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QUASICONFORMAL AND QUASIREGULAR HARMONIC ANALOGUES OF KOEBE'S THEOREM AND APPLICATIONS

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Abstract. We show two versions of the Koebe theorem: one for quasiregular harmonic functions and another for quasiconformal functions. We also give an elementary proof of a version of the Koebe one-quarter theorem for holomorphic functions. As an application, we show the harmonic analogue of the Koebe one-quarter theorem and that holomorphic functions (more generally, quasiregular harmonic functions) and their modulus have similar behaviour in a certain sense.

1. Introduction

The harmonic analogue of the Koebe one-quarter theorem proved by Clunie and Sheil-Small [Cl-Sh] (cf. Theorem CS) attracted certain attention. In particular, the method of the proof was not expected by some experts in univalent functions. It provides at least a partial motivation to study their approach and, in particular, raises the following question.

Question A. Which property of holomorphic (more generally harmonic) univalent functions is essential for the validity of a version of the Koebe one-quarter theorem?

Among other things, the study of the approach in the paper mentioned [Cl-Sh] leads to a version of the Koebe theorem for quasiconformal functions (see Theorem 3.1 below, which can be considered as the main result of this paper). Theorem CS and Lemma 6.1 are derived as an application of Theorem 3.1, and it also leads to the classical Koebe's one-quarter theorem; cf. Theorem K. Note that Theorem 3.1 is strictly related to results in subsections V.6.7 in the book [LV]; see subsection 3.2 for further comments. Using the method of extremal length (more precisely, the Grötzsch theorem; see Section 2 and [Ahl]) we prove Theorem 3.1 in Section 3. We refer to this result as the version of Koebe's theorem for quasiconformal mappings.

In connection with the above indicated subject and applications of Bloch and Koebe theorem in [AMM], we have also found a new version of the Koebe one-quarter theorem, which holds for holomorphic functions, and which has an independent

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interest; in detail, using the subordination principle, we give a very short proof of a new version of the Koebe one-quarter theorem for holomorphic functions (see Section 4, Lemma 4.1 and Theorem 4.1, which we call the Koebe lemma (theorem) for holomorphic functions, respectively).

In Section 6, combining the decomposition property for quasiregular functions with the Koebe lemma for holomorphic functions and Lemma 6.1, we derive the Koebe lemma for quasiregular harmonic mappings; cf. Lemma 6.2. As an application of the Koebe lemma for quasiregular harmonic mappings, we immediately obtain Theorem 6.1, which we call the Koebe theorem for quasiregular harmonic mappings.

Theorem 6.1 is applied to show Dyakonov's theorem [Dyk], stated as Theorem A in Section 5, as well as the corresponding generalization for quasiregular harmonic functions.

2. Extremal module

For $r > 0$, by Δ_r, we denote the disc $\{z : |z| < r\}$ in the complex plane and by $\Delta = \Delta_1$ the unit disc in the complex plane.

The proof of Theorem 3.1 is based on the extremal property of the Grötzsch annulus, which we need to discuss first.

2.1. Grötzsch annulus. If Ω is a double-connected domain, by $M(\Omega)$ we denote the modulus of Ω; and for a given family of curves Γ by $M(\Gamma)$ the modulus of the family of curves Γ. If Ω is a double-connected domain and E_1 and E_2 the components of $\partial\Omega$, the extremal distance $d_\Omega(E_1, E_2)$ between E_1 and E_2 is the modulus of Ω. Note that if Γ is the family of curves which joins the components E_1 and E_2, then $d_\Omega(E_1, E_2) = 1/M(\Gamma)$, and if Γ' is the family of curves which separates the components E_1 and E_2, then $M(\Omega) = M(\Gamma')$.

Let $0 < r < 1$ and c be any continuum that contains $\{0, r\}$, and let Γ, Γ_0 and Γ_c be the families of closed curves in the unit disc that separate $\{0, r\}$, $s = [0, r]$ and c from the circumference, respectively. There is an extremal metric ρ_0 for the family Γ_0, which is obviously symmetric with respect to s.

Given any $\gamma \in \Gamma$, we obtain a path γ_0 of equal ρ_0- length by reflecting part of γ across s. Hence, we conclude $M(\Gamma) = M(\Gamma_0)$. Note the crucial role of symmetry.

Since Γ_c is a smaller family than Γ, we find

$$M(\Gamma_c) \leq M(\Gamma) = M(\Gamma_0).$$

In view of conformal invariance, we get:

Theorem 2.1. (Grötzsch) *Of all the continua that join the point $R > 1$ to ∞, the segment $[R, +\infty]$ has the greatest extremal distance from the unit circle.*

2.2. Teichmüller theorem. Another extremal problem of similar nature was solved by Teichmüller; notice that we do not use the Teichmüller theorem in our proof of Theorem 3.1.

Theorem 2.2. (Teichmüller) *Of all double-connected domains that separate the pair $\{-1, 0\}$ from a pair $\{\omega_0, \infty\}$ with $|\omega_0| = R$, the one with the greatest modulus is the complement of the segments* $[-1, 0]$ *and* $[R, +\infty]$.

In [Ahl], the proof makes use of Koebe's one-quarter theorem and the distortion theorem.

Define

$$k(z) = z(1+z)^{-2},\ k_R(z) = 4Rz(1+z)^{-2},\ k_R^+(\rho) = 4R\,\rho(1-\rho)^{-2},$$
$$k_R^-(\rho) = 4R\,\rho(1+\rho)^{-2} \text{ and } k^+ = k_1^+.$$

Let f be a conformal mapping of Δ normalized by $f(0) = 0$, which omits ω_0. Then, by Koebe's one-quarter theorem $|f'(0)| \le 4R$ and hence by the distortion theorem,

$$|f(z)| \le k^+(|z|)|f'(0)| \le 4\,R\,k^+(|z|) = k_R^+(|z|). \tag{2.1}$$

Let Ω be a double-connected domain that separates the pair $\{-1, 0\}$ from the pair $\{\omega_0, \infty\}$ and E_1 be the bounded component of the complement. By Riemann's mapping theorem there exists a univalent function ϕ from the unit disc onto $\Omega \cup E_1$ such that $\phi(0) = 0$. Let ω be the inverse image of -1; that is, $\phi(\omega) = -1$. We note further that the modulus of Ω is the extremal distance between $\phi^{-1}(E_1)$ and the unit circle and we already know, by the Grötzsch theorem, that this extremal distance is greatest when $\phi^{-1}(E_1)$ is the line segment between 0 and ω. Let $\omega^* = k_R^{-1}(-1)$. Then k_R maps $\Delta \setminus [0, \omega^*]$ onto the complement of the segments $[-1, 0]$ and $[R, +\infty]$.

On the other hand, by Koebe's one-quarter theorem $|\phi'(0)| \le 4R$ and hence by the distortion theorem,

$$|\phi(z)| \le k^+(|z|)|\phi'(0)| \le 4Rk^+(|z|) = k_R^+(|z|), \quad z \in \Delta. \tag{2.2}$$

Since $k_R(\omega^*) = -1$, it follows that $\omega^* = -|\omega^*|$. Hence $k_R^+(|\omega^*|) = |k_R(\omega^*)| = |-1| = |\phi(\omega)| \le k_R^+(|\omega|)$, which implies $|\omega| \ge |\omega^*|$. Hence the modulus is indeed the maximum for the Teichmüller annulus.

Since there is a proof of Koebe's one-quarter theorem by means of the Grötzsch theorem, it seems natural to try to prove the Teichmüller theorem more directly (without Koebe's one-quarter theorem and the distortion theorem). We can do this, for example, using the symmetry if $\omega_0 > 0$.

Let Γ be the family of curves which separates the pair $\{-1,\ 0\}$ from $\{R,\ +\infty\}$ and let Γ_0 be the family of all closed curves which separates $[-1, 0]$ and $[R, +\infty)$, and let $\Lambda = [-1, 0] \cup [R, +\infty)$. There is an extremal metric ρ_0 for the family Γ_0, which is obviously symmetric with respect to Λ.

Given any $\gamma \in \Gamma$, we obtain a γ_0 of equal ρ_0 length by reflecting part of γ across Λ. Hence, we conclude $M(\Gamma) = M(\Gamma_0)$. Note the crucial role of symmetry.

3. Koebe's one-quarter theorem and generalizations

3.1. Proof of Koebe's one-quarter theorem for univalent functions. It is interesting that we can prove Koebe's one-quarter theorem for univalent functions using a method based on the Grötzsch theorem. It also gives motivation for a proof of Theorem 3.1.

First, we prove the estimate (3.3) (see below).

It is convenient to introduce the notation $\mathbf{C}^R$, $R \geq 0$, for the complex plane cut along $[R, +\infty)$ and $\mathbf{C}^R_r = \mathbf{C}^R \setminus \overline{\Delta_r}$ for the complement of the closed disc $\overline{\Delta_r}$ and $[R, +\infty)$ (known as the Grötzsch annulus).

Let $\tau^R_- = (k^+_R)^{-1}$ and $\tau^R_+ = (k^-_R)^{-1}$. It is clear that $\tau_- \leq \tau_+$. It is also convenient to use the short notation $\tau_+ = \tau^R_+$ and $\tau^R_- = \tau_-$ if the meaning of R is clear from the context.

Note that τ_+ maps $[0, R]$ onto $[0, 1]$ and τ_- maps $[0, +\infty)$ onto $[0, 1)$; k_R maps the disk $\Delta_{\tau_-(r)}$ into Δ_r, $r > 0$, and $\Delta_r \subset k_R(\Delta_{\tau_+(r)})$, $0 < r < R$.

For $\delta > 0$, the function k_δ maps Δ onto C^δ, the plane cut along $[\delta, +\infty)$.

Hence, for $0 < r < R$,

$$\frac{1}{2\,\pi} \ln \frac{1}{\tau_+(r)} \leq M(\mathbf{C}^R_r) \leq \frac{1}{2\,\pi} \ln \frac{1}{\tau_-(r)}. \tag{3.1}$$

Since $\tau'_-(0) = \frac{1}{4R}$ and hence $\tau_-(\rho) = \frac{\rho}{4R} + o(\rho)$, $\rho \to 0$, it follows that

$$M(\mathbf{C}^R_\rho) \leq \frac{1}{2\,\pi} \ln \frac{4R}{\rho + o(\rho)}, \quad \rho \to 0, \tag{3.2}$$

and since $\tau_+(\rho) = \frac{\rho}{4R} + o(\rho)$, $\rho \to 0$, we also obtain the opposite estimate. Hence,

$$M(\mathbf{C}^R_\rho) = \frac{1}{2\,\pi} \ln \frac{4R}{\rho + o(\rho)}, \quad \rho \to 0. \tag{3.3}$$

Now, we introduce a particulary interesting class of conformal mappings of the disc, the class $\mathscr{S}$.

By $\mathscr{S}$, we denote the family of holomorphic univalent functions f on the unit disc, with normalization $f(0) = 0$ and $f'(0) = 1$.

Theorem K. (Koebe's one-quarter theorem) *If $f \in \mathscr{S}$ then $f(\Delta) \supset \Delta_{1/4}$.*

Proof. Let $f \in \mathscr{S}$, $0 < \varepsilon < 1$, $A_\varepsilon = \Delta \setminus \overline{\Delta_\varepsilon}$, $G_\varepsilon = f(A_\varepsilon)$, and $\delta = \delta_f = \mathrm{dist}\big(0, \partial \mathrm{f}(\Delta)\big)$. Since the modulus is a conformal invariant, it follows that $M(G_\varepsilon) = \dfrac{1}{2\pi} \ln \dfrac{1}{\varepsilon}$.

For $0 < \varepsilon < 1$, let $\varepsilon_* = \min\{|f(z)| : |z| = \varepsilon\}$. Then, by (3.3),

$$M(C^\delta_\varepsilon) = \frac{1}{2\pi} \ln \frac{4\delta}{\varepsilon + o(\varepsilon)}.$$

By the monotonous principle for modulus and Grötzsch's theorem, $M(G_\varepsilon) \leq M(C^\delta_{\varepsilon_*})$ and hence, since $\varepsilon_* = \varepsilon + 0(\varepsilon)$,

$$\ln \frac{1}{\varepsilon} \leq \ln \frac{4\delta}{\varepsilon + 0(\varepsilon)}.$$

Now passing to the limit when $\varepsilon \to 0$, it follows that $\frac{1}{4} \leq \delta_f$; that is, $f(\Delta) \supset \Delta_{1/4}$. $\square$

3.2. Remarks on generalizations of Koebe's one-quarter theorem for quasiconformal mappings. First, we need to introduce some notations.

It is well known that if f is a quasiconformal mapping defined on a region G, then the function f_z is nonzero a.e. in G. The function

$$\mu_f = \frac{f_{\bar z}}{f_z}$$

is therefore a well defined bounded measurable function on G, called the complex dilatation or the Beltrami coefficient of f.

Let μ be a complex dilatation on Δ. For $z \in \Delta$, we define $\mu^+(z) = \operatorname{ess\,sup}\{|\mu(\zeta)| : |\zeta| = |z|\}$, where ess sup is taken with respect to the arc length (the angular measure) on the circle of the radius $|z|$. Set

$$\tau = \tau(f) = \tau_f = \int_0^1 \frac{2\mu^+(t)}{1+\mu^+(t)}\,\frac{dt}{t}, \quad \text{and} \quad \delta = \delta(f) = \delta_f = \operatorname{dist}\big(f(0), \partial f(\Delta)\big).$$

Note that μ^+ is a radial function. Define

$$I(r) = \frac{1}{2\pi} \iint\limits_{|z|\le r} \frac{D(z)-1}{|z|^2}\,dx\,dy,$$

where $D = D_f$ denotes the dilatation of f.

If f is a quasiconformal mapping on Δ with $f(0) = 0$, then

$$|\partial f(0)| \le 4\,\delta\, e^{I(1)}. \tag{3.4}$$

This inequality is proved in the book [LV], pp. 219–233, as a corollary of Theorem V.6.1 in the same book, and there the reader is referred to Pfluger and Juve for further generalizations of Koebe's one-quarter theorem for quasiconformal mappings; see also Astala and Gehring [AsGe].

Using (3.4), one can derive a harmonic analogue of the Koebe one-quarter theorem with constant $c = 2^{-10}$.

Now, we are going to prove Theorem 3.1.

3.3. A new version of Koebe one-quarter theorem for qc. By A^c we denote $\mathbf{C} \setminus A$. Suppose that f is a quasiconformal mapping on Δ_r for all $0 < r < 1$, conformal at 0, and $f(0) = 0$; for $0 < \varepsilon < 1$ define $G_\varepsilon = f(A_\varepsilon)$.

Let ρ be a non-negative Borel measurable function defined on Δ; set

$$D^+ = \frac{1+\mu^+}{1-\mu^+} \quad \text{and} \quad I^+(\rho) = I_f^+(\rho;\varepsilon) = \iint\limits_{A_\varepsilon} \rho^2(z) D^+(z)\,dx\,dy.$$

It is convenient to use the notation $p = \partial f$, $q = \overline{\partial} f$, $L_f(z) = |p(z)| + |q(z)|$ and $l_f(z) = |p(z)| - |q(z)|$.

First we prove an estimate of $M(G_\varepsilon)$ from below by choosing an admissible metric for the family of curves Γ_ε in A_ε, which join the components of A_ε^c, and then we apply Grötzsch's theorem in order to get an estimate of $M(G_\varepsilon)$ from above.

In order to get an estimate of $M(G_\varepsilon)$ from below, we consider the family of curves Γ_ε^* in G_ε, which join the components of G_ε^c. Note that Γ_ε^* is the complementary family to the family which separates the components of G_ε^c.

Step 1. (Proof of inequality (3.7)) Let ρ be admissible for the family of curves Γ_ε in A_ε, which join the components of A_ε^c, and let ρ_* be defined by

$$\rho_*(w) = \frac{\rho(f^{-1}(w))}{l_f(f^{-1}(w))}, \quad w \in G_\varepsilon,$$

where recall $l_f = |p| - |q|$. We shortly write $l_f(z)\rho_*(w) = \rho(z)$, where $w = f(z)$.

If $\gamma_* \in \Gamma_\varepsilon^*$, then for $\gamma := f^{-1} \circ \gamma_*$,

$$\int_{\gamma_*} \rho_*(w)\,|dw| \geq \int_\gamma \rho_*(f(z))\,l_f(z)\,|dz| = \int_\gamma \rho(z)\,|dz|.$$

Thus ρ_* is admissible for the family of curves Γ_ε^* in G_ε, which join the components of G_ε^c. Hence

$$M(\Gamma_\varepsilon^*) \leq \iint\limits_{G_\varepsilon} \rho_*^2(w)\,du\,dv = \iint\limits_{A_\varepsilon} \frac{\rho^2(z)}{l_f^2(z)} J_f(z)\,dx\,dy,$$

and since

$$\frac{J_f(z)}{l_f^2(z)} = D_f(z) \leq D_f^+(z),$$

it follows that

$$\frac{1}{M(G_\varepsilon)} = M(\Gamma_\varepsilon^*) \leq I^+(\rho) \tag{3.5}$$

for every ρ, which is admissible for the family of curves Γ_ε.

Let

$$s_\varepsilon = \int_\varepsilon^1 \frac{1-\mu^+}{1+\mu^+}\frac{dt}{t},$$

and let ρ_1 be defined by $\rho_1(z)D^+(z) = \frac{1}{|z|}$ and ρ_0 be the normalization of ρ_1 defined by

$$\rho_0(z) = \frac{\rho_1(z)}{s_\varepsilon} = \frac{1}{s_\varepsilon\,|z|\,D^+(z)}.$$

Since

$$\int_\gamma \rho_1 |dz| \geq \int_\varepsilon^1 \frac{1}{tD^+(t)}\,dt = s_\varepsilon,$$

for every $\gamma \in \Gamma_\varepsilon$, it follows that ρ_0 is admissible for the family of curves Γ_ε.

Using the polar coordinates,

$$I_f^+(\rho_0;\varepsilon) = \frac{2\pi}{s_\varepsilon^2}\int_\varepsilon^1 \frac{1}{tD^+(t)}\,dt = \frac{2\pi}{s_\varepsilon}.$$

Hence, by (3.5),

$$M(G_\varepsilon) = \frac{1}{M(\Gamma^*_\varepsilon)} \geq \frac{1}{I^+(\rho_0)} = \frac{s_\varepsilon}{2\pi}, \tag{3.6}$$

and, therefore, it yields

$$2\pi M(G_\varepsilon) \geq \ln\frac{1}{\varepsilon} - \tau_\varepsilon, \tag{3.7}$$

where $\tau_\varepsilon = \int_\varepsilon^1 \frac{2\mu^+}{1+\mu^+}\frac{dt}{t}$.

It is also convenient to rewrite this inequality in the form

$$M(A_\varepsilon) \leq M(G_\varepsilon) + \tau^*_\varepsilon,$$

where $\tau^*_\varepsilon = \frac{1}{2\pi}\int_\varepsilon^1 \frac{2\mu^+}{1+\mu^+}\frac{dt}{t}$.

Step 2. (Application of the Grötzsch theorem) For $0 < \varepsilon < 1$, let $\varepsilon_* = \min\{|f(z)| : |z| = \varepsilon\}$, and let $a_1 = f'(0)$. Since f is a quasiconformal mapping on Δ, which is conformal at 0, then $\varepsilon_* = |a_1|\varepsilon + o(\varepsilon)$.

Let $\tau = \tau(f) = \int_0^1 \frac{2\mu^+(t)}{1+\mu^+(t)}\frac{dt}{t}$ and let $\delta = \delta_f = \operatorname{dist}\big(0, \partial f(\Delta)\big)$. We are going to prove

$$|a_1|e^{-\tau} \leq 4\delta.$$

By the monotonous principle for modulus and the Grötzsch theorem,

$$M(G_\varepsilon) \leq M(C^\delta_{\varepsilon_*})$$

and hence, since $\varepsilon_* = |a_1|\varepsilon + o(\varepsilon)$,

$$\ln\frac{1}{\varepsilon} - \tau_\varepsilon \leq \ln\frac{4\delta}{|a_1|\varepsilon + o(\varepsilon)}, \quad \varepsilon \to 0,$$

and passing to the limit when $\varepsilon \to 0$, it follows that

$$-\tau \leq \ln\frac{4\delta_f}{|a_1|} \quad \text{and, therefore,} \quad \frac{|a_1|}{4}e^{-\tau} \leq \delta_f.$$

Thus, we can summarize the above considerations:

Theorem 3.1. *Suppose the above notation and*

a) f *is a quasiconformal mapping on* Δ_r *for all* $0 < r < 1$, *and*
b) $f(0) = 0$ *and* f *is conformal at* 0.

Then

$$|f'(0)| \leq 4\,\delta_f\, e^{\tau(f)}. \tag{3.8}$$

Note that τ does not converge in general. If τ converges, by Theorem 6.1 in [LV] f is conformal at 0.

If f is a conformal mapping, then μ_f equals 0 on Δ and, therefore, $\tau(f) = 0$. Hence, Koebe's one-quarter theorem follows from the estimate (3.8).

Note that the integral τ does not converge in general, and the theorem has useful content only when $\tau = \int_0^1 \frac{2\mu^+(t)}{1+\mu^+(t)}\frac{dt}{t}$ converges.

It also seems that we can get a version of Theorem 3.1 if we instead of the hypothesis a) suppose that

a_1) f is an ACL homeomorphism on Δ.

Having Theorem 3.1 in mind, it is clear that one can prove the corresponding version of the Bloch theorem for quasiregular mappings by means of the Bloch theorem for holomorphic functions and the decomposition property for quasiregular functions.

3.4. Application to harmonic univalent functions. $\delta_h \geq \frac{1}{16}$. Now, we discuss an application of Theorem 3.1 to harmonic univalent mappings.

First of all, we observe that if h is a conformal mapping normalized by $h(0) = 0$ and $h'(0) = 1$, then $\mu \equiv 0$ and, therefore, $\tau = 0$, and hence we obtain Koebe's one-quarter theorem for univalent mappings.

Let h be a harmonic mapping in the Euclidean sense on Δ. Then $h = f + \overline{g}$, where f and g are analytic on Δ.

By Σ, we denote the family of harmonic univalent, orientation-preserving functions with normalization $h(0) = 0$, $f'(0) = 1$ and $g'(0) = 0$.

Theorem CS. [Clunie, Sheil-Small] *If $h \in \Sigma$, then $\delta_h \geq \frac{1}{16}$.*

Proof. Note $\mu = \mu_h = \dfrac{\overline{g'}}{f'}$. Since $\mu(0) = 0$ and $|\mu(z)| \leq 1$, by the Schwarz lemma, $|\mu(z)| \leq |z|$ for $z \in \Delta$, and we first obtain $\mu^+(t) \leq t$, $0 \leq t < 1$, and, therefore,

$$\tau = \tau(f) = \tau_f = \int_0^1 \frac{2\mu^+(t)}{1+\mu^+(t)} \frac{dt}{t} \leq 2 \int_0^1 \frac{1}{1+t}\, dt = 2 \ln 2.$$

Hence, Theorem CS follows from Theorem 3.1. □

Notice that further study of the application of Theorem 3.1 to harmonic mappings with the non-negative Gaussian curvature of the target is a subject of a forthcoming paper. The method of the extremal length also works for some classes of mappings which boundedly distort some extremal length related metrics but which need not be quasiconformal. Some distortion results for such mappings were proved in [FMV, 3.21, 3.24, 4.21].

4. Koebe's one-quarter theorem for holomorphic functions

We will use the following notation. If $r > 0$ and a is a complex number,

$$B(a; r) = \{z \in \mathbf{C} : |z - a| < r\}$$

is the open circular disc with centre at a and radius r. We also use the notation $\Delta_r = B(0, r)$ and $\Delta = \Delta_1$.

If $B = B(a; r)$ is a disc, we say that f is holomorphic (harmonic) on $\overline{B}$ if there is an open set Ω such that $\overline{B} \subset \Omega$ and f is holomorphic (harmonic) on Ω, respectively.

The next example shows that holomorphic functions in general do not enjoy the property described by the Koebe theorem for univalent functions and it indicates a completely new phenomenon.

Example 1. The example $f_n(z) = \frac{1}{n}(e^{nz} - 1)$ shows that if f is a holomorphic function on the unit disc Δ, $f(0) = 0$ and $|f'(0)| \geq 1$, then there is not an absolute constant $s > 0$ such that the disc Δ_s belongs to $f(\Delta)$.

This example shows that the hypothesis $f \in S$, which is roughly speaking the injectivity of f, is essential for the validity of the classical Koebe's one-quarter theorem. Therefore, it seems that the following question is natural: does there exists an appropriate generalization of this theorem to the functions which are not injective?

After writing several versions of this paper, the author has found a very simple proof of the following result, which seems to be an appropriate generalization of the Koebe one-quarter theorem (with the best constant $\frac{1}{4}$). In order to state the theorem, we need to introduce some notations.

If $\theta \in \mathbf{R}$ and a is a complex number, we write Λ_θ, Λ_a^θ and $\Lambda(a)$ instead of $\{\rho e^{i\theta} : \rho \geq 0\}$, $\{a + \rho e^{i\theta} : \rho \geq 0\}$ and $\{\rho\, a : \rho \geq 0\}$, respectively.

By $\omega = \omega_f$, we denote the modulus of continuity of f.

Lemma 4.1. (Koebe lemma for holomorphic functions) *Suppose that f is a holomorphic function on the closed unit disc $\overline{\Delta}$, $f(0) = 0$ and $|f'(0)| \geq 1$. Then for every $\theta \in \mathbb{R}$ there exists a point w on the half-line Λ_θ which belongs to $f(\overline{\Delta})$, such that $|w| \geq \frac{1}{4}$.*

Proof. On the contrary, suppose there exists a θ for which the theorem is not true. Without loss of generality, we can assume that

$(*)$ $f(\overline{\Delta})$ does not intersect $\Lambda = [1/4, \infty)$.

First note that the Koebe function k, which is defined by $k(z) = z(1 + z)^{-2}$, maps Δ onto $\mathbf{C} \setminus \Lambda$. Since, $k(0) = f(0) = 0$ and $k'(0) = 1$, by the subordination principle, $|f'(0)| \leq 1$. If $|f'(0)| = 1$, then $f = k$ and, therefore, $f(1) = k(1) = 1/4$, which is a contradiction with the hypothesis $(*)$. Hence $|f'(0)| < 1$, which is again a contradiction with the hypothesis $|f'(0)| \geq 1$. □

As an application of Lemma 4.1, we immediately obtain the following result, which we call the Koebe theorem for holomorphic functions.

Theorem 4.1. (Koebe theorem for holomorphic functions) *Suppose that f is a holomorphic function on $\overline{B}$, where $B = B(a; r)$, and $f(a) = b$. Then, for every $\theta \in \mathbf{R}$, there exists a point w on the half-line Λ_b^θ which belongs to $f(\overline{B})$, such that $|w - b| \geq \rho_f(a)$, where $\rho_f(a) := \frac{1}{4}r|f'(a)|$. In particular, there exists a point $\omega \in f(\overline{B})$ such that $|\omega| - |b| = |\omega - b| \geq \rho_f(a)$.*

Proof. If $f'(a) \neq 0$, applying Lemma 4.1 to the function $s \cdot \big(f(rz + a) - b\big)$, where $s = \dfrac{1}{rf'(a)}$, immediately gives the result. □

5. Applications

As an application of the estimates obtained in Section 4, we prove Dyakonov's theorem; see also [P], [MM1], [MM2] and [Ka1].

In the discussion that follows, we suppose that $\alpha \in (0,1)$. By $\mathrm{Lip}(\alpha) = \mathrm{Lip}(\alpha; A)$, we denote the family of Hölder functions with exponent α and the multiplier constant A.

Roughly speaking, the Koebe theorem for holomorphic functions states that holomorphic functions have the same dilatation in all directions and indicates a similar behaviour of holomorphic functions and their modulus in a certain sense and leads, via the crucial estimate (5.3), to what we call geometrically a visual proof of Dyakonov's theorem stated here as:

Theorem A. *Suppose that f is a holomorphic function on Δ. Then f belongs to* $\mathrm{Lip}(\alpha)$ *if and only if $|f|$ belongs to* $\mathrm{Lip}(\alpha)$.

Namely, if f belongs to $\mathrm{Lip}(\alpha)$, then, by the triangle inequality, $|f|$ belongs to $\mathrm{Lip}(\alpha)$.

The proof of the opposite result is more delicate as the next example indicates:

Example 2. Let $R = R_r = (-r, r) \times (-\infty, +\infty) = \{z : -r < \operatorname{Re} z < r\}$, $r > 0$, φ a conformal mapping of Δ onto R such that $\varphi(0) = 0$ and $\phi(z) = \phi_n(z) = \exp(\varphi(z)/n)$. Then $\omega_{\phi_n}(1) \geq 2$ and $\phi_n(\Delta) = A_n$, where $A_n = \{w : \exp(-r/n) < |w| < \exp(r/n)\}$, and, therefore, $\omega_{|\phi_n|}(1) \leq 2r/n + o(1/n) \to 0$, when $n \to \infty$.

Thus, the example shows the following: even if we consider the family $\mathscr{H}^\infty$ of bounded holomorphic functions on the unit disc, there is no absolute constant c such that $\omega_f \leq c\,\omega_{|f|}$ for every $f \in \mathscr{H}^\infty$.

Note that, by the triangle inequality, $\omega_{|f|} \leq \omega_f$, and ϕ_n does not belong to $\bigcup_{0<\alpha<1} \mathrm{Lip}(\alpha)$.

The following lemma, which is an immediate corollary of the key estimate (5.3), gives, for a holomorphic function f, the estimate of the modulus of the derivative by means of the modulus of continuity of $|f|$. It immediately yields, with the gradient growth lemma (see below), the proof that

$(**)$ if $|f|$ belongs to $\mathrm{Lip}(\alpha)$, then f belongs to $\mathrm{Lip}(\alpha)$.

Lemma 5.1. *Let f be a holomorphic function on Δ. Then,*

$$(1-|z|)|f'(z)| \leq 4\,\omega(1-|z|),\ z \in \Delta, \tag{5.1}$$

where $\omega = \omega_{|f|}$ is the modulus of continuity of $|f|$.

In addition, if $|f|$ belongs to $\mathrm{Lip}\,(\alpha; A)$ *on Δ, then*

$$|f'(z)| \leq 4\,A \cdot (1-|z|)^{\alpha-1},\ z \in \Delta. \tag{5.2}$$

Proof. Let $z \in \Delta$, $r = s\cdot(1-|z|)$, $0 < s < 1$, $B = B(z;r)$, $w = f(z)$ and $\tilde{B} = f(B)$. We suppose for a moment that s is a fixed constant. By Theorem 4.1, there is a point w_1, which belongs to $\tilde{B} \cap \Lambda(w)$, such that

$$|w_1 - w| \geq \frac{1}{4}|f'(z)|r. \tag{5.3}$$

That is obvious if $w = 0$. If $w \neq 0$, then $\tilde{B} \cap \Lambda(w)$ contains two points, w_1 and w_2. It is clear that we can choose the numeration of those points such that $0 \in \Lambda(w, w_2) := \{w + \rho(w_2 - w) : \rho \geq 0\}$ and, therefore, $|w_1 - w| = |w_1| - |w|$.

Let z_1 be a preimage of w_1. Since

$$|w_1 - w| = ||w_1| - |w|| = ||f(z_1)| - |f(z)|| \text{ and } |z_1 - z| \leq r,$$

then

$$s \cdot (1 - |z|)|f'(z)| \leq 4\,\omega\,[s \cdot (1 - |z|)]\,,\ z \in \Delta. \tag{5.4}$$

Hence, if $s \to 1_-$, then (5.1) follows from (5.3).

Note that if $|f|$ belongs to $\mathrm{Lip}(\alpha)$, then (5.1) reduces to (5.2). □

Now, an application of the next lemma (for the proof, see, for example, [R], Lemma 6.4.8) immediately yields the proof of $(**)$.

Lemma 5.2. (Gradient growth lemma) *If $u\colon \Delta \to \mathbf{C}$ and*

$$|\mathrm{grad}\, u(z)| \leq A\,(1 - |z|)^{\alpha - 1}, \quad z \in \Delta,$$

for some $A > 0$ and $\alpha \in (0, 1)$, then $u \in \mathrm{Lip}(\alpha)$ with the multiplier constant $M_\alpha = (1 + 2\,\alpha^{-1})A$.

Thus, if $|f|$ belongs to $\mathrm{Lip}(\alpha)$, then it follows from the inequality (5.2) and Lemma 5.2 that f belongs to $\mathrm{Lip}\,\alpha$, that is, $(**)$ is proved.

6. Koebe theorem for quasiregular harmonic functions

In this section, we will show that a version of Theorem 4.1 holds for quasiregular harmonic functions. For basic definitions and results we refer to the books [Ah2], [LV] and [Ri].

First, we need to introduce some notations and results:

Every harmonic function f in Δ can be written in the form $f = \bar{g} + h$, where g and h are holomorphic functions in Δ. For $f \in S_H^0$ (see [Cl-Sh] for the notation), $|g'(z)| \leq |z||h'(z)|$.

Lemma 6.1. *Let f be a diffeomorphism of Δ,*

1) *$|\bar{\partial} f(z)| \leq |z||\partial f(z)|$ for $z \in \Delta$, and*
2) *$f(z) = z + O(|z|^\beta)$ for some $\beta > 1$ as $z \to 0$.*

Then $f(\Delta) \supset \Delta_{1/16}$.

Clunie and Sheil-Small proved the lemma for harmonic mappings $f = \bar{g} + h \in S_H^0$. For the proof of the lemma in general, one can repeat their approach using $\partial f(z)$ and $\overline{\bar{\partial} f(z)}$, respectively, instead of h' and g'. The details are left to the interested reader. The lemma also appears in [He-Sc], [He-Po].

The lemma is true if the hypothesis 2) is replaced by the hypothesis

3) *f is conformal at* 0 *and* $f'(0) = 1$.

We now outline an argument which shows that the lemma is also an immediate corollary of Theorem 3.1.

Let $\mu = \mu_f$. By the hypotheses of the lemma, $|\mu(z)| \le |z|$, $z \in \Delta$, and we obtain first $\mu^+(t) \le t$, $0 \le t < 1$, and, therefore, $\tau \le 2 \ln 2$. Hence, since the hypothesis 2) implies the hypothesis 3), an application of Theorem 3.1 gives $2^{-4} \le \delta$, that is, $f(\Delta) \supset \Delta_{1/16}$.

Lemma 6.2. *Suppose that f is a K-quasiregular harmonic mapping on the unit disc $\overline{\Delta}$, $f(0) = 0$ and $|\operatorname{grad} f(0)| \ge 1$. Then for every $\theta \in \mathbf{R}$ there exists a point w on the half-line Λ_θ which belongs to $f(\Delta)$, such that $|w| \ge c$, where $c = c(K) = \frac{1}{K}2^{-6}$.*

We call this result the Koebe lemma for quasiregular harmonic mappings with constant c.

Proof. Let us first verify the decomposition property for quasiregular harmonic functions (shortly qrh): if f is a K-quasiregular harmonic mapping, then $f = F \circ g$, where F is an analytic function from Δ and g is a K-quasiconformal mapping from Δ onto itself.

It is known that there is a quasiconformal mapping g from Δ onto itself such that g is a solution of the *Beltrami* equation

$$g_{\bar z} = \mu g_z$$

(see [Ah2], [LV]).

Let $F = f \circ g^{-1}$. Then we have for μ_F (see [Ah2], [LV]) that

$$\mu_F \circ g = \frac{g_z}{\overline{g}_{\bar z}} \cdot \frac{\mu_f - \mu_g}{1 - \mu_f \overline{\mu}_g} = 0$$

and we conclude that F is a holomorphic function.

Before we proceed with the proof, we give some remarks which are useful for understanding it.

Notes: Let us first observe that if g is euclidean harmonic, then $L_g(0) \le 2$. The following simple proof was suggested by the referee:

Let K_r be a positively oriented circle of radius r with centre at the origin. Then $2\pi i \partial g(0) = \int_{K_r} g(z) z^{-2}\, dz$ for $r \in (0,1)$. Hence by $g(\Delta) = \Delta$, we find $|\partial g(0)| \le 1$; therefore, $L_g(0) \le 2|\partial g(0)| \le 2$.

Note that g is harmonic with respect to the metric $d\tilde s^2 = \tilde\rho(\zeta)|d\zeta|^2$, where $\tilde\rho = |F'|^2$. Although g is not in general harmonic with respect to the euclidean metric, it turns out that one can estimate the distortion of g at 0. Namely, if g is conformal at 0, since $\mu_f = \mu_g$, we can apply Lemma 6.1 and it yields $L_g(0) \le 16$ (see Case 1 below).

We now continue the proof. Without loss of generality, one can suppose that $g(0) = 0$, and let $dg(0) = p^*\, dz + q^*\, d\bar z$.

Case 1: $q^* = 0$. Now it is convenient to use the notation f^* instead of g and to write $f = F \circ f^*$ and f in the form $f = \overline{g} + h$, where h and g are holomorphic functions on $\overline{\Delta}$. Since $\mu_f = \frac{\overline{g'}}{h'}$, $\mu_f = \mu_{f^*}$ and $\mu_{f^*}(0) = 0$, $f^*/|p^*|$ satisfies the conditions of Lemma 6.1, and hence $f^*(\Delta)$ contains an open disc of radius $\frac{|p^*|}{16}$ and, therefore, $\frac{1}{16}|p^*| \leq 1$, that is $|p^*| \leq 16$.

Hence, since $L_f(0) = |F'(0)|\, L_{f^*}(0)$, and $L_{f^*}(0) = l_{f^*}(0) = |p^*|$, it follows that $L_f = |F'(0)||p^*| \leq 16|F'(0)|$. By the hypothesis, $1 \leq L_f(0)$ and, therefore, $1 \leq 16|F'(0)|$, that is $|F'(0)| \geq 2^{-4}$. Hence, it follows from Lemma 4.1 (Koebe lemma for holomorphic functions) that the theorem holds for f with constant $c_0 = 2^{-6}$.

Note that the function $f_0^* = L \circ f^*$ satisfies conditions of Lemma 6.1 in general, where $L = (A^*)^{-1}$ and $A^*(\zeta) = p^*\zeta + q^*\overline{\zeta}$, and, in particular, if f^* is Euclidian harmonic, then $|p^*| \leq 1$.

Case 2: $q^* \neq 0$. Let $df(0) = p\,dz + q\,d\bar{z}$ and $A = df(0)$, that is $A(\zeta) = p\zeta + q\,\overline{\zeta}$, and let $B = A^{-1}$ and $f_0 = B \circ f$. Since f_0 is harmonic and $df_0(0) = \mathrm{Id}$, it satisfies the conditions of the previous case. Thus the theorem holds for f_0 with constant $c_0 = 2^{-6}$.

By the hypothesis, $L_f(0) \geq 1$, and, therefore, $l_f(0) \geq 1/D(0)$, where $D(0) = L_f(0)/l_f(0)$ is the dilatation of f at 0. Hence, since $f = A \circ f_0$ and $A(\Delta_{c_0})$ contains an open disc of radius $r_0 = l_f(0) \cdot c_0 = (|p| - |q|) \cdot c_0$, it follows that the theorem holds for f with constant $l_f(0)2^{-6}$ and, therefore, with constant $2^{-6}/D(0)$. Since $K \geq D(0)$, it holds with constant $c(K)$. $\square$

As an application of the Koebe lemma for quasiregular harmonic mappings, we immediately obtain the following result, which we call the Koebe theorem for quasiregular harmonic mappings.

Theorem 6.1. (Koebe theorem for quasiregular harmonic mappings) *Suppose that f is a K-quasiregular harmonic mapping on $\overline{B}$, where $B = B(a;r)$; let $D = f(B)$ and $f(a) = b$. Then, for every $\theta \in \mathbf{R}$, there exists a point w on the half-line Λ_b^θ which belongs to $f(\overline{B})$, such that $|w-b| \geq R_f(a)$, where $R_f(a) := c(K)r|\operatorname{grad} f(a)|$.*

In particular, there exists a point $\omega \in f(\overline{B})$ such that $|\omega| - |b| = |\omega - b| \geq R_f(a)$.

Now, it is clear that one can prove a version of Theorem A for quasiregular harmonic mappings by means of the Koebe theorem for quasiregular harmonic mappings, using a similar procedure to the one in the case of holomorphic functions. It is left to the interested reader as an exercise (see also [MM1]).

The following example shows that the constant 1/16 is sharp for quasiconformal mappings with dilatation $|\mu(z)| \leq |z|$.

Example 3. Let $A(z) = 4z/(1+|z|)^2$, $k^0(z) = z(1-z)^{-2}$ and $f = k^1 = k^0 \circ A/4$. By a straightforward calculation, one can verify that $k^1(\Delta)$ contains the disc of radius 1/16, that 1/16 is on the boundary of $k^1(\Delta)$ and that $|f_{\bar{z}}| = |z||f_z|$.

Concerning the harmonic Koebe theorem, Clunie and Sheil-Small in the paper [Cl-Sh] mentioned that they had first found a proof with a weaker constant, and then Hinkkanen suggested to use the metric which gave 1/16, the best constant which the method could produce.

There is a conjecture that the best constant is 1/6 in the harmonic analogy of the Koebe one-quarter theorem (see [Du]).

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The referee mentioned that Lemma 6.1 appears in the paper [He-Po], found numerous misprints and gave advice which greatly improved the exposition.

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