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GLOBAL ESTIMATES FOR THE SCHRÖDINGER MAXIMAL OPERATOR

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Abstract. The Schrödinger equation, $i\partial_t u + \Delta u = 0$, with initial datum f contained in a Sobolev space $H^s(\mathbf{R}^n)$, has solution $e^{it\Delta}f$. We give sharp conditions under which $\sup_t |e^{it\Delta}f|$ is bounded from $H^s(\mathbf{R})$ to $L^q(\mathbf{R})$ for all q, and give sharp conditions under which $\sup_{0<t<1}|e^{it\Delta}f|$ is bounded from $H^s(\mathbf{R})$ to $L^q(\mathbf{R})$ for all $q \neq 2$. In higher dimensions, we show that $\sup_t |e^{it\Delta}f|$ and $\sup_{0<t<1}|e^{it\Delta}f|$ are bounded from $H^s(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$ only if $s \geq \frac{1}{2} - \frac{1}{2(n+1)}$.

1. Introduction

The Schrödinger equation, $i\partial_t u + \Delta u = 0$, in $\mathbf{R}^{n+1}$, with initial datum f contained in a Sobolev space $H^s(\mathbf{R}^n)$, has solution $e^{it\Delta}f$ which can be formally written as

$$e^{it\Delta}f(x) = \int \widehat{f}(\xi)e^{2\pi i(x\cdot\xi - 2\pi t|\xi|^2)}\,d\xi. \tag{1}$$

We will consider the Schrödinger maximal operators S^* and S^{**}, defined by

$$S^*f = \sup_{0<t<1}|e^{it\Delta}f| \quad \text{and} \quad S^{**}f = \sup_{t\in\mathbf{R}}|e^{it\Delta}f|.$$

The minimal regularity of f under which $e^{it\Delta}f$ converges almost everywhere to f, as t tends to zero, has been studied extensively. By standard arguments, the problem reduces to the minimal value of s for which

$$\|S^*f\|_{L^q(\mathbf{B}^n)} \leq C_{n,q,s}\|f\|_{H^s(\mathbf{R}^n)} \tag{2}$$

holds, where $\mathbf{B}^n$ is the unit ball in $\mathbf{R}^n$.

In two dimensions, that is one spatial dimension, Carleson [4] (see also [10]) showed that (2) holds when $s \geq 1/4$. Dahlberg and Kenig [6] showed that this is sharp in the sense that it is not true when $s < 1/4$.

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In three dimensions, significant contributions have been made by Bourgain [1, 2], Moyua, Vargas and Vega [12, 13], and Tao and Vargas [21, 22]. The best known result is due to Lee [11] who showed that (2) holds when $s > 3/8$.

In higher dimensions, Sjölin [15] and Vega [23, 24] independently showed that (2) holds when $s > 1/2$. It is conjectured that, in all dimensions, the minimal value of s for which (2) holds is $1/4$.

Replacing the unit ball $\mathbf{B}^n$ in (2) by the whole space $\mathbf{R}^n$, we consider the global estimates

$$\|S^* f\|_{L^q(\mathbf{R}^n)} \leq C_{n,q,s} \|f\|_{H^s(\mathbf{R}^n)} \tag{3}$$

and

$$\|S^{**} f\|_{L^q(\mathbf{R}^n)} \leq C_{n,q,s} \|f\|_{H^s(\mathbf{R}^n)}. \tag{4}$$

In one spatial dimension, Kenig, Ponce and Vega [9] proved that (4) holds when $q = 4$ and $s = 1/4$. This was extended by Gülkan [7] who proved that (4) holds when $q \in [4, \infty)$ if and only if $s \geq 1/2 - 1/q$, and it is well known that (4) holds when $q = \infty$ if and only if $s > 1/2$ (see [19]). Sjölin [16] proved that if $q = 2$, then (4) does not hold for any s, and we will show that this is also the case when $q \in (2, 4)$. Thus, we have the following theorem.

Theorem 1. *Let $n = 1$. Then (4) holds if and only if $q \in [4, \infty)$ and $s \geq 1/2 - 1/q$, or $q = \infty$ and $s > 1/2$.*

The following theorem extends a result of Vega [23, 8] (see also [17]) by the endpoint $s = 1/q$ in the range $q \in (2, 4)$.

Theorem 2. *Let $n = 1$ and $q \in (2, \infty)$. Then (3) holds if and only if $s \geq \max\{1/q,\ 1/2 - 1/q\}$.*

Vega [23, 8] (see also [16]) proved that (3) holds when $q = 2$ and $s > 1/2$, and this is not true when $q = 2$ and $s < 1/2$, or for any value of s when $q < 2$. As in Theorem 1, when $q = \infty$, (3) holds if and only if $s > 1/2$ (see [19]). Thus, in order to have complete results in Theorem 2, the only case that remains undecided is $q = 2$, $s = 1/2$.

In higher dimensions, we show that (3) holds only if

$$s \geq \frac{n}{2(n+1)}.$$

We note that the minimal s is thus strictly greater than $1/4$ when $n \geq 2$. A plausible conjecture is that these are indeed the minimal values of s that can appear in (3).

Throughout, C will denote an absolute constant whose value may change from line to line.

2. The positive results

First, we consider one spatial dimension, and extend the argument of Carleson as in [14]. We employ the Kolmogorov–Seliverstov–Plessner method and the following two lemmas. The first is proved by a very slight modification of a lemma due to

Sjölin [20]; the details are omitted. The second is proved by refining the ideas of Carleson.

Lemma 1. *Let* $x, t \in \mathbf{R}$ *and* $\alpha \in [1/2, 1)$. *Then there is a constant* C *such that*

$$\left|\int_{\mathbf{R}} \frac{e^{2\pi i(x\xi - t\xi^2)}}{(1+|\xi|)^\alpha}\, d\xi\right| \le \frac{C}{|x|^{1-\alpha}}.$$

Lemma 2. *Let* $x \in \mathbf{R}$, $t \in [-1, 1]$ *and* $\alpha \in [1/2, 1]$. *Then there is a constant* C *such that*

$$\left|\int_{\mathbf{R}} \frac{e^{2\pi i(x\xi - t\xi^2)}}{(1+|\xi|)^\alpha}\, d\xi\right| \le \frac{C}{|x|^{\alpha}}.$$

Proof. Splitting the integral in two and taking the complex conjugate if necessary we can suppose that $x > 0$, and consider the integral over $(0, \infty)$. When $x \le 4$ and $\alpha < 1$, we are done by Lemma 1, so we can suppose that $x \ge 4$ and $1/x \le C/x^\alpha$.

When $t \le 0$, there exist c_1, $c_2 \in (0, \infty)$ such that

$$\left|\int_0^\infty \frac{e^{2\pi i(x\xi - t\xi^2)}}{(1+|\xi|)^\alpha}\, d\xi\right| \le \left|\int_0^{c_1} \cos\big(2\pi(x\xi - t\xi^2)\big)\, d\xi\right| + \left|\int_0^{c_2} \sin\big(2\pi(x\xi - t\xi^2)\big)\, d\xi\right|,$$

by the Bonnet form of the second mean value theorem for integrals. The derivative of the phase, $x - 2t\xi$, is monotone, and bounded below by x, so by van der Corput's lemma,

$$\left|\int_0^\infty \frac{e^{2\pi i(x\xi - t\xi^2)}}{(1+|\xi|)^\alpha}\, d\xi\right| \le \frac{C}{x} \le \frac{C}{x^\alpha},$$

and we are done.

Now we suppose that $t > 0$, and make the change of variables $\xi \to \xi + 1$, so that

$$\left|\int_0^\infty \frac{e^{2\pi i(x\xi - t\xi^2)}}{(1+|\xi|)^\alpha}\, d\xi\right| = \left|\int_1^\infty \frac{e^{2\pi i((x+2t)\xi - t\xi^2)}}{\xi^\alpha}\, d\xi\right|.$$

As $x + 2t > x$, it will suffice to show that

$$\left|\int_1^\infty \frac{e^{2\pi i(x\xi - t\xi^2)}}{\xi^\alpha}\, d\xi\right| \le \frac{C}{x^\alpha}.$$

Changing variables again, $\xi \to \sqrt{t}\xi$, and denoting $2A = x/\sqrt{t}$, we are required to show that

$$\frac{1}{\sqrt{t}^{1-\alpha}} \left|\int_{\sqrt{t}}^\infty \frac{e^{2\pi i(2A\xi - \xi^2)}}{\xi^\alpha}\, d\xi\right| \le \frac{C}{x^\alpha}.$$

Note that $A > 2$, as we have that $x \ge 4$.

Consider first the integral over $(\sqrt{t}, A/2)$. By the change of variables, $\xi \to A\xi$, we are required to show that

$$\frac{1}{x^{1-\alpha}}\left|\int_{x/2}^{A^2/2}\frac{e^{2\pi i(2\xi-\xi^2/A^2)}}{\xi^\alpha}\,d\xi\right|\le\frac{C}{x^\alpha}.$$

The derivative of the phase, $2-2\xi/A^2$, is bounded below by one on $(x/2, A^2/2)$, so that, by the mean value theorem and van der Corput's lemma,

$$\frac{1}{x^{1-\alpha}}\left|\int_{x/2}^{A^2/2}\frac{e^{2\pi i(2\xi-\xi^2/A^2)}}{\xi^\alpha}\,d\xi\right|\le\frac{C}{x}\le\frac{C}{x^\alpha},$$

and we are done.

Finally, we are required to show that

$$\frac{1}{\sqrt{t}^{1-\alpha}}\left|\int_{A/2}^{\infty}\frac{e^{2\pi i(2A\xi-\xi^2)}}{\xi^\alpha}\,d\xi\right|\le\frac{C}{x^\alpha}.$$

By the mean value theorem, and the fact that modulus of the second derivative of the phase is bounded below by one,

$$\frac{1}{\sqrt{t}^{1-\alpha}}\left|\int_{A/2}^{\infty}\frac{e^{2\pi i(2A\xi-\xi^2)}}{\xi^\alpha}\,d\xi\right|\le\frac{C\sqrt{t}^{2\alpha-1}}{x^\alpha}\left|\int_{A/2}^{c}e^{2\pi i(2A\xi-\xi^2)}\,d\xi\right|\le\frac{C}{x^\alpha},$$

and we are done. □

The following theorem is an endpoint improvement of result of Vega [23, 8] (see also [17]) in the range $(2,4)$.

Theorem 3. *Let $n=1$. If $q\in[4,\infty)$ and $s\ge 1/2-1/q$, then (4) holds. If $q\in(2,\infty)$ and $s\ge\max\{1/q,\,1/2-1/q\}$, then (3) holds.*

Proof. By duality, it will suffice to show that

$$\left|\int_{\mathbf{R}}e^{it(x)\Delta}f(x)w(x)\,dx\right|^2\le C_q\|f\|^2_{H^s(\mathbf{R})}\|w\|^2_{L^{q'}(\mathbf{R})}$$

for all positive $w\in L^{q'}(\mathbf{R})$, where the measurable function t maps into $\mathbf{R}$ when we are considering the bound (4) and into $(0,1)$ when we consider (3).

By Fubini's theorem and the Cauchy–Schwarz inequality, the left hand side of this inequality is bounded by

$$\int_{\mathbf{R}}|\widehat{f}(\xi)|^2(1+|\xi|)^{2s}\,d\xi\int_{\mathbf{R}}\left|\int_{\mathbf{R}}e^{2\pi i(x\xi-t(x)\xi^2)}w(x)\,dx\right|^2\frac{d\xi}{(1+|\xi|)^{2s}}.$$

Thus, by writing the squared integral as a double integral, it will suffice to show that

$$\int_{\mathbf{R}}\int_{\mathbf{R}}\int_{\mathbf{R}}e^{2\pi i((x-y)\xi-(t(x)-t(y))\xi^2)}w(x)w(y)\,dx\,dy\frac{d\xi}{(1+|\xi|)^{2s}}\le C_p\|w\|^2_{L^{q'}(\mathbf{R})}.\tag{5}$$

By Lemma 1, we have

$$\left|\int_{\mathbf{R}} \frac{e^{2\pi i((x-y)\xi-(t(x)-t(y))\xi^2)}}{(1+|\xi|)^{2s}}\,d\xi\right| \le \frac{C}{|x-y|^{1-2s}}$$

when t takes values in $\mathbf{R}$, and $2s \in [1/2, 1)$, and by Lemmas 1 and 2, we have

$$\left|\int_{\mathbf{R}} \frac{e^{2\pi i((x-y)\xi-(t(x)-t(y))\xi^2)}}{(1+|\xi|)^{2s}}\,d\xi\right| \le \frac{C}{|x-y|^{\max\{2s,1-2s\}}}$$

when t takes values in $(0,1)$. Thus, by Fubini's theorem, the left hand side of (5) is bounded by a constant multiple of

$$\int_{\mathbf{R}}\int_{\mathbf{R}} \frac{w(x)w(y)}{|x-y|^{1-2s}}\,dx\,dy$$

in the first case, and

$$\int_{\mathbf{R}}\int_{\mathbf{R}} \frac{w(x)w(y)}{|x-y|^{\max\{2s,1-2s\}}}\,dx\,dy$$

in the second. Finally, by Hölder's inequality and the Hardy–Littlewood–Sobolev inequality, these are bounded by

$$\|w\|_{L^{q'}(\mathbf{R})}\left\|\int_{\mathbf{R}} \frac{w(x)}{|x-\cdot\,|^{1-2s}}\,dx\right\|_{L^q(\mathbf{R})} \le C_q\|w\|^2_{L^{q'}(\mathbf{R})},$$

where $s = 1/2 - 1/q$ and $q \ge 4$ when we are considering the bound in (4), and

$$\|w\|_{L^{q'}(\mathbf{R})}\left\|\int_{\mathbf{R}} \frac{w(x)}{|x-\cdot\,|^{\max\{2s,1-2s\}}}\,dx\right\|_{L^q(\mathbf{R})} \le C_q\|w\|^2_{L^{q'}(\mathbf{R})},$$

where $s = \max\{1/q,\ 1/2-1/q\}$ and $q > 2$ when we consider (3). □

In higher dimensions, we simply interpret the known results. By modifying very slightly the proof of Theorem 2.2 in [21] due to Tao and Vargas, the following result is proved using bilinear restriction estimates.

Theorem 4. *Let* $q \in (2+\frac{4}{n+1}, \infty]$, $p \in (\max\{q, \frac{2q}{nq-2(n+1)}\}, \infty]$, *and* $s > n(\frac{1}{2}-\frac{1}{q}) - \frac{2}{p}$. *Then there exists a constant* $C_{n,q,p,s}$ *such that*

$$\|e^{it\Delta}f\|_{L^q(\mathbf{R}^n, L^p(\mathbf{R}))} \le C_{n,q,p,s}\|f\|_{H^s(\mathbf{R}^n)}.$$

As usual, we define ∂_t^α by $\widehat{\partial_t^\alpha g}(\tau) = (2\pi|\tau|)^\alpha \widehat{g}(\tau)$, where $\alpha > 0$. Observing that $\partial_t^\alpha e^{it\Delta}f = e^{it\Delta}f_\alpha$, where $\widehat{f_\alpha}(\xi) = (4\pi^2|\xi|^2)^\alpha \widehat{f}(\xi)$, and applying the Sobolev imbedding theorem with $\alpha > 1/p$, we recover their theorem in the following corollary.

Corollary 1. *If* $q \in (2+\frac{4}{n+1}, \infty]$ *and* $s > n(1/2-1/q)$, *then (3) and (4) hold.*

We will see below that these kind of global bounds do not hold when $q < 2$. Thus, for completeness, we provide sufficient conditions, albeit not sharp, for the remaining values of q in (3).

Theorem 5. *If* $q \in [2, 2+\frac{4}{n+1}]$ *and* $s > 3/q - 1/2$, *then (3) holds.*

Proof. Carbery [3] and Cowling [5] independently proved that if $q = 2$ and $s > 1$, then (3) holds. Considering H^s to be a weighted L^2 space, we can interpolate between this and the bound in Corollary 1 to get the result. □

3. The negative results

First of all, we consider one spatial dimension and complete the proof of Theorem 1. The novelty in the following is that if $n = 1$ and $q \in (2, 4)$, then (4) cannot hold for any value of s.

Theorem 6. *Let $n = 1$. If (4) holds, then $q \in [4, \infty)$ and $s \geq 1/2 - 1/q$, or $q = \infty$ and $s > 1/2$.*

The following theorem is due to Sjölin [17], but it will also follow easily from the following proof of Theorem 6.

Theorem 7. *Let $n = 1$. If (3) holds then $q \in [2, \infty)$ and $s \geq \max\{1/q, 1/2 - 1/q\}$, or $q = \infty$ and $s > 1/2$.*

Proof. By a change of variables,

$$S^{**}f(x) = \sup_{t\in\mathbf{R}} \left| \frac{1}{2\pi} \int \widehat{f}\left(\frac{\xi}{2\pi}\right) e^{i(x\xi - t\xi^2)}\, d\xi \right|.$$

Define $A = [N, N + N^\lambda]$, where $N \gg 1$ and $\lambda \in (-\infty, 1]$, and consider f_A defined by $\widehat{f_A}(\xi/2\pi) = e^{-iN^{-\lambda}\xi}\chi_A(\xi)$. We will show that for a range of values of x, a time $t(x)$ can be chosen so that the phase,

$$\phi_x(\xi) = (x - N^{-\lambda})\xi - t(x)\xi^2,$$

is roughly constant on A. With the phase roughly constant, we have

$$S^{**}f_A(x) \geq C \left| \int_A e^{i((x-N^{-\lambda})\xi - t(x)\xi^2)}\, d\xi \right| \geq C|A|.$$

As A is an interval of length N^λ, in order to insure that the phase is roughly constant, we impose the condition $|\phi_x'(\xi)| \leq N^{-\lambda}$ on A. This insures that for all N and λ, there exists a θ_x such that

$$\theta_x - 1/2 \leq \phi_x(\xi) \leq \theta_x + 1/2.$$

As $\phi_x'(\xi) = x - N^{-\lambda} - 2t(x)\xi$, the condition can be rewritten as

$$\frac{x - 2N^{-\lambda}}{2\xi} \leq t(x) \leq \frac{x}{2\xi}$$

for all $\xi \in A$. Define a and b by

$$a(x) = \sup_{\xi\in A} \frac{x - 2N^{-\lambda}}{2\xi} \quad \text{and} \quad b(x) = \inf_{\xi\in A} \frac{x}{2\xi}.$$

To be able to choose the time $t(x)$ we require that $a(x) \leq b(x)$. This is clear when $x \in [0, 2N^{-\lambda}]$, so we suppose that $x > 2N^{-\lambda}$. Now, when $x > 2N^{-\lambda}$,

$$a(x) = \frac{x - 2N^{-\lambda}}{2N} \quad \text{and} \quad b(x) = \frac{x}{2(N + N^{\lambda})},$$

so that we can choose a $t(x)$ when

$$\frac{x - 2N^{-\lambda}}{2N} \leq \frac{x}{2(N + N^{\lambda})}.$$

This condition can be rewritten as $x \leq 2N^{-\lambda} + 2N^{1-2\lambda}$, so we will consider the set $E = [0, N^{1-2\lambda}]$.

As $S^{**} f_A \geq C|A|$ on E, we see that

$$\|S^{**} f_A\|_{L^q(\mathbf{R})} \geq C|A||E|^{1/q}.$$

On the other hand,

$$\|f_A\|_{H^s(\mathbf{R})} \leq C \left(\int_A (1 + |\xi|)^{2s} \right)^{1/2} \leq C|A|^{1/2}(1 + N + N^{\lambda})^s,$$

so that, as $\|S^{**} f_A\|_{L^q(\mathbf{R})} \leq C\|f_A\|_{H^s(\mathbf{R})}$, we have

$$|A||E|^{1/q} \leq C|A|^{1/2}(1 + N + N^{\lambda})^s.$$

Recalling that $|A| = N^{\lambda}$ and $|E| = N^{1-2\lambda}$, we see that

$$N^{\frac{\lambda}{2}} N^{\frac{1-2\lambda}{q}} \leq CN^s,$$

so that, letting N tend to infinity,

$$s \geq \frac{1}{q} + \lambda \left(\frac{1}{2} - \frac{2}{q} \right)$$

for all $\lambda \in (-\infty, 1]$. When $q < 4$, we let λ tend to $-\infty$ to obtain a contradiction for all s. Letting $\lambda = 1$ we recover the fact that $s \geq 1/2 - 1/q$.

Finally, by a well–known counterexample (see [19]), $s > 1/2$ is necessary when $q = \infty$, and we are done.

In order to prove results for S^*, we have the added requirement that

$$[a(x), b(x)] \cap (0, 1) \neq \emptyset$$

for all $x \in E$. We have that $a(x) < 1$ when

$$\frac{x - 2N^{-\lambda}}{2N} < 1,$$

which we rewrite as

$$x < 2N + 2N^{-\lambda}.$$

When $\lambda < 0$, this is an added restriction so we reanalyze in this case. Redefining a smaller $E = [0, 2N + 2N^{-\lambda}]$, we see that

$$N^{\lambda/2}(N + N^{-\lambda})^{1/q} \leq CN^s$$

for all $\lambda \in (-\infty, 0]$, so that, letting N tend to infinity,

$$s \geq \frac{1}{q} + \frac{\lambda}{2} \tag{6}$$

and

$$s \geq \lambda \left(\frac{1}{2} - \frac{1}{q} \right). \tag{7}$$

When $q < 2$, we see by (7) that, letting λ tend to $-\infty$, we have a contradiction for all s. If we let $\lambda = 0$ in (6), we see that $s \geq 1/q$, and from before, when $\lambda = 1$, we have that $s \geq 1/2 - 1/q$.

Again, by the well-known counterexample (see [19]), $s > 1/2$ is necessary when $q = \infty$, and so we are done. □

Remark 1. We note that taking $\lambda = 1/2$ in the above proof, $E = [0, 1]$, the time $t(x)$ can be chosen to be a member of $(0, 1)$ for all $x \in E$, and $s \geq 1/4$ for all q, so we recover the fact that $s \geq 1/4$ is necessary in (2). It is easy to generalise this to higher dimensions. Indeed, it can be shown that g defined by

$$\widehat{g} = \sum_{j=2}^{\infty} 2^{-\alpha j} \chi_{[2^{2j}, 2^{2j}+2^{j-3}] \times [1, 9/8]^{n-1}},$$

where $\alpha \in (2s+1/2, 1)$ and $s < 1/4$, is a member of $H^s(\mathbf{R}^n)$ such that $e^{it\Delta} g$ diverges on the set $[8/9, 1]^n$ as t tends to zero.

We now consider higher dimensions. A corollary of the following theorems is that the minimal value of s that can appear in (3) or (4) is greater than or equal to $\frac{1}{2} - \frac{1}{2(n+1)}$. Again, both theorems will follow from the same proof.

It can be seen by scaling that if $q < 2$ or $s < n(1/2 - 1/q)$, then (4) does not hold. The novelty in Theorem 8 is that if $q \in (2, 2 + 2/n)$, then (4) cannot hold for any value of s. That q cannot equal 2 is due to Sjölin [16].

Theorem 8. *If (4) holds, then $q \in [2 + \frac{2}{n}, \infty)$ and $s \geq n(1/2 - 1/q)$, or $q = \infty$ and $s > n/2$.*

Theorem 9. *If (3) holds, then $q \in [2, \infty)$ and $s \geq \max\{1/q, n(1/2 - 1/q)\}$, or $q = \infty$ and $s > n/2$.*

Proof. We consider S^{**} and argue as in the proof of Theorem 6. Define A by

$$A = [N, N + N^{\lambda}]^n,$$

where $N \gg 1$ and $\lambda \in (-\infty, 1]$, and consider f_A defined by $\widehat{f_A}(\xi/2\pi) = e^{-i\widetilde{N}_\lambda \cdot \xi} \chi_A(\xi)$, where $\widetilde{N}_\lambda = (N^{-\lambda}, \dots, N^{-\lambda})$.

In order to show that the phase in (1) is roughly constant on A, we will need that the partial derivatives of the phase are small. More precisely, we require that

$$|x_j - N^{-\lambda} - 2t(x)\xi_j| \leq N^{-\lambda},$$

for all $j = 1, \dots, n$. Rewriting this condition, for each x we need to choose a $t(x)$ so that

$$\frac{x_j - 2N^{-\lambda}}{2\xi_j} \le t(x) \le \frac{x_j}{2\xi_j}$$

for all $\xi \in A$ and $j = 1, \dots, n$. Define a and b by

$$a(x) = \sup_{1\le j\le n} \sup_{\xi\in A} \frac{x_j - 2N^{-\lambda}}{2\xi_j} \quad \text{and} \quad b(x) = \inf_{1\le j\le n} \inf_{\xi\in A} \frac{x_j}{2\xi_j}.$$

To be able to choose the time $t(x)$ we need that $a(x) \le b(x)$. As before, we require that $x_j \ge 0$ and

$$\frac{x_j - 2N^{-\lambda}}{2N} \le \frac{x_k}{2(N + N^{\lambda})},$$

for all $j, k = 1 \dots n$. We rewrite this as

$$0 \le x_j \le 2N^{-\lambda} + \frac{N}{N + N^{\lambda}} x_k$$

for all $j, k = 1 \dots n$. Now, the set E defined by these conditions, is the convex solid body with vertices $(0, \dots, 0)$, $2(N^{1-2\lambda} + N^{-\lambda})(1, \dots, 1)$, and $2N^{-\lambda} e_j$ for all $j = 1, \dots, n$, where e_j are the standard basis vectors. Thus,

$$|E| \ge C N^{-\lambda(n-1)} N^{1-2\lambda}.$$

As $S^{**} f_A \ge C|A|$ on E, we see that

$$\|S^{**} f_A\|_{L^q(\mathbf{R}^n)} \ge C|A||E|^{1/q}.$$

As before,

$$\|f_A\|_{H^s(\mathbf{R}^n)} \le C \left(\int_A (1 + |\xi|)^{2s} \right)^{1/2} \le C|A|^{1/2} (1 + N + N^{\lambda})^s,$$

so that, as $\|S^{**} f_A\|_{L^q(\mathbf{R}^n)} \le C\|f_A\|_{H^s(\mathbf{R}^n)}$, we have

$$|A||E|^{1/q} \le C|A|^{1/2}(1 + N + N^{\lambda})^s.$$

Recalling that $|A| = N^{n\lambda}$ and $|E| \ge C N^{1-(n+1)\lambda}$, we see that

$$N^{\frac{n\lambda}{2}} N^{\frac{1-(n+1)\lambda}{q}} \le C N^s$$

for all $\lambda \in (-\infty, 1]$, so that

$$s \ge \frac{1}{q} + \lambda \left(\frac{n}{2} - \frac{n+1}{q} \right).$$

When $q < 2 + 2/n$, we let λ tend to $-\infty$ to obtain a contradiction for all s, and letting $\lambda = 1$ we recover the fact that $s \ge n(1/2 - 1/q)$. We also note for later that by letting $\lambda = 0$, we have $s \ge 1/q$.

By a well-known counterexample (see [19]), $s > n/2$ is necessary when $q = \infty$, so we have finished the proof of Theorem 8.

In order to prove results for S^*, we have the added requirement that

$$[a(x), b(x)] \cap (0,1) \neq \emptyset$$

for all $x \in E$. Now, we can ensure that $a(x) < 1$ when

$$\frac{x_j - 2N^{-\lambda}}{2N} < 1$$

for all $j = 1 \dots n$, which we rewrite as

$$x_j < 2N^{-\lambda} + 2N.$$

When $\lambda < 0$, this is an added restriction so we reanalyze the case when λ tends to negative infinity. As before, we consider the set E defined by

$$0 \leq x_j \leq 2N^{-\lambda} + \min\left\{\frac{Nx_k}{N+N^\lambda}, 2N\right\}$$

for all $j, k = 1 \dots n$. It is clear from here that

$$|E| \geq CN^{-\lambda n},$$

so that, as before,

$$N^{n\lambda/2}N^{-n\lambda/q} \leq CN^s.$$

Letting N tend to infinity, we have

$$s \geq n\lambda\left(\frac{1}{2} - \frac{1}{q}\right),$$

so that when $q < 2$, we can let λ tend to $-\infty$ to obtain a contradiction for all s.

From before we have that $s \geq n(1/2-1/q)$ and $s \geq 1/q$ are necessary conditions, and by the well-known counterexample (see [19]), $s > n/2$ is necessary when $q = \infty$, and so we are done. □

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