$ is defined to be equal to $A_{(s,t)}$ on the interval $[s, t]$ and equal to the identity on the complement of $[s, t]$. Then the cross-ratio of the image quadruple $f_{(s,t)}(Q)$ considered as a function of two variables $s \in [y, z]$ and $t \in (z, +\infty)$ decreases in s for each fixed t and increases in t for each fixed s.*

Lemma 4. ([5]) *With the same notation as in the previous lemma, suppose $s \in [b, c]$ and $t \in [d, \infty]$. Then the cross-ratio of the image quadruple $f_{(s,t)}(Q)$ is increasing in s for each fixed t and also increasing in t for each fixed s.*

Corollary 1. *With the same notation as in the previous two lemmas, suppose $s \in [a, b]$ and $t \in [c, d]$. Then the cross-ratio of the image quadruple $f_{(s,t)}(Q)$ is decreasing in s for each fixed t and also decreasing in t for each fixed s.*

Proof. Let Q' denote the quadruple $\{d, a, b, c\}$. Then

$$cr(f_{(s,t)}(Q)) = \frac{1}{cr(f_{(s,t)}(Q'))}.$$

Apply the previous lemma to $cr(f_{(s,t)}(Q'))$, we complete the proof. □

Now we start to deal with the second case: Suppose that a point a on $\mathbf{S}^1$ is not the endpoint of any geodesic in $\mathscr{L}$ but it is an accumulation point of the geodesics in $\mathscr{L}$. Let $\{l_n\}_{n=0}^{\infty}$ be an infinite sequence of the geodesics in $\mathscr{L}$ shrinking to a. We want to show under condition (1), $E(l_n)$ shrinks to a point on $\mathbf{S}^1$ as $n \to \infty$. Again we work with the upper half plane $\mathbf{H}$ and assume $a = 0$.

Let us first illustrate how condition (1) works in a special situation in which we assume further that for any $n \ge 0$, the geodesic l_n connecting $-1/e^n$ to $1/e^n$ is either a geodesic in $\mathscr{L}$ or contained in a gap of $\mathscr{L}$. For each $n > 1$, let Q_n be the quadruple $\{-1/e^{n-1}, -1/e^n, 1/e^n, 1/e^{n-1}\}$ and let $E(Q_n)$ be the quadruple consisting of the four endpoints of $E(l_{n-1})$ and $E(l_n)$. Let y_n denote the cross ratio of $E(Q_n)$ and let x_n denote the hyperbolic distance between $E(l_{n-1})$ and $E(l_n)$. Finally, let β_n be the common perpendicular geodesic segment between l_{n-1} and l_n and let $\sigma_n = \sigma(\beta_n)$. By condition (1),

$$e^{\sigma_n} \le e^{\frac{2}{3}\ln\ln\frac{1}{\delta(\beta_n)}+C} = e^{\frac{2}{3}\ln\ln e^n + C} = O(n^{2/3}).$$

Let E_n be an earthquake map which induces the measure $\sigma|_{\beta_n}$. Then

$$cr(E(Q_n)) = cr(E_n(Q_n)).$$

Let E'_n be an earthquake map which has only one leaf at the geodesic connecting $-1/e^n$ to $1/e^{n-1}$ and has the weight on this leaf equal to σ_n. By Corollary 1,

$$cr(E_n(Q_n)) \geq cr(E'_n(Q_n)).$$

One can easily work out $cr(E'_n(Q_n))$ by mapping $-1/e^n$ to 0 and $1/e^{n-1}$ to ∞ through conjugation by a Möbius transformation, that is,

$$cr(E'_n(Q_n)) = \frac{cr(Q_n)}{e^{\sigma_n}} = \frac{C_0}{e^{\sigma_n}},$$

where $C_0 = cr(Q_n) = \frac{(e-1)^2}{4e}$. Therefore,

$$cr(E_n(Q_n)) \geq cr(E'_n(Q_n)) = \frac{C_0}{e^{\sigma_n}} \geq \frac{C_0}{O(n^{2/3})}.$$

Let $\phi(x) = (\frac{e^{x/2}-e^{-x/2}}{2})^2$. It is an increasing function of x. By Lemma 2,

$$x_n = \phi^{-1}(y_n) = \phi^{-1}(cr(E_n(Q_n))) \geq \phi^{-1}(\frac{C_0}{O(n^{2/3})}) \geq \frac{1}{M}\frac{1}{n^{1/3}}$$

for a constant $M > 0$ and all sufficiently large n's. Hence $\sum_{n=1}^{\infty} x_n$ diverges. This implies that the distance between $E(l_0)$ and $E(l_n)$ goes to ∞ and thus that $E(l_n)$ shrinks to a point as $n \to \infty$.

Proposition 2. *Let $(\sigma, \mathscr{L})$ be an earthquake measure satisfying condition (1) and let E be a generalized earthquake map inducing the measure σ. Suppose that a point a on $\mathbf{S}^1$ is not an endpoint of any geodesic in $\mathscr{L}$ but it is an accumulation point of the geodesics in $\mathscr{L}$. If l_0 is a leaf in $\mathscr{L}$, then E maps the hyperbolic half plane bounded by l_0, containing a as a boundary point, onto a half plane bounded by $E(l_0)$.*

Proof. We work with the upper half plane $\mathbf{H}$ again. Without loss of generality, we may assume that $a = 0$ and l_0 is a geodesic connecting -1 to 1. Let $\mathscr{L}_0$ be the collection of the leaves in $\mathscr{L}$ connecting points on $[-1, 0)$ to points on $(0, 1]$. The main part of the proof is to arrange the leaves in $\mathscr{L}_0$ into groups such that the measures of the leaves in the groups are commensurable with the Euclidean lengths of the shortest geodesics in the groups and they are also quantitatively related in a useful manner.

Let $I_1^- = [-1, -1/2]$ and $I_1^+ = [1/2, 1]$. Denote by $\mathscr{L}_1^-$ the collection of the leaves in $\mathscr{L}_0$ connecting points on I_1^- to points on $(0, 1]$ and $\mathscr{L}_1^+$ the collection of the leaves in $\mathscr{L}_0$ connecting points on I_1^+ to points on $[-1, 0)$. Let $\mathscr{L}_1 = \mathscr{L}_1^- \cup \mathscr{L}_1^+$. Then each leaf in $\mathscr{L}_0 \setminus \mathscr{L}_1$ connects a point on $(-1/2, 0)$ to a point on $(0, 1/2)$. Let l_1 be the leaf in $\mathscr{L}_1$ such that all leaves in $\mathscr{L}_1$ are between l_0 and l_1. Assume that l_1 connects u_1 to v_1 with $-1 \leq u_1 < 0$ and $0 < v_1 \leq 1$. We may assume that either $u_1 = -1/2$ and $v_1 < 1/2$ or $u_1 > -1/2$ and $v_1 = 1/2$ by adding a leaf with 0 weight to $\mathscr{L}_1$ if necessary.

Now we construct $\mathscr{L}_n$ inductively for $n \geq 2$. Let $w_{n-1} = \min\{|u_{n-1}|, v_{n-1}\}$, $I_n^- = [u_{n-1}, -w_{n-1}/2]$ and $I_n^+ = [w_{n-1}/2, v_{n-1}]$. Denote by $\mathscr{L}_n^-$ the collection of the leaves in $\mathscr{L}_0 \setminus \cup_{k=1}^{n-1}\mathscr{L}_k$ connecting points on I_n^- to points on $(0, v_{n-1}]$ and by $\mathscr{L}_n^+$ the collection of the leaves in $\mathscr{L}_0 \setminus \cup_{k=1}^{n-1}\mathscr{L}_k$ connecting points on I_n^+ to points on $[u_{n-1}, 0)$. Let $\mathscr{L}_n = \mathscr{L}_n^- \cup \mathscr{L}_n^+$. Then the leaves in $\mathscr{L}_0 \setminus \cup_{k=1}^{n}\mathscr{L}_k$ connect points on $(-w_{n-1}/2, 0)$ to point on $(0, w_{n-1}/2)$. Suppose l_n is the leaf in $\mathscr{L}_n$ such that the leaves in $\mathscr{L}_n$ are between l_{n-1} and l_n and l_n connects u_n to v_n with $u_{n-1} \leq u_n < 0$ and $0 < v_n \leq v_{n-1}$. We may assume that either $u_n = -w_{n-1}/2$ and $v_n < w_{n-1}/2$ or $u_n > -w_{n-1}/2$ and $v_n = w_{n-1}/2$ by adding a leaf with 0 weight to $\mathscr{L}_n$ if necessary.

Let $s_n = \frac{1}{2}\max\{|u_n|, v_n\}$. In the following, we first show that condition (1) implies

$$\sigma(\mathscr{L}_n) \leq 2 \ln \ln \sqrt{\frac{36}{35}\frac{1}{s_n}} + C' \tag{5}$$

for a constant $C' > 0$ and all $n \geq 1$.

In order to prove this inequality, we need to treat four different situations according to whether $|u_{n-1}| \leq v_{n-1}$ or $|u_{n-1}| > v_{n-1}$ and whether $|u_n| \leq v_n$ or $|u_n| > v_n$. We will show details to handle the two cases in which $|u_{n-1}| \leq v_{n-1}$. The proofs for the other two cases are very similar and will be skipped.

Case 1: $|u_{n-1}| \leq v_{n-1}$ and $|u_n| \leq v_n$.

As arranged in the inductive construction of l_n, in this case $v_n = |u_{n-1}|/2$. Let α_n be the semicircle centered at u_{n-1} and connecting $v_n/2$ to $2u_{n-1} - v_n/2$, β_n be the segment on α_n between l_{n-1} and l_n, and $\delta(\beta_n)$ be the Euclidean distance from β_n to the real line. Let y_n be the y-coordinate of the intersection point between α_n and the geodesic γ_n connecting 0 to v_n. Then

$$\delta(\beta_n) \geq y_n.$$

Let $\tilde{l}_{n-1}$ be the geodesic passing through u_{n-1} and perpendicular to α_n, $\tilde{l}_n$ be the geodesic passing through v_n and perpendicular to α_n, and $\tilde{\beta}_n$ be the segment on α_n between $\tilde{l}_{n-1}$ and $\tilde{l}_n$. Then the hyperbolic length $l(\beta_n)$ is less than $l(\tilde{\beta}_n)$.

Since $u_{n-1} = -2v_n$, through a rescaling by $z \mapsto z/v_n$, we may assume that α_n connects $-9/2$ to $1/2$ and γ_n connects 0 to 1. Then we can easily work out the y-coordinate of their intersection point, that is, $\frac{1}{2}\sqrt{\frac{99}{100}}$. Therefore,

$$\delta(\beta_n) \geq y_n = \frac{v_n}{2}\sqrt{\frac{99}{100}} = s_n\sqrt{\frac{99}{100}}.$$

In the rescaled coordinate system, $\tilde{l}_{n-1}$ connects -2 to ∞. Through Lemma 1, we can work out that $\tilde{l}_n$ connects $1/12$ to 1. Hence

$$cr(\{-2, \frac{1}{12}, 1, \infty\}) = \frac{25}{11}.$$

Set $x = l(\tilde{\beta}_n)$. By Lemma 2,

$$\frac{(e^x-1)^2}{4e^x} = \frac{25}{11}.$$

Let $\phi(x) = \frac{(e^x-1)^2}{4e^x} = (e^{x/2} - e^{-x/2})^2/4$. Since ϕ is an increasing function of x and $\phi(2) < 25/11 < \phi(3)$, $2 < x < 3$. Hence

$$l(\beta_n) < l(\tilde{\beta}_n) = x < 3.$$

Therefore, condition (1) implies

$$\sigma(\beta_n) \le 3\Big(\frac{2}{3}\ln\ln\frac{1}{y_n} + C\Big) \le 2\ln\ln\sqrt{\frac{100}{99}\frac{1}{s_n}} + 3C. \tag{6}$$

This implies inequality (5).

Case 2: $|u_{n-1}| \le v_{n-1}$ and $|u_n| > v_n$.

In this case, $u_n = u_{n-1}/2$. Let α_n be the semicircle centered at u_{n-1} and connecting $u_n/2$ to $2u_{n-1} - u_n/2$ and let β_n be the segment on α_n between l_{n-1} and l_n. Let y_n be the y-coordinate of the intersection point between α_n and the geodesic γ_n connecting u_n to 0. Using a similar argument, rescaling as in Case 1, we can work out that

$$\delta(\beta_n) \ge y_n = \frac{|u_n|}{2}\sqrt{\frac{35}{36}} = s_n\sqrt{\frac{35}{36}}.$$

Let $\tilde{l}_{n-1}$ be the geodesic passing through u_{n-1} and perpendicular to α_n, $\tilde{l}_n$ be the geodesic passing through 0 and perpendicular to α_n, and $\tilde{\beta}_n$ be the segment on α_n between $\tilde{l}_{n-1}$ and $\tilde{l}_n$. Then

$$l(\beta_n) < l(\tilde{\beta}_n).$$

Again, rescaling as in Case 1, we can find that $\tilde{l}_n$ connects to $\frac{7}{8}u_n$ to 0. Let $\tilde{Q}_n$ be the quadruple consisting of the endpoints of $\tilde{l}_{n-1}$ and $\tilde{l}_n$, that is $\tilde{Q}_n = \{u_{n-1}, \frac{7}{8}u_n, 0, \infty\}$. Then $cr(\tilde{Q}_n) = 9/7$. Again let $x = l(\tilde{\beta}_n)$. Then

$$\frac{(e^x-1)^2}{4e^x} = cr(\tilde{Q}_n) = \frac{9}{7}.$$

Since $\phi(x) = \frac{(e^x-1)^2}{4e^x}$ is an increasing function of x and $\phi(1) < 9/7 < \phi(2)$, $1 < x < 2$. It follows that

$$l(\beta_n) < l(\tilde{\beta}_n) = x < 2.$$

Therefore, condition (1) implies

$$\sigma(\beta_n) \le 2\Big(\frac{2}{3}\ln\ln\frac{1}{y_n} + C\Big) \le \frac{4}{3}\ln\ln\sqrt{\frac{36}{35}\frac{1}{s_n}} + 2C. \tag{7}$$

The other two cases are the mirror images of the above two cases with respect to the imaginary axis. Therefore with similar arguments, we can also obtain inequalities (6) and (7) respectively. All these together imply inequality (5).

Let $s_n = e^{-t_n}$ for each $n > 0$. The arguments of the proofs of Proposition 1 and the special situation of Proposition 2 lead to the idea of proving Proposition 2 by dividing the sequence $\{t_n\}_{n=1}^{\infty}$ into the following two situations.

Situation 1: $t_n = O(n \ln n)$.

The proof for this case follows from the same idea to handle the special situation of Proposition 2. Let Q_n be the quadruple consisting of the endpoints of l_{n-1} and l_n, that is, $Q_n = \{u_{n-1}, u_n, v_n, v_{n-1}\}$. Using the arrangement in the inductive construction of $l_n' s$, it is easy to check that $cr(Q_n) \geq 1/8$. Arguing as in dealing with the special situation of Proposition 2, we obtain

$$cr(E(Q_n)) \geq \frac{cr(Q_n)}{e^{\sigma(\mathscr{L}_n)}} \geq \frac{1}{8e^{\sigma(\mathscr{L}_n)}}.$$

By inequality (5),

$$cr(E(Q_n)) \geq \frac{1}{8e^{C'}(\ln\sqrt{\frac{36}{35}\frac{1}{s_n}})^2} = \frac{1}{8e^{C'}(\varepsilon + t_n)^2},$$

where $\varepsilon = \ln\sqrt{\frac{36}{35}}$. Let x_n denote the hyperbolic distance between $E(l_{n-1})$ and $E(l_n)$. By Lemma 2, if t_n is sufficiently large then

$$x_n = \phi^{-1}(cr(E(Q_n))) \geq \phi^{-1}(\frac{1}{8e^{C'}(\varepsilon + t_n)^2}) \geq \frac{1}{2\sqrt{2}e^{C'/2}(\varepsilon + t_n)}.$$

Hence $\sum_{n=1}^{\infty} x_n$ diverges. This implies that the hyperbolic distance between $E(l_0)$ and $E(l_n)$ goes to ∞ and thus that $E(l_n)$ shrinks to a point as $n \to \infty$.

Situation 2: $\{\frac{t_n}{n \ln n}\}_{n=1}^{\infty}$ is unbounded.

The proof for this case is motivated by the proof of Proposition 1. The above sequence has a subsequence converging to ∞. Without loss of generality, we may assume that it converges to ∞ as $n \to \infty$. Now we let Q_n be the quadruple consisting of the endpoints of l_0 and l_n, that is, $Q_n = \{-1, u_n, v_n, 1\}$. Clearly, for each $n > 0$,

$$cr(Q_n) = \frac{(u_n + 1)(1 - v_n)}{2(v_n - u_n)} \geq \frac{1}{8(v_n - u_n)}.$$

Since $s_n = \frac{1}{2}\max\{|u_n|, v_n\}$ and $s_n = e^{-t_n}$, $v_n - u_n \leq 4s_n$ and then

$$cr(Q_n) \geq \frac{1}{32 s_n} = \frac{1}{32}e^{t_n}.$$

By inequality (5),

$$\sigma\Big(\bigcup_{k=1}^{n} \mathscr{L}_k\Big) \leq n\tau_n,$$

where $\tau_n = 2\ln\ln\sqrt{\frac{36}{35}\frac{1}{s_n}} + C' = 2\ln(\varepsilon + t_n) + C'$ and $\varepsilon = \sqrt{\frac{36}{35}}$. Then

$$cr(E(Q_n)) \geq \frac{cr(Q_n)}{e^{\sigma(\cup_{k=1}^n \mathscr{L}_k)}} \geq \frac{cr(Q_n)}{e^{n\tau_n}} \geq \frac{e^{t_n}}{32e^{n\tau_n}}.$$

Now let x_n denote the hyperbolic distance between $E(l_0)$ and $E(l_n)$. In order to know $x_n \to \infty$ as $n \to \infty$, we only need to show $cr(E(Q_n)) \to \infty$ as $n \to \infty$. It is sufficient to show that $e^{t_n}/e^{n\tau_n} \to \infty$ as $n \to \infty$. Furthermore, it is sufficient to show $(e^{t_n}/e^{n\tau_n})^{1/n} > 1$ when n is large enough. Clearly,

$$\Big(\frac{e^{t_n}}{e^{n\tau_n}}\Big)^{1/n} = \frac{e^{t_n/n}}{e^{\tau_n}} = \frac{e^{t_n/n}}{e^{C'}(\varepsilon + t_n)^2} = \frac{1}{e^{C'}}\Big(\frac{e^{t_n/2n}}{\varepsilon + t_n}\Big)^2.$$

Let $t_n = k_n n \ln n$. Then

$$\frac{e^{t_n/2n}}{\varepsilon + t_n} = \frac{n^{k_n/2}}{\varepsilon + k_n n \ln n}.$$

Now it is clear that $\frac{e^{t_n/2n}}{\varepsilon + t_n}$ goes to ∞ as both k_n and n go to ∞. It follows that $cr(E(Q_n))$ and then x_n go to ∞ as n goes to ∞. This implies that $E(l_n)$ shrinks to a point as $n \to \infty$.

Situations 1 and 2 complete the proof. □

Propositions 1 and 2 imply our Theorem 1, and then Theorem D and Theorem 1 imply our main theorem.

Remark. In the course of proving Proposition 1 and the special case of Proposition 2, it is clear that the constant 2/3 in condition (1) can be easily relaxed, but the proof of Proposition 2 relies on this constant and it doesn't seem to be easy to raise it.

Remark. Along an earthquake curve E_t determined by $t\sigma$, $t \geq 0$, with σ satisfying condition (1), the earthquakes may become generalized earthquakes for $t > 1$. But interestingly, if an earthquake measure σ satisfies

$$\sigma(\beta) \leq \ln\ln\ln\frac{1}{\delta(\beta)} + C \tag{8}$$

for a constant C and any geodesic arc β transversal to the lamination $\mathscr{L}$ of σ of hyperbolic length ≤ 1 and sufficiently close to the boundary $\mathbf{S}^1$ in the Euclidean metric, then an earthquake curve E_t determined by $t\sigma$, $t \geq 0$, is a curve of earthquakes for all $t \geq 0$. The differentiability of such a curve on t will be studied in the forthcoming note by Hu and/or Su.

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Received 15 August 2005