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A REMARK ON QUASICONFORMAL DIMENSION DISTORTION ON THE LINE

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Abstract. The general dimension distortion result of Astala says that a one dimensional set goes to a set of dimension at least $1-k$ under a k-quasiconformal mapping. An improved version for rectifiable sets appears in recent work of Astala, Clop, Mateu, Orobitg and Uriarte-Tuero in connection with quasiregular removability problems. We give an alternative proof of their result establishing a bound of the form $1-ck^2$, provided that either the initial or the target set lies on a straight line. The bound $1-k^2$ holds under the additional assumption that the line stays fixed.

1. Introduction

A homeomorphism $f\colon \Omega \to \Omega'$ between planar domains is called *k-quasiconformal* if it lies in the Sobolev class $W^{1,2}_{\mathrm{loc}}(\Omega)$ and satisfies the Beltrami equation

$$\bar\partial f(z) = \mu(z)\partial f(z) \quad \text{a.e. } z \in \Omega,$$

with a measurable coefficient $\|\mu\|_\infty \le k < 1$.

1.1. Remark. Most commonly, such a map is called a K-quasiconformal map in the literature, with $K = \frac{1+k}{1-k}$. However, we shall work with the definition above, since the L^∞-norm of the Beltrami coefficient has a natural role in connection with holomorphic motions. We make an exception in this introductory part and use the traditional form in Corollaries 1.8 and 1.9. In any case, the reader should think of both dilatations, $0 \le k < 1$ and $K \ge 1$, simultaneously.

Astala [A1], in his seminal paper, gave a complete description of dimension distortion of general sets under planar quasiconformal mappings. Here and in the sequel, dimension always refers to Hausdorff dimension.

1.2. Theorem. ([A1]) *Let $f\colon \Omega \to \Omega'$ be k-quasiconformal and suppose $E \subset \Omega$ is compact. Then*

$$\frac{1-k}{1+k}\left(\frac{1}{\dim(E)} - \frac{1}{2}\right) \le \frac{1}{\dim(f(E))} - \frac{1}{2} \le \frac{1+k}{1-k}\left(\frac{1}{\dim(E)} - \frac{1}{2}\right). \tag{1.3}$$

This inequality is best possible.

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It is expected that, say, for subsets of the real line the range for dimension distortion should be more restrictive. In fact, this is the case for *quasicircles*, these are quasiconformal images of the unit circle (or a straight line).

1.4. Theorem. ([BP]) *For every k-quasicircle Γ for k close to* 0,

$$\dim \Gamma \le 1 + 37k^2.$$

1.5. Remark. Note that (1.3) would give the bound $1+k$. The result above provides a bound of the form $1+ck^2$, an improvement for small values of k. We could choose $c = 60$ to obtain a valid bound for all values of k. In fact, $\dim\Gamma \le 1+k^2$ due to Smirnov's unpublished result. This is conjectured to be sharp. The fact, that the order k^2 is sharp was proven in [BP].

The $(1+ck^2)$-type estimate for quasicircles reflects back to the dimension distortion of subsets of the line, as well, allowing us to improve the general estimate (1.3) in the case of the jump to dimension one. Throughout these notes $c \ge 1$ will denote a fixed positive absolute constant, such that $\dim\Gamma \le 1+ck^2$ holds, for every k-quasicircle Γ, i.e. we can choose $c = 60$ or even $c = 1$ in view of Smirnov's result.

The following type of result (and in particular Corollary 1.9) is a crucial step in [ACMOU] for their improved version of Painlevé removability for bounded K-quasiregular mappings ($K > 1$): sets of σ-finite Hausdorff measure at the critical dimension are always removable.

1.6. Theorem. *Let $f\colon \mathbf{C} \to \mathbf{C}$ be a k-quasiconformal map with $0 < k < 1/\sqrt{8c}$ and $E \subset \mathbf{R}$. Then $\dim fE < 1$ provided that $\dim E \le 1 - 8ck^2$. Conversely, if $\dim E = 1$ then $\dim fE > 1 - 8ck^2$.*

This result (with unspecified constant) is due to [ACMOU]. Discussing their results with the authors I found a more direct proof to this kind of improved quasiconformal dimension distortion. The purpose of this paper is to present this alternative proof of Theorem 1.6 which has its own interest. Our approach relies on the area distortion argument of Astala, we shall follow the presentation in [A1]. This approach allows for further generalizations and improvements, see [APS].

1.7. Remark. The borderline dimension for the jump to one dimension is $2/(K+1) = 1-k$ in the general case. Thus Theorem 1.6 is really an improvement for small values of k and then it is easy to establish some improvement for every k in the sense of Corollary 1.9.

Let us mention two immediate corollaries of Theorem 1.6 from [ACMOU].

1.8. Corollary. ([ACMOU]) *Let $E \subset \mathbf{R}$ be a compact set. For every $1 < K < K_0$ there exists a positive number $\varepsilon(K)$, such that if*

$$\dim E \le \frac{2}{K+1} + \varepsilon(K),$$

then E is removable for bounded K-quasiregular mappings.

1.9. Corollary. ([ACMOU]) *Let* $E \subset \mathbf{R}$ *of dimension* 1 *and* $K > 1$*. Then for any* K*-quasiconformal map* $f\colon \mathbf{C} \to \mathbf{C}$,

$$\dim fE \geq \frac{2}{K+1} + \delta(K),$$

where $\delta(K) > 0$ *and depends (continuously) only on* K*.*

Section 2 is devoted to the proof of Theorem 1.6, while in Section 3 we discuss related results concerning quasisymmetric maps of the line.

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2. Improved distortion

The key idea in [A1] was to look at quasiconformal mappings as holomorphic motions. Recall that a function $\Phi\colon \mathbf{D} \times E \to \overline{\mathbf{C}}$ is a *holomorphic motion* of a set $E \subset \overline{\mathbf{C}}$ if

- for any fixed $z \in E$, the map $\lambda \mapsto \Phi(\lambda, z)$ is holomorphic in $\mathbf{D}$ (the open unit disk),
- for any fixed $\lambda \in \mathbf{D}$, the map $z \mapsto \Phi_\lambda(z) = \Phi(\lambda, z)$ is an injection, and
- the mapping Φ_0 is the identity on E.

A fundamental result about holomorphic motions is the extended version of the λ-lemma by Slodkowski [S], which says that every holomorphic motion extends to a global motion $\Phi\colon \mathbf{D} \times \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ and $\Phi_\lambda\colon \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ is a $|\lambda|$-quasiconformal mapping.

Let us define the dimension $t(k)$ for $0 < t < 2$ and $k < 1/\sqrt{4c}$ by the formula

$$\frac{1}{t(k)} - \frac{1}{2} = \frac{1}{3}\left[\left(\frac{1}{t} - \frac{1}{2}\right) + \frac{1 - 4ck^2}{1 + 4ck^2}\right]. \tag{2.1}$$

Recall that $c > 0$ is an absolute constant. Due to the assumption that $k < 1/\sqrt{4c}$, we see that $0 < t(k) < 2$ as $0 < t < 2$ and that $t(k)$ is continuous and strictly increasing in both t and k. We will see that a t-dimensional set on a line goes to a set of dimension at most $t(k)$ under a k-quasiconformal mapping. In the following theorem we establish the corresponding estimate under *conformality assumption* on finite union of disks.

2.2. Theorem. *Let* $f\colon \mathbf{C} \to \mathbf{C}$ *be a* k*-quasiconformal homeomorphism of* $\mathbf{C}$ ($k < k_0 = 1/\sqrt{4c}$)*, conformal outside* $\mathbf{D}$*, normalized by* $f(z) = z + o(1)$ ($z \to \infty$)*. Assume that* f *is conformal on some finite union of disjoint disks* $E = \bigcup_{i=1}^n B(z_i, r_i) \subset \mathbf{D}$*, where* $z_i \in \mathbf{R}$*. Then for any* $0 < t < 2$,

$$\sum_{i=1}^n (|f'(z_i)| r_i)^{t(k)} \leq C \left(\sum_{i=1}^n r_i^t\right)^{\frac{1}{3}\frac{t(k)}{t}}, \tag{2.3}$$

where C is a positive constant (may be chosen to be 64). The exponent $0 < t(k) < 2$ is determined by formula (2.1). For the value $t = 1 - 8ck^2$, $t(k) < 1$, provided that k is nonzero.

In the other direction, we have

$$\sum_{i=1}^{n} r_i^{t(k)} \le C \left(\sum_{i=1}^{n} (|f'(z_i)| r_i)^t \right)^{\frac{1}{3} \frac{t(k)}{t}}. \tag{2.4}$$

Proof. We closely follow the ideas of [A1]. There, the author considers invariant measures on holomorphically moving Cantor sets and applies the Ruelle–Bowen thermodynamic formalism. The main observation is that we can make use of the quasicircle dimension estimate of Theorem 1.4 in this framework in a natural way. Let us discuss the proof in detail.

Embed the map f into a holomorphic motion in a standard manner. Denote by μ the complex dilatation of f and define $\mu_\lambda = \frac{\lambda\mu}{k}$ for every $\lambda \in \mathbf{D}$. This Beltrami coefficient satisfies $\|\mu_\lambda\|_\infty \le |\lambda| < 1$ and thus we have a principal solution f_λ by the measurable Riemann mapping theorem. Principal solution refers to the unique homeomorphic solution with asymptotics at infinity $f_\lambda(z) = z + o(1)$. By uniqueness, for $\lambda = k$, we get back our original map, $f_k = f$ and $f_0 = \mathrm{id}$. Since μ and hence μ_λ vanish on E, the complex derivatives $f'_\lambda(z_i)$ exist and are nonzero. We shall use the important fact,

(2.5) the function $\lambda \mapsto f'_\lambda(z_i)$ is holomorphic [AB, Theorem 3].

By Koebe's 1/4-theorem

$$D_i(\lambda) = B(f_\lambda(z_i), 1/4|f'_\lambda(z_i)|r_i) \subset f_\lambda(B(z_i, r_i))$$

and $f_\lambda(\mathbf{D}) \subset B(f_\lambda(0), 4)$. Here $D_i(\lambda) - f_\lambda(0) = \psi_{i,\lambda} D_i(0)$, where

$$\psi_{i,\lambda}(z) = f'_\lambda(z_i)(z - z_i) + (f_\lambda(z_i) - f_\lambda(0)).$$

The coefficients of the similarities $\psi_{i,\lambda}$ vary holomorphically in λ, thus $\{D_i(\lambda) - f_\lambda(0)\}_1^n$ is a holomorphic family of disjoint disks contained in $B(0,4)$. Choosing additional similarities $\phi_i\colon B(0,4) \to D_i(0)$, $\phi_i(z) = \frac{1}{16} r_i z + z_i$, set $\gamma_{i,\lambda} = \psi_{i,\lambda} \circ \phi_i$. These contractions generate a holomorphic family of Cantor sets $C_\lambda \subset B(0,4)$ as described in [A1]. There is a natural identification of the points of C_λ with sequences of $\{1, \dots, n\}^{\mathbf{N}}$. This correspondence gives a bijective map $\Phi_\lambda\colon C_0 \to C_\lambda$. Here $\Phi_0 = \mathrm{id}$ and $\Phi_\lambda(z)$ depends holomorphically on λ and thus $\Phi_\lambda(z)$ is a holomorphic motion. By the extended λ-lemma of [S], it extends to a global $\Phi_\lambda\colon \mathbf{C} \to \mathbf{C}$ $|\lambda|$-quasiconformal mapping. Observe that $C_0 \subset \mathbf{R}$ since $D_i(0) = B(z_i, 1/4r_i)$'s are centered on the real line. This shows that C_λ is contained in a $|\lambda|$-quasicircle and thus has dimension at most $1 + c|\lambda|^2$ according to Theorem 1.4. On the other hand the dimension s of the self-similar Cantor set C_λ is determined by the formula [H]

$$\sum_{i=1}^{n} \left(\frac{1}{16} |f'_\lambda(z_i)| r_i \right)^s = 1.$$

We certainly have

$$\sum_{i=1}^{n}\left(\frac{1}{16}|f_\lambda'(z_i)|r_i\right)^{1+c|\lambda|^2}\le 1. \tag{2.6}$$

We are going to use this fact to obtain some improvement on the dimension distortion.

For a probability distribution $\{p_i\}_{i=1}^n$ define the function

$$u(\lambda)=2\sum p_i\log(a|f_\lambda'(z_i)|r_i)-\sum p_i\log p_i,$$

where we write a for $1/16$ for simplicity. This is a harmonic function by (2.5) and we have the estimate

$$\begin{aligned}
u(\lambda)&=\frac{2}{1+c|\lambda|^2}\left[(1+c|\lambda|^2)\sum p_i\log(a|f_\lambda'(z_i)|r_i)-\sum p_i\log p_i\right]\\
&\quad+\frac{1-c|\lambda|^2}{1+c|\lambda|^2}\sum p_i\log p_i\\
&\le\frac{2}{1+c|\lambda|^2}\log\left(\sum(a|f_\lambda'(z_i)|r_i)^{1+c|\lambda|^2}\right)+\frac{1-c|\lambda|^2}{1+c|\lambda|^2}\sum p_i\log p_i\\
&\le\frac{1-c|\lambda|^2}{1+c|\lambda|^2}\sum p_i\log p_i
\end{aligned}$$

in terms of Jensen's inequality for the concave logarithm function and (2.6).

In order to make use of this estimate for the growth of u, apply Harnack's inequality in the disk $\{|\lambda|<2k\}$ $(k<1/2)$,

$$u(k)\le\frac{1}{3}u(0)+\frac{2}{3}\,\frac{1-4ck^2}{1+4ck^2}\sum p_i\log p_i. \tag{2.7}$$

For dimension estimate, write

$$\begin{aligned}
&\sum p_i\log(a|f'(z_i)|r_i)-\frac{1}{t(k)}\sum p_i\log p_i\\
&=\frac{1}{2}u(k)+\left(\frac{1}{2}-\frac{1}{t(k)}\right)\sum p_i\log p_i\\
&\overset{(2.7)}{\le}\frac{1}{3}\sum p_i\log(ar_i)+\left[\frac{1}{3}\,\frac{1-4ck^2}{1+4ck^2}-\frac{1}{6}+\frac{1}{2}-\frac{1}{t(k)}\right]\sum p_i\log p_i\\
&=\frac{1}{3}\left(\sum p_i\log(ar_i)-\frac{1}{t}\sum p_i\log p_i\right)\\
&\quad+\left[\frac{1}{3}\left(\frac{1}{t}-\frac{1}{2}+\frac{1-4ck^2}{1+4ck^2}\right)+\frac{1}{2}-\frac{1}{t(k)}\right]\sum p_i\log p_i\\
&\overset{(J)}{\le}\frac{1}{3t}\log\left(\sum(ar_i)^t\right).
\end{aligned}$$

In the last step we see that due to the definition of $t(k)$ in (2.1), the expression in the square brackets is zero, while (J) refers to another application of Jensen's inequality. With a proper choice of the weights p_i we actually have equality in Jensen's inequality, namely put $p_i = (|f'(z_i)|r_i)^{t(k)}/\sum(|f'(z_i)|r_i)^{t(k)}$ to arrive at the following form of (2.3)

$$\frac{1}{t(k)}\log\Big(\sum(a|f'(z_i)|r_i)^{t(k)}\Big) \le \frac{1}{3t}\log\Big(\sum(ar_i)^t\Big).$$

Our setting is not symmetric with respect to the inverse mapping, however, invoking Harnack's inequality the other way around one obtains (2.4) in an analogous way. It remains to observe that in case of $t(k) = 1$, t reads as

$$t = \frac{1+4ck^2}{1+12ck^2} > 1 - 8ck^2 \qquad (k \neq 0). \qquad \square$$

Our estimates are only interesting as $k \to 0$. In particular, we often will make the assumption $k < k_0$ with $k_0 = 1/\sqrt{4c}$, this is the range where $t(k)$ is defined at all. We will need the following standard *deformation lemma* from [A2, Lemma 4.2]. For the sake of completeness we sketch here a short proof based on holomorphic motions.

2.8. Lemma. *Let f be a k-quasiconformal mapping on $\overline{\mathbf{C}}$ fixing 0, 1 and ∞. Then for each $\varepsilon > 0$ there is a number $\varrho = \varrho(k,\varepsilon) \in (0,1)$ and a k_ε-quasiconformal mapping φ on $\mathbf{C}$ such that*

(a) $\varphi(z) = f(z)$ *if* $1 \le |z|$,
(b) $\varphi(z) = z$ *if* $|z| \le \varrho$,

and $k_\varepsilon \to k$ as $\varepsilon \to 0$.

Proof. We consider the associated holomorphic motion $\{f_\lambda(z)\}$ as in Theorem 2.2 with the exception that the homeomorphic solution f_λ is now normalized by the condition that it fixes 0, 1 and ∞. Consider the following modified motion of the set $\{|z| \le \varrho\} \cup \{|z| \ge 1\}$ for some $0 < \varrho < 1$,

$$\Phi_\lambda(z) = \begin{cases} f_\lambda(z) & \text{if } |z| \ge 1, \\ z & \text{if } |z| \le \varrho. \end{cases}$$

Classical distortion properties of quasiconformal mappings assure that the image of the unit circle $f_\lambda(S^1)$ will remain disjoint from the disk $\{|z| \le \rho\}$ as long as $|\lambda| < \lambda_0 = \lambda_0(\varrho) < 1$, where $\lambda_0(\varrho) \to 1$ as $\varrho \to 0$. In other words, $\Phi_\lambda(z)$ is a holomorphic motion parametrized by the disk $\{|\lambda| < \lambda_0\}$. The extension of Φ_k provided by the extended λ-lemma gives a (k/λ_0)-quasiconformal deformation of f described in the statement of the lemma. $\square$

2.9. Lemma. *Assume that $f\colon \mathbf{C} \to \mathbf{C}$ is a k-quasiconformal mapping $(k < k_0)$ fixing 0, 1 and ∞. Let $B_i = B(z_i, r_i)$ $(z_i \in \mathbf{R})$ disjoint disks in $\mathbf{D}$. Then for every* *sufficiently small* $\varepsilon > 0$ *we have*

$$\sum(\operatorname{diam} fB_i)^{t(k_\varepsilon)} \le C(k,\varepsilon)\left(\sum r_i^t\right)^{\frac{1}{3}\frac{t(k_\varepsilon)}{t}},$$

with $k_\varepsilon \to k$ *as* $\varepsilon \to 0$*. Similarly, in the other direction*

$$\sum r_i^{t(k_\varepsilon)} \le C(k,\varepsilon)\left(\sum(\operatorname{diam} fB_i)^t\right)^{\frac{1}{3}\frac{t(k_\varepsilon)}{t}}.$$

Proof. Apply Lemma 2.8 to deform f in disks B_i and outside $\mathbf{D}$. We obtain a k_ε-quasiconformal map $\varphi\colon \mathbf{C} \to \mathbf{C}$ which agrees with f in $\mathbf{D} \setminus \bigcup B_i$, identity outside $B(0, 1/\varrho)$ and a τ_i similarity inside $B(z_i, \varrho r_i)$. Here τ_i is determined by $\tau_i(z_i) = f(z_i)$ and $\tau_i(z_i + r_i) = f(z_i + r_i)$. Moreover we have a good control on the diameters of the corresponding sets,

$$\begin{aligned} |\varphi'(z_i)|r_i = |\tau_i'|r_i &= |f(z_i + r_i) - f(z_i)| \\ &\le \operatorname{diam} fB_i \lesssim |f(z_i + r_i) - f(z_i)| = |\varphi'(z_i)|r_i, \end{aligned} \tag{2.10}$$

up to a constant depending only on k, as quasiconformal maps distort circles in a uniform manner.

Conjugating with an additional similarity $u(z) = (1/\varrho)z$, $(u^{-1}\circ\varphi\circ u)$ is identical outside $\mathbf{D}$ and similarity in disks $B(\varrho z_i, \varrho^2 r_i)$. We may apply Theorem 2.2 to find

$$\sum(|(u^{-1}\circ\varphi\circ u)'(\varrho z_i)|\varrho^2 r_i)^{t(k_\varepsilon)} \le C\left(\sum(\varrho^2 r_i)^t\right)^{\frac{1}{3}\frac{t(k_\varepsilon)}{t}},$$

$$\sum(\varrho^2 r_i)^{t(k_\varepsilon)} \le C\left(\sum(|(u^{-1}\circ\varphi\circ u)'(\varrho z_i)|\varrho^2 r_i)^t\right)^{\frac{1}{3}\frac{t(k_\varepsilon)}{t}}.$$

Combining with (2.10), the desired estimates follow. □

Proof of Theorem 1.6. We shall prove the following claim.

Let $f\colon \mathbf{C} \to \mathbf{C}$ *be a* k*-quasiconformal map with* $k < k_0$ *and* $E \subset \mathbf{R}$*. Then*

$$\begin{aligned} \dim E \le t &\Rightarrow \dim fE \le t(k), \\ \dim fE \le t &\Rightarrow \dim E \le t(k). \end{aligned}$$

Theorem 1.6 follows now from the fact that for $t = 1-8ck^2$, $t(k) < 1$. The claim follows from Lemma 2.9 by a standard covering argument. We sketch the proof in the second case, distorting the dimension downwards. The first case is similar.

First of all, we may clearly assume that $E \subset [-1/2, 1/2]$ and f fixes 0, 1 and ∞. Suppose that $\dim fE = t$, what we need to prove is that $\dim E \le t(k)$. Choose an exponent $t' > t$. Making use of a basic covering theorem we can find a countable family of disjoint disks $D_i = B(w_i, \varrho_i)$ such that $fE \subset \bigcup 5D_i$, and $\sum \varrho_i^{t'}$ is arbitrary small. Furthermore, we may assume that $w_i \in fE$. Set $z_i = f^{-1}(w_i) \in E$ and $r_i = \operatorname{dist}(z_i, \partial f^{-1}(D_i))$. In this way $B_i = B(z_i, r_i) \subset f^{-1}(D_i)$, so the disks B_i are disjoint, centered on the real line and $\cup B_i \subset \mathbf{D}$ may be assumed, as well.

Now the uniform bound of Lemma 2.9 (with a fixed $\varepsilon > 0$) holds for this possibly infinite family of disks, too

$$\sum r_i^{t'(k_\varepsilon)} \le C(k,\varepsilon)\Big(\sum (\operatorname{diam} fB_i)^{t'}\Big)^{\frac{1}{3}\frac{t'(k_\varepsilon)}{t'}}. \tag{2.11}$$

Observe that $\{f^{-1}(5D_i)\}$ gives a cover of E with sets of size

$$\operatorname{diam} f^{-1}(5D_i) \lesssim r_i,$$

up to a constant depending only on k by distortion properties of quasiconformal maps. While the right-hand side of (2.11) can be made arbitrary small with a proper choice of the family $\{D_i\}$, since $\operatorname{diam} fB_i \le 2\varrho_i$. We conclude that $\dim E \le t'(k_\varepsilon)$, letting $\varepsilon \to 0$ and $t' \to t$, $\dim E \le t(k)$ follows. □

3. Distortion of quasisymmetric functions

In this section we make the assumption that our map fixes the real line. In other words, we consider *quasisymmetric maps* of $\mathbf{R}$, where the quasisymmetricity is measured by the dilatation of (the best) quasiconformal extension. This assumption allows us to sharpen our estimates and obtain the aesthetically appealing (and possibly sharp) bound $1 - k^2$ for distortion of 1-dimensional sets. This is a dual result to Smirnov's $(1 + k^2)$-bound on the dimension of quasicircles, apparently known to him. In fact, we rely on some of the ideas of him developed for the quasicircle estimate. We are grateful to him for allowing us to include this result here.

3.1. Theorem. *Let $f\colon \mathbf{C} \to \mathbf{C}$ be a k-quasiconformal map for which $f(\mathbf{R}) = \mathbf{R}$. Then for a 1-dimensional set $E \subset \mathbf{R}$,*

$$\dim fE \ge 1 - k^2.$$

Standard covering arguments reduce the theorem to the following statement. We sketch the details after the proof of Lemma 3.2.

3.2. Lemma. *Let there be given a sequence of finite families of disjoint disks $\{B_{i,j} = B(z_{i,j}, r_{i,j})\}_{i=1}^{n_j}$ $(j = 1, 2, \dots)$ in the unit disk $\mathbf{D}$, such that in every collection $z_i \in \mathbf{R}$, for any $t < 1$ $\sum_i r_i^t \to \infty$, $r_i \le \delta_j$ and $\delta_j \to 0$ as $j \to \infty$. Consider a sequence of k-quasiconformal maps $f_j\colon \mathbf{C} \to \mathbf{C}$, $f_j(\bar z) = \overline{f_j(z)}$, f_j conformal outside $\mathbf{D}$, normalized by $f_j(z) = z + o(1)$ $(z \to \infty)$. Assume that f_j is conformal on the disks $B_{i,j}$ belonging to the level j. Then*

$$\sum_{i=1}^{n_j}\left(\frac{1}{16}|f_j'(z_i)|r_i\right)^{1-k^2-\eta_j} \ge 1.$$

Here $\eta_j \to 0$ as $j \to \infty$ for some subsequence.

Proof. For every j embed the map $f = f_j$ into the standard holomorphic motion $f_\lambda(z)$ as in Theorem 2.2. In this way $f_0 = \mathrm{id}$, $f_k = f_{(j)}$. Since the level j is fixed for a while we will not explicitly write the dependence on j. As μ the complex dilatation of f is symmetric with respect to the real axis, we have $\mu_\lambda(\bar z) = \overline{\mu_{\bar\lambda}(z)}$. This inherits to the solutions, $f_\lambda(\bar z) = \overline{f_{\bar\lambda}(z)}$. In particular, for purely imaginary λ

$$|f'_{-\lambda}(z_i)| = |f'_\lambda(z_i)|, \tag{3.3}$$

while for real values of λ the map f_λ is symmetric with respect to the real axis.

Recall from the proof of Theorem 2.2 that the disks

$$D_i(\lambda) = B(f_\lambda(z_i), 1/4|f'_\lambda(z_i)|r_i)$$

are disjoint and included in a disk of radius 4. Hence comparing their area gives (with $a = 1/16$)

$$\sum (a|f'_\lambda(z_i)|r_i)^2 \le 1.$$

Moreover if λ is real then all the disks $D_i(\lambda)$ are centered on the real line as f_λ preserves the real axis. In this case, we have

$$\sum (a|f'_\lambda(z_i)|r_i) \le 1.$$

As before, consider the harmonic function for a given probability distribution $\{p_i\}_{i=1}^n$,

$$u(\lambda) = u_j(\lambda) = 2\sum p_i \log(a|f'_\lambda(z_i)|r_i) - \sum p_i \log p_i.$$

Jensen's inequality and the estimates above tell us that u is negative for every $\lambda \in \mathbf{D}$ and $u(\lambda) \le \sum p_i \log p_i$ for real valued λ. Due to (3.3) u is even on the imaginary axis, $u(-\lambda) = u(\lambda)$ for $\lambda \in i\mathbf{R}$.

Choose a sequence $t_l \to 1-$ as $l \to \infty$. For a fixed l, $\sum_i r_{i,j}^{t_l} \to \infty$ as $j \to \infty$ by assumption. So there exits a subsequence j_l such that $\sum_i (ar_{i,j_l})^{t_l} \ge 1$ for every l. For a level $j = j_l$, set the weights

$$p_{i,j} = p_i = \frac{r_i^{t_l}}{\sum r_i^{t_l}}.$$

Then

$$\begin{aligned} u_{j_l}(0) &= 2\sum p_i \log(ar_i) - \sum p_i \log p_i \\ &= \frac{2}{t_l}\left(\sum p_i \log(ar_i)^{t_l} - \sum p_i \log p_i\right) + \left(\frac{2}{t_l} - 1\right)\sum p_i \log p_i \\ &= \frac{2}{t_l}\log\left(\sum (ar_i)^{t_l}\right) + \left(\frac{2}{t_l} - 1\right)\sum p_i \log p_i \\ &\ge \left(\frac{2}{t_l} - 1\right)\sum p_i \log p_i. \end{aligned} \tag{3.4}$$

Since the family $-\frac{u_{j_l}(\lambda)}{u_{j_l}(0)}$ form a normal family of harmonic functions, there exists a harmonic function u_0 such that $u_{j_l} \to u_0$ locally uniformly as $j_l \to \infty$ through a subsequence. For this limit function we have

- $u_0(\lambda) \le 0$ $(\lambda \in \mathbf{D})$,
- $u_0(-\lambda) = u_0(\lambda)$ for $\lambda \in i\mathbf{R}$,

- $u_0(\lambda) \le -1$ for $\lambda \in \mathbf{R}$ and $u_0(0) = -1$.

The last one follows from (3.4) and the fact that $u_j(\lambda) \le \sum p_i \log p_i$ if $\lambda \in \mathbf{R}$.

Now the second item tells us that $\frac{\partial}{\partial y} u_0(0) = 0$ and the third one says $\frac{\partial}{\partial x} u_0(0) = 0$. In this case we have a squared-type Harnack inequality (see Lemma 3.6) of the form

$$u_0(\lambda) \ge \frac{1+|\lambda|^2}{1-|\lambda|^2} u_0(0).$$

Put $\lambda = k$, then

$$u_j(k) \ge \left(\frac{1+k^2}{1-k^2} + \varepsilon_j \right) u_j(0), \tag{3.5}$$

with $\varepsilon_j \to 0$ ($j \to \infty$) for a subsequence.

The usual manipulation with Jensen's inequality provides the desired estimate (here $j = j_l$ and $j(k)$ denotes an exponent depending on k and j to be chosen later).

$$\begin{aligned}
&\frac{1}{j(k)} \log \left(\sum (a|f_j'(z_i)r_i)^{j(k)} \right) \\
&\ge \sum p_i \log(a|f_j'(z_i)|r_i) - \frac{1}{j(k)} \sum p_i \log p_i \\
&= \frac{1}{2} u_j(k) + \left(\frac{1}{2} - \frac{1}{j(k)} \right) \sum p_i \log p_i \\
&\overset{(3.5)}{\ge} \frac{1}{2} \left(\frac{1+k^2}{1-k^2} + \varepsilon_j \right) u_j(0) + \left(\frac{1}{2} - \frac{1}{j(k)} \right) \sum p_i \log p_i \\
&\overset{(3.4)}{\ge} \left[\frac{1}{2} \left(\frac{1+k^2}{1-k^2} + \varepsilon_j \right) \left(\frac{2}{t_l} - 1 \right) + \left(\frac{1}{2} - \frac{1}{j(k)} \right) \right] \sum p_i \log p_i = 0.
\end{aligned}$$

We define $j(k)$ by the expression in the square brackets, so that it will be zero. Since $\varepsilon_j \to 0$ and $t_l \to 1$ as $j_l \to \infty$ (for a subsequence) we see that $j(k) = 1 - k^2 - \eta_j$ where $\eta_j \to 0$ for some subsequence. □

Proof of Theorem 3.1. Let $E \subset [-1/2, 1/2]$ with $\dim E = 1$. Assume to the contrary that $\dim fE < 1 - k^2$ for some k-quasiconformal map which is, as we may assume, symmetric with respect to the real axis. We can find a sequence of finite families of disjoint disks $\{B_i^j = B(z_i, r_i)\}$ such that $z_i \in \mathbf{R}$, $\sup r_i \to 0$, for any $t < 1$, $\sum r_i^t \to \infty$ ($j \to \infty$) and $\sum (\operatorname{diam} fB_i)^d \to 0$ ($j \to \infty$), with a fixed exponent $d < 1 - k^2$. Choose $\varepsilon_0 > 0$ so that also $d < 1 - k_{\varepsilon_0}^2$. Now deform f according to the family on the level j to obtain a k_{ε_0}-quasiconformal map φ_j which is identical outside the unit disk and similarity in $B(\varrho z_i, \varrho^2 r_i)$, here $\varrho = \varrho(\varepsilon_0, k)$. Moreover, $\operatorname{diam} fB_i \asymp |\varphi_j'(\varrho z_i)| r_i$. Apply Lemma 3.2 to the sequence φ_j, we have a contradiction as $j \to \infty$.

Note that we have skipped one detail, we need to make sure that the deformation preserves the symmetry with respect to the line. This is not a serious problem, e.g. with a little modification in the spirit of the proof of Lemma 2.8 one could omit the conformality assumption. On the other hand, it is easy to see that the explicit deformation described in [A2, Lemma 4.2] provides a symmetric deformation. □

3.6. Lemma. (Squared-type Harnack's inequality) *Suppose that the function* $u \le 0$ *is harmonic in* $\mathbf{D}$ *and* $\nabla u(0) = 0$. *Then we have an improved Harnack's inequality of the form*

$$\frac{1+|z|^2}{1-|z|^2}u(0) \le u(z) \le \frac{1-|z|^2}{1+|z|^2}u(0).$$

Proof. The proof is a slight modification of the complex analytic proof of the standard Harnack's inequality. There is a holomorphic function $f\colon \mathbf{D} \to \{w : \Re w < 0\}$ such that $f = u + iv$, where v is real-valued harmonic function. We may assume, that $f(0) = -1$, that is $u(0) = -1$ and $v(0) = 0$. In virtue of the Cauchy–Riemann equations $f'(0) = 0$, since $(\nabla u)(0) = 0$. Map the left half-plane onto the unit disk by the linear fractional transformation $\frac{w+1}{w-1}$, this takes -1 to 0. The composed function maps the unit disk into the unit disk and vanishes at the origin with double multiplicity. We have a squared-type Schwarz lemma in this situation,

$$\left|\frac{f(z)+1}{f(z)-1}\right| \le |z|^2.$$

Observe the following geometric fact for $u = \Re w$,

$$\frac{u+1}{u-1} \le \left|\frac{w+1}{w-1}\right|.$$

Combining the two estimates leads us to

$$u(z) \ge -\frac{1+|z|^2}{1-|z|^2}.$$

Noting that $u(0) = -1$, this is the left hand side of the inequality in (3.6). The argument for the right hand side follows similar lines, one just needs to replace the linear fractional transformation $\frac{w+1}{w-1}$ by its negative. □

3.7. Remark. The order k^2 in Theorem 1.6 and Theorem 3.1 is sharp. Answering a question of Hayman and Hinkkanen, Tukia [T] constructed a k-quasisymmetric map of the unit interval which does not preserve one-dimensional sets. It is actually even more singular, mapping a set of less-than-one dimensional complement to a less-than-one dimensional set. Moreover, the quasisymmetricity k can be arbitrary close to 0. An analysis of the example shows that the dimension distortion is of the type $1 - Ck^2$ as $k \to 0$, with $C > 0$.

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