AHLFORS-REGULAR CURVES IN METRIC SPACES

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Abstract. We discuss 1-Ahlfors-regular connected sets in a general metric space and prove that such sets are 'flat' on most scales and in most locations. Our result is quantitative, and when combined with work of Hahlomaa, gives a characterization of 1-Ahlfors regular subsets of 1-Ahlfors-regular curves in metric spaces. Our result is a generalization to the metric space setting of the analyst's (geometric) traveling salesman theorems of Jones, Okikiolu, and David and Semmes, and it can be stated in terms of average Menger curvature.

1. Introduction

We will state our new results in subsection 1.3, but first, we will give some basic definitions and notation, as well as a description of some known results.

1.1. Basic definitions and notation

Hausdorff length. For a set K we denote by $\mathscr{H}^1(K)$ the one dimensional Hausdorff measure, which we call *Hausdorff length*.

$\lesssim$ **and** $\sim$**.** Given two functions a and b into $\mathbf{R}$ we say

$$a \lesssim b$$

with constant C, when there exists a constant $C = C_{a,b}$ such that

$$a \leq Cb.$$

We say that $a \sim b$ if we have $a \lesssim b$ and $b \lesssim a$. We will allow the constants behind the symbols $\sim$ *and* $\lesssim$ to depend on the 1-Ahlfors-regularity constant (which will be defined later) and the constant A in the definition of $\mathscr{G}^K$ (see equation (1.1)).

Balls and nets, multiresolution families. Let $\mathscr{M}$ be a metric space with metric $\mathrm{dist}(\cdot, \cdot)$. A ball B is a set

$$B = \mathrm{Ball}(x, r) := \{y : \mathrm{dist}(x, y) \leq r\}$$

for some $x \in \mathscr{M}$ and some $r > 0$. The set

$$\{y : \mathrm{dist}(x, y) \leq \lambda r\}$$

will then be denoted by λB.

We say that $X \subset K$ is an *ε-net* for K if

(i) for all $x_1, x_2 \in X$ we have $\mathrm{dist}(x_1, x_2) > \varepsilon$,
(ii) for all $y \in K$ there exists $x \in X$ such that $\mathrm{dist}(x, y) \leq \varepsilon$.

Hence $K \subset \bigcup\limits_{x \in X} \mathrm{Ball}(x, \varepsilon)$ for an ε-net X for K.

Fix a set K. Denote by X_n^K a sequence of 2^{-n}-nets for K. Set

$$(1.1) \qquad \mathscr{G}^K = \{\mathrm{Ball}(x, A2^{-n}) : x \in X_n^K, n \text{ an integer}\}$$

for a constant $A > 1$. Note that we do not assume in this essay that $X_n \subset X_{n+1}$. We call $\mathscr{G}^K$ a *multiresolution family*. Also note that $\mathscr{G}^K$ depends on K.

Remark 1.1. One of the results we quote (Theorem 1.4), for which we use this definition of $\mathscr{G}^K$, requires the additional property that $X_n \subset X_{n+1}$. To get this we may construct the sets X_n inductively, however we then require some starting point, which we denote by $n = n_0$. For Theorem 1.4 we also require n_0 to be sufficiently negative, namely we need $2^{-n_0} \geq \mathrm{diam}(K)$.

Lipschitz functions, rectifiable sets, rectifiable curves. A function $f \colon \mathbf{R}^k \to \mathscr{M}$ is said to be *C-Lipschitz* if for any $x, y \in \mathbf{R}^k$ such that $x \neq y$,

$$\frac{\mathrm{dist}(f(x), f(y))}{\|x - y\|} \leq C.$$

A function $f \colon \mathbf{R}^k \to \mathscr{M}$ is said to be *Lipschitz* if it is *C-Lipschitz* for some $C > 0$. A set is called *k-rectifiable* if it is contained in a countable union of images of Lipschitz functions $f_j \colon \mathbf{R}^k \to \mathscr{M}$, except for a set of k-dimensional Hausdorff measure zero. For more details see [Mat95], where one can also find an excellent discussion of rectifiability in the setting of $\mathbf{R}^d$, part of which carries over to other metric spaces.

A set is called a *rectifiable curve* if it is the image of a Lipschitz function defined on $\mathbf{R}$.

Geodesic metric spaces. A minimizing geodesic is a map $\tau \colon I \to M$, where $I \subset \mathbf{R}$ is an interval, and τ preserves distances. A metric space is said to be geodesic if any two points are the two endpoints of a minimizing geodesic.

Ahlfors-regularity. Given a set $K \subset \mathscr{M}$ we say that K is *k-Ahlfors-regular* if there is a constant $C > 0$ so that for all $x \in K$ and $0 < r < \mathrm{diam}(K)$ we have

$$\frac{r^k}{C} \leq \mathscr{H}^k|_K(\mathrm{Ball}(x, r)) < Cr^k.$$

We say that a connected set $\Gamma \subset \mathscr{M}$ is a 1-Ahlfors-regular curve with constant C if there is a $C > 0$ and a surjective C-Lipschitz function $\gamma \colon [0, 1] \to \Gamma$ such that for any $x \in \Gamma$ and $0 < r < \mathrm{diam}(\Gamma)$ we have

$$\mathscr{H}^1(\gamma^{-1}\mathrm{Ball}(x, r)) \leq Cr.$$

(In this case we automatically have $\frac{r}{C} \leq \mathscr{H}^1(\gamma^{-1}\mathrm{Ball}(x, r))$.) A 1-Ahlfors-regular curve is often called an Ahlfors-regular curve.

The Jones β numbers. Assume we have a set K lying in $\mathbf{R}^d$. Consider a ball B. We define the Jones β_∞ number as

$$\beta_{\infty,K}(B) \;=\; \frac{1}{\operatorname{diam}(B)} \inf_{L \text{ line}} \sup_{x \in K \cap B} \operatorname{dist}(x, L)$$

$$=\; \frac{\text{radius of thinnest cylinder containing } K \cap B}{\operatorname{diam}(B)}.$$

Hence if $\hat{K} \supset K$ then $\beta_{\infty,\hat{K}}(B) \geq \beta_{\infty,K}(B)$. Note that β_∞ is scale independent. This quantity has L^p variants. Given a locally finite measure μ and $1 \leq p < \infty$, one defines

$$\beta_{p,\mu}(B) = \frac{1}{\operatorname{diam}(B)} \inf_{L \text{ line}} \left(\int_B \operatorname{dist}(y, L)^p \frac{d\mu(y)}{\mu(B)} \right)^{1/p}.$$

Clearly

$$(1.2) \qquad\qquad \beta_{p,\mu} \leq \beta_{\infty,\operatorname{supp}(\mu)}$$

when the left hand side is defined. We define $\beta_{\infty,\mu} = \beta_{\infty,\operatorname{supp}(\mu)}$.

Menger curvature and other useful quantities. Let $x_1, x_2, x_3 \in \mathscr{M}$ be three distinct points. Take $x_1', x_2', x_3' \in \mathbf{C}$ such that $\operatorname{dist}(x_i, x_j) = |x_i' - x_j'|$ for $1 \leq i, j \leq 3$. If x_1', x_2', x_3' are collinear then define

$$c(x_1, x_2, x_3) := 0.$$

Otherwise, let R be the radius of the circle going through x_1', x_2', x_3'. In this case define

$$c(x_1, x_2, x_3) := \frac{1}{R}.$$

In any case, $c(\cdot)$ is called the *Menger curvature*.

For an ordered triple $(x_1, x_2, x_3) \in \mathscr{M}^3$ we define

$$\partial_1(x_1, x_2, x_3) := \operatorname{dist}(x_1, x_2) + \operatorname{dist}(x_2, x_3) - \operatorname{dist}(x_1, x_3).$$

Let $\{x_1, x_2, x_3\} \subset \mathscr{M}$ be an unordered triple. Assume without loss of generality $\operatorname{dist}(x_1, x_2) \leq \operatorname{dist}(x_2, x_3) \leq \operatorname{dist}(x_1, x_3)$. Define

$$\partial(\{x_1, x_2, x_3\}) := \partial_1(x_1, x_2, x_3),$$

or equivalently

$$\partial(\{x_1, x_2, x_3\}) = \min_{\sigma \in S_3} \partial_1\left(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}\right),$$

where S_3 is the permutation group on $\{1, 2, 3\}$. Hence we have for all $\{x, y, z\} \subset \mathscr{M}$

$$\partial(\{x, y, z\}) \leq \operatorname{diam}\{x, y, z\}$$

as well as

$$0 \leq \partial(\{x, y, z\}) \leq \partial_1(x, y, z) \leq 2 \operatorname{diam}\{x, y, z\}$$

where non-negativity follows from the triangle inequality.

Remark 1.2. If

$$(1.3) \qquad \operatorname{dist}(x,y) \le \operatorname{dist}(y,z) \le \operatorname{dist}(x,z) \le A \cdot \operatorname{dist}(x,y)$$

then

$$c^2(x,y,z) \operatorname{diam}\{x,y,z\}^3 \sim \partial(\{x,y,z\})$$

with constant depending only on A. Moreover, in a *Euclidean* space,

$$(1.4) \qquad \beta^2_{\infty,\{x,y,z\}}(\operatorname{Ball}(x,\operatorname{diam}\{x,y,z\})) \operatorname{diam}\{x,y,z\} \sim \partial(\{x,y,z\})$$

with constant depending only on A.

See [Hah05] for the first part of the above remark. The second part of the remark follows from the Pythagorean theorem.

We define $\beta_2(B)$ by

$$(1.5) \quad \beta_2^2(B) \operatorname{radius}(B) = \iiint\limits_{(B\cap\Gamma)^3} \partial(\{x,y,z\}) \operatorname{radius}(B)^{-3} \, d\mathscr{H}^1(z) \, d\mathscr{H}^1(y) \, d\mathscr{H}^1(x).$$

Note that $0 \le \beta_2(B) \lesssim 1$ (where the constant depends only on the 1-Ahlfors-regularity constant).

1.2. $\mathbf{R}^d$, Hilbert spaces, metric spaces. We briefly mention some results. For more details and historical background see [Dav], [Paj02], the introduction of [DS93], or the survey [Schar].

Theorem 1.3. [Jon90, Oki92, Sch] *Let H be $\mathbf{R}^d$ or an infinite dimensional Hilbert space. For any connected set Γ and any $K \subset H$ such that $K \subset \Gamma \subset H$ we have*

$$(1.6) \qquad \sum_{\mathscr{G}^K} \beta^2_{\infty,\Gamma}(B) \operatorname{diam}(B) \lesssim \mathscr{H}^1(\Gamma).$$

This was first proven for $\mathbf{R}^d$ with $d = 2$ by Jones using complex analysis, and then extended to all d by Okikiolu, using geometric methods. The constant that comes out of Okikiolu's proof depends exponentially on the dimension d, but in [Sch] it was shown that the constants do not depend on the dimension and moreover, that the theorem holds for an infinite dimensional Hilbert space. The following converse theorem gives a very good reason to care about the left hand side of inequality (1.6).

Theorem 1.4. [Jon90, Sch] *Let H be $\mathbf{R}^d$ or an infinite dimensional Hilbert space. Suppose A in the definition of $\mathscr{G}^K$ is large enough, and assume $\mathscr{G}^K$ satisfies the conditions of Remark 1.1. Given a set $K \subset H$, there exists a connected set $\Gamma_0 \supset K$ such that the length of Γ_0 satisfies*

$$(1.7) \qquad \mathscr{H}^1(\Gamma_0) \lesssim \operatorname{diam}(K) + \sum_{\mathscr{G}^K} \beta^2_{\infty,K}(B) \operatorname{diam}(B).$$

This theorem was shown by Jones for $\mathbf{R}^d$ ([Jon90]) and, with some modifications, the proof essentially carries over to the setting of an infinite dimensional Hilbert

space (see [Sch]). Theorem 1.4 also has analogues for general metric spaces (see [Hah05, Hah07]) and for Heisenberg groups (see [FFPar]).

We especially mention the following metric space generalization of Theorem 1.4 for the category of 1-Ahlfors-regular sets.

Theorem 1.5. [Hah] *Let K be a 1-Ahlfors-regular set in a complete geodesic metric space $\mathcal{M}$ with metric* $\mathrm{dist}(\cdot, \cdot)$. *Assume further that for all $z \in K$ and $R > 0$*

$$\iiint c^2(x_1, x_2, x_3) d\mathcal{H}^1|_K(x_3) \, d\mathcal{H}^1|_K(x_2) \, d\mathcal{H}^1|_K(x_1) \leq C_0 R$$

where the integral on the left hand side is over all triples $x_1, x_2, x_3 \in K \cap \mathrm{Ball}(z, R)$ such that

$$A \cdot \mathrm{dist}(x_i, x_j) \geq \mathrm{diam}\{x_1, x_2, x_3\}.$$

Then there is a 1-Ahlfors-regular connected set $\Gamma_0 \supset K$, whose constant depends only on C_0 and on the 1-Ahlfors-regularity constant of K.

The proof for this theorem is essentially contained in [Hah07]. Other results of this type and a relevant counterexample are discussed in the survey [Schar].

Before we go on, let us mention an older result which is a special case of a much bigger theorem by David and Semmes.

Theorem 1.6. [DS91] *Let $K \subset \mathbf{R}^d$ be a 1-Ahlfors-regular set and $1 \leq q \leq \infty$. Then K is contained in a connected 1-Ahlfors-regular set if and only if for all $z \in K$ and $0 < R < \mathrm{diam}(K)$*

$$(1.8) \qquad \int_0^R \int_{\mathrm{Ball}(z,R)} \beta_{q,\mathcal{H}^1|_K}(\mathrm{Ball}(x,t))^2 \, d\mathcal{H}^1|_K(x) \frac{dt}{t} \lesssim R.$$

Remark 1.7. Note that the left hand side of inequality (1.8) can be discretized as a multiresolution sum as in the left hand side of inequality (1.6).

The purpose of this paper is to prove the converse of Theorem 1.5, and thus to obtain a metric space analogue of Theorem 1.6.

1.3. New results. In Section 2 we show the following.

Theorem 1.8. *Let $\Gamma \subset \mathcal{M}$ be a connected 1-Ahlfors-regular set in a metric space. Then*

$$(1.9) \quad \iiint_{\Gamma^3} \partial(\{x, y, z\}) \mathrm{diam}\{x, y, z\}^{-3} \, d\mathcal{H}^1|_\Gamma(z) \, d\mathcal{H}^1|_\Gamma(y) \, d\mathcal{H}^1|_\Gamma(x) \lesssim \mathcal{H}^1(\Gamma).$$

The constant behind the symbol $\lesssim$ depends only on the 1-Ahlfors-regularity constant of Γ.

It follows from Theorem 1.8 that

$$(1.10) \qquad \iiint c^2(x, y, z) \lesssim \mathcal{H}^1(\Gamma)$$

where the integral is taken over triples $x, y, z \in \Gamma$ such that $\operatorname{dist}(x, y) \leq \operatorname{dist}(y, z) \leq \operatorname{dist}(x, z) \leq A \cdot \operatorname{dist}(x, y)$. The constant behind the symbol $\lesssim$ depends only on the choice of A (which can be given any value greater then 1) and the 1-Ahlfors-regularity constant of Γ.

On route we show

Theorem 1.9. *Let* $\Gamma \subset \mathscr{M}$ *be a connected 1-Ahlfors-regular set in a metric space. Let* $K \subset \Gamma$ *and let* $\hat{\mathscr{G}}^K$ *be a multiresolution family as in equation (1.1). Then we have*

$$(1.11) \quad \sum_{B \in \hat{\mathscr{G}}^K} \int_B \int_B \int_B \partial(\{x, y, z\}) \operatorname{radius}(B)^{-3} \, d\mathscr{H}^1|_\Gamma(z) \, d\mathscr{H}^1|_\Gamma(y) \, d\mathscr{H}^1|_\Gamma(x)$$
$$\lesssim \mathscr{H}^1(\Gamma).$$

The constant behind the symbol $\lesssim$ *depends only on the 1-Ahlfors-regularity constant of* Γ *and the constant* A *in the definition of* $\hat{\mathscr{G}}^K$.

In Section 3 we use these theorems to prove the following.

Theorem 1.10. *Let* $\Gamma \subset \mathscr{M}$ *be a connected 1-Ahlfors-regular set in a metric space. Let* $z \in \Gamma$ *and* $R > 0$. *Then*

$$(1.12) \quad \iiint\limits_{(\Gamma \cap \operatorname{Ball}(z, R))^3} \partial(\{x, y, z\}) \operatorname{diam}\{x, y, z\}^{-3} \, d\mathscr{H}^1|_\Gamma(z) \, d\mathscr{H}^1|_\Gamma(y) \, d\mathscr{H}^1|_\Gamma(x) \lesssim R.$$

The constant behind the symbol $\lesssim$ *depends only on the 1-Ahlfors-regularity constant of* Γ.

Theorem 1.11. *Let* $\Gamma \subset \mathscr{M}$ *be a connected 1-Ahlfors-regular set in a metric space. Let* $K \subset \Gamma$ *and let* $\hat{\mathscr{G}}^K$ *be a multiresolution family as in equation (1.1). Then we have for every* $z \in \Gamma$ *and* $R > 0$

$$(1.13) \quad \sum_{\substack{B \in \hat{\mathscr{G}}^K \\ B \subset \operatorname{Ball}(z, R)}} \iint\limits_B \int_B \partial(\{x, y, z\}) \operatorname{radius}(B)^{-3} \, d\mathscr{H}^1|_\Gamma(z) \, d\mathscr{H}^1|_\Gamma(y) \, d\mathscr{H}^1|_\Gamma(x) \lesssim R.$$

The constant behind the symbol $\lesssim$ *depends only on the 1-Ahlfors-regularity constant of* Γ *and the constant* A *in the definition of* $\hat{\mathscr{G}}^K$.

2. Proof of Theorems 1.8 and 1.9

2.1. Preliminaries, notation and definitions. Assume $\Gamma \subset \mathscr{M}$ is a connected 1-Ahlfors-regular set. If $\mathscr{H}^1(\Gamma) = \infty$ then there is nothing to prove. Hence we may assume $\mathscr{H}^1(\Gamma) < \infty$. Since the statements of the theorems are invariant under isometry, we may replace $\mathscr{M}$ by $\ell_\infty(\Gamma)$ without loss of generality. This follows from the Kuratowski embedding (see [Hei03]). Thus we may assume that $\mathscr{M}$ is complete, and that

$$\text{diam}(\text{Ball}(x, r)) \sim \text{radius}(\text{Ball}(x, r)) = r.$$

Lemma 2.1. *Assume $\Gamma \subset \mathscr{M}$ is connected. Then $\mathscr{H}^1(\Gamma) = \mathscr{H}^1(\Gamma^{closure})$.*

Lemma 2.2. *Assume $\Gamma \subset \mathscr{M}$ is a closed connected set with $\mathscr{H}^1(\Gamma) < \infty$. Then Γ is compact.*

Proofs of these lemmas can be found in the appendix of [Sch] (where they are stated for a Hilbert space, but the proofs work in the category of a complete Metric space).

We will denote by $\mathbf{T}$ the one dimensional torus $\mathbf{R}/\mathbf{Z}$.

Lemma 2.3. *Let $\Gamma \subset \mathscr{M}$ be a compact connected set of finite $\mathscr{H}^1$ measure. Then there is a Lipschitz function $\gamma\colon \mathbf{T} \to \mathscr{M}$ such that $\text{Image}(\gamma) = \Gamma$ and $\|\gamma\|_{\text{Lip}} \leq 32\mathscr{H}^1(\Gamma)$. Further, if Γ is 1-Ahlfors-regular, then*

$$(2.1) \qquad \frac{R}{C} \leq \mathscr{H}^1(\gamma^{-1}(\text{Ball}(x, R))) \leq CR \quad \forall x \in \Gamma, \ 0 < R \leq \text{diam}(\Gamma).$$

i.e. γ will be witness to the fact that Γ is an 1-Ahlfors-regular curve. Here C is a constant depending only on the 1-Ahlfors-regularity constant of the set Γ.

The proof of this lemma is a modification of a proof in the appendix of [Sch]. This modification is given in the appendix of this paper.

Fix $\gamma\colon \mathbf{T} \to \Gamma$ as assured by the above lemma. We may assume without loss of generality that γ is an arc-length parameterization (by re-parameterizing by arc-length and by globally scaling the metric so that the total arc-length is 1). This also gives us that $\text{diam}(\Gamma) \leq 1$. We will use this fixed γ throughout this essay.

Let $\tau = \gamma|_{[a,b]}$. We denote by $\ell(\tau)$ the arc-length of τ. We will also use ℓ as a measure on $\mathscr{M}$ obtained as the push-forward by γ of the Lebesgue measure on $\mathbf{T}$. By (2.1), for any integrable function f, we have that $\int f \, d\ell \sim \int f \, d\mathscr{H}^1|_\Gamma$.

As $K \subset \Gamma$ in the formulation of theorem 1.9 is fixed, we denote by $\widehat{\mathscr{G}} = \widehat{\mathscr{G}^K}$. Clearly

$$\iiint_{\Gamma^3} \partial(\{x_1, x_2, x_3\}) \, \text{diam}\{x_1, x_2, x_3\}^{-3} \, d\mathscr{H}^1(x_3) \, d\mathscr{H}^1(x_2) \, d\mathscr{H}^1(x_1)$$

$$\lesssim \sum_{B \in \widehat{\mathscr{G}}^\Gamma} \iiint_{(B \cap \Gamma)^3} \partial(\{x_1, x_2, x_3\}) \, \text{diam}(B)^{-3} \, d\mathscr{H}^1(x_3) \, d\mathscr{H}^1(x_2) \, d\mathscr{H}^1(x_1).$$

Hence Theorem 1.9 implies Theorem 1.8.

To prove Theorem 1.9 we will show

$$(2.2) \qquad \sum_{B \in \widehat{\mathscr{G}}} \iiint_{(B \cap \gamma(\mathbf{T}))^3} \partial(\{x_1, x_2, x_3\}) \operatorname{diam}(B)^{-3} \, d\ell(x_3) \, d\ell(x_2) \, d\ell(x_1) \lesssim \ell(\gamma),$$

or equivalently,

$$\sum_{B \in \widehat{\mathscr{G}}} \beta_2^2(B) \operatorname{diam}(B) \lesssim \mathscr{H}^1(\Gamma).$$

Remark 2.4. We may consider the isometric embedding e

$$\mathscr{M} = \ell_\infty(\Gamma) \xrightarrow{e} \ell_\infty(\Gamma) \times \{(0,0)\} \subset \ell_\infty(\Gamma) \times [-1,1]^2$$

and obtain a sequence of maps $\gamma_n \colon \mathbf{T} \to \ell_\infty(\Gamma) \times [-1,1]^2$ such that γ_n is one-to-one, $\gamma_n \to \gamma$ uniformly, $\|\gamma_n\|_{\mathrm{Lip}} \le (1 + 2^{-n}) \|\gamma\|_{\mathrm{Lip}}$, and γ_n gives a 1-Ahlfors-regular curve with constant uniformly comparable to that of γ. This means that we may assume without loss of generality that γ in inequality (2.2) is one-to-one. This will be useful for the proof of Lemma 2.11.

We define

$$(2.3) \qquad\qquad \mathscr{G} = \{B \in \widehat{\mathscr{G}} : \mathscr{H}^1|_\Gamma(4B) < \tfrac{1}{6}\ell(\Gamma)\}.$$

We first consider $\widehat{\mathscr{G}} \smallsetminus \mathscr{G}$.

Lemma 2.5. $\displaystyle\sum_{B \in \widehat{\mathscr{G}} \smallsetminus \mathscr{G}} \beta_2^2(B) \operatorname{diam}(B) \lesssim \ell(\gamma)$.

Proof. Set $L = \ell(\gamma)$.

Consider balls $B \in \widehat{\mathscr{G}}$ with $\mathscr{H}^1(4B) \ge \frac{L}{6}$ and $\operatorname{radius}(B) \le AL$. There are at most C such balls at each scale, and at most C' scales. The constants C, C' depend only on the Ahlfors regularity constant of Γ and the constant A.

Consider now balls $B \in \widehat{\mathscr{G}}$ with $\operatorname{radius}(B) > AL$. There is at most one ball B of each scale, and

$$\beta_2^2(B) \operatorname{diam}(B) \lesssim \operatorname{diam}(B)^{-3} L^3 \operatorname{diam}(B) \sim L \frac{L^2}{\operatorname{diam}(B)^2}.$$

Summing over all scales we get

$$\sum_{B \in \widehat{\mathscr{G}} \smallsetminus \mathscr{G}} \beta_2^2(B) \operatorname{diam}(B) \lesssim L. \qquad \square$$

We need some more notation. Let $E \subset \mathscr{M}$ be a closed set such that $\Gamma \cap (\mathscr{M} \smallsetminus E) \ne \emptyset$. We define

$$\Lambda(E) := \{\tau = \gamma|_{[a,b]} : [a,b] \subset \mathbf{T}; [a,b] \text{ a connected component of } \gamma^{-1}(\Gamma \cap E)\}.$$

We will freely use $\tau \in \Lambda(E)$ as both a parameterization of an arc (given by restriction of γ), and its image. In particular, we will denote by $\operatorname{diam}(\tau)$ the diameter of the image of τ.

Let $B \in \mathscr{G}$ be a ball. For $\tau \in \Lambda(B)$ we denote by τ^i the extension of τ to an arc in $\Lambda(2^i B)$. We set

$$(2.4) \qquad\qquad \Lambda^i(B) := \{\tau^i : \tau \in \Lambda(B)\}.$$

We will only use $i \in \{0, 1, 2\}$.

Let $\tau \colon [a, b] \to \Gamma$ be a sub-arc of γ (and hence an arc-length parameterization). We define the quantity $\tilde{\beta}(\tau)$ by

$$\tilde{\beta}^2(\tau) \operatorname{diam}(\tau) := \ell(\tau)^{-3} \int\limits_a^b \int\limits_x^b \int\limits_y^b \partial_1(\gamma(x), \gamma(y), \gamma(z)) dz dy dx.$$

(This is how we define the Jones β number of an arc).

The constant ε_2 below will be set in Section 2.2 and will depend on the 1-Ahlfors-regularity constant. Consider $\tau \in \Lambda^2(B)$. We call τ *almost flat* iff

$$\tilde{\beta}(\tau) \leq \varepsilon_2 \beta_2(B).$$

We denote the collection of *almost flat* arcs in $\Lambda^2(B)$ by

$$S_B := \{\tau \in \Lambda^2(B) : \tilde{\beta}(\tau) \leq \varepsilon_2 \beta_2(B)\}.$$

Set

$$\mathscr{G}_2 := \{B \in \mathscr{G} : \Lambda^2(B) \subset S_B\},$$
$$\mathscr{G}_1 := \mathscr{G} \setminus \mathscr{G}_2.$$

We note that $B \in \mathscr{G}_1$ implies the existence of an arc $\tau_B \in \Lambda^2(B)$ with $\tau_B \notin S_B$. We will make use of this special (possibly non-unique) arc later on.

We will have Theorem 1.9 if we prove

$$(2.5) \qquad\qquad \sum_{B \in \mathscr{G}_i} \beta_2^2(B) \operatorname{diam}(B) \lesssim \ell(\Gamma)$$

for $i \in \{1, 2\}$. We prove inequality (2.5) for $i = 1$ in Section 2.2, and for $i = 2$ in Section 2.3.

2.2. Non-flat arcs. In this subsection we prove inequality (2.5) for $i = 1$.

We have $\gamma \colon \mathbf{T} \to \Gamma$. Identify $\mathbf{T}$ with $[0, 1]$ for the purpose of defining $\mathscr{D}^0$—a dyadic decomposition of $\mathbf{T}$ given by the standard dyadic decomposition of $[0, 1]$. We also define $\mathscr{D}^1$—the dyadic decomposition of $\mathbf{T}$ corresponding to the rotation of $\mathbf{T}$ by $\frac{1}{3}$ of a full rotation, i.e. $x \to (x + \frac{1}{3}) \mod 1$, and then using the standard dyadic decomposition of $[0, 1]$. The reason for these two filtrations is the following remark, which earns this (standard) idea the name 'the one third trick'.

Remark 2.6. Given a (possibly non-dyadic) interval $J \subset \mathbf{T}$ such that $\operatorname{diam}(J) < \frac{1}{6}$ there exits an interval I such that $I \in \mathscr{D}^0 \cup \mathscr{D}^1$, with $J \subset I$ and $\operatorname{diam}(I) \leq 6 \operatorname{diam}(J)$.

We also define the arcs (mappings) $\gamma^0 \colon [0,1] \to \Gamma$ and $\gamma^1 \colon [0,1] \to \Gamma$ using the above identifications of $[0,1]$ with $\mathbf{T}$. They should be thought of as two ways of *cutting* γ at a point. We define $\gamma^i(x,y,z) := (\gamma^i(x), \gamma^i(y), \gamma^i(z))$.

Let $B \in \mathscr{G}_1$. Let $\tau = \tau_B \notin S_B$. Let I be a dyadic interval (assured by remark 2.6) such that $\gamma^i(I) \supset \tau$ and $\operatorname{diam}(I) \leq 6\ell(\tau) \lesssim \operatorname{diam}(\tau)$, where $i = i(\tau)$ is one of 0 or 1. Note that the mapping $\tau \to I$ is at most K_1-to-1 for some constant K_1 depending only on the 1-Ahlfors-regularity constant of Γ and the constant A in equation (1.1). Assume that we have $i(\tau) = 0$. For numbers $r, v \in [0,1]$ we will look at the mapping $\psi^{v,r} \colon [0,1] \to [0,1]$ given by $\psi^{v,r}(t) = v + rt \mod 1$. Note that there are exactly 2^k choices of $\tilde{v}$ and corresponding $\tilde{I} \in \mathscr{D}^0$ (of size 2^{-k}) with $\psi^{\tilde{v},r}(\tilde{I}) = [v, v + 2^{-k}r]$.

For an interval $I \subset [0,1]$ write $I = [a(I), b(I)]$.

Remark 2.7. When doing addition $\mod 1$, we have (by change of variable) for any I' with $\operatorname{diam}(I') = 2^{-k}$

$$\sum_{\substack{I \in \mathscr{D}^0 \\ \operatorname{diam}(I)=2^{-k}}} \operatorname{diam}(I)^{-3} \int_{a(I)}^{b(I)} \int_{x}^{b(I)} \int_{y}^{b(I)} \partial_1 \circ \gamma^0(x,y,z)\, dz\, dy\, dx$$

$$\leq \operatorname{diam}(I')^{-3} \int_{v=0}^{1} \int_{r=0}^{1} \int_{y \in v+rI'} \partial_1 \circ \gamma^0(v+ra(I'), y, v+rb(I'))\, dy \cdot \operatorname{diam}(I')\, dr\, dv$$

giving

$$\sum_{\substack{I \in \mathscr{D}^0 \\ \operatorname{diam}(I)=2^{-k}}} \operatorname{diam}(I)^{-3} \int_{a(I)}^{b(I)} \int_{x}^{b(I)} \int_{y}^{b(I)} \partial_1 \circ \gamma^0(x,y,z)\, dz\, dy\, dx$$

$$\leq \sum_{\substack{I \in \mathscr{D}^0 \\ \operatorname{diam}(I)=2^{-k}}} \operatorname{diam}(I)^{-3}$$

$$\cdot \int_{v=0}^{1} \int_{r=0}^{1} \int_{y \in v+rI} \partial_1 \circ \gamma^0(v+ra(I), y, v+rb(I))\, dy \cdot \operatorname{diam}(I)\, dr\; \operatorname{diam}(I)\, dv.$$

Let $I' = [a,b] \in \mathscr{D}^0$. Define

$$(2.6) \qquad \partial_d(\gamma^0 \psi^{v,r}(I')) := \partial_1(\gamma^0(v+ra), \gamma^0(v+r\tfrac{a+b}{2}), \gamma^0(v+rb)).$$

Lemma 2.8. *Let $I \in \mathscr{D}^0$. Let $v, r \in [0,1]$ be chosen such that $\psi^{v,r}(I) = [x,z] \ni y$. Then*

$$\partial_1 \circ \gamma^0(x,y,z) \leq \sum_{\substack{I' \in \mathscr{D}, I' \subset I \\ y \in \psi^{v,r}(I')}} \partial_d(\gamma^0 \psi^{v,r}(I')).$$

Proof. This is just the triangle inequality reiterated. $\qquad\square$

Lemma 2.9. *Let $r, v \in [0, 1]$ be fixed. Then*

$$\sum_{I' \in \mathscr{D}^0} \partial_d(\gamma^0 \psi^{v,r}(I')) \lesssim \mathscr{H}^1(\Gamma)$$

Proof. We have that $v + r\{I' \in \mathscr{D}^0\}$ is a dyadic filtration contained in **T**. The sum in the statement of the lemma is therefore a sum of a telescoping series, whose partial sums are bounded by the arc-length of γ. $\qquad\square$

Now,

$$\sum_{I \in \mathscr{D}^0} \operatorname{diam}(I)^{-3} \int\limits_{a(I)}^{b(I)} \int\limits_x^{b(I)} \int\limits_y^{b(I)} \partial_1 \circ \gamma^0(x, y, z) dz\, dy\, dx$$

$$\leq \sum_{I \in \mathscr{D}^0} \operatorname{diam}(I)^{-3} \int\limits_{v=0}^1 \int\limits_{r=0}^1 \int\limits_{y \in v+rI} \partial_1 \circ \gamma^0(v + ra(I), y, v + rb(I))\, dy$$
$$\cdot \operatorname{diam}(I)\, dr \cdot \operatorname{diam}(I)\, dv$$

$$\leq \sum_{I \in \mathscr{D}^0} \operatorname{diam}(I)^{-3} \int\limits_{v=0}^1 \int\limits_{r=0}^1 \sum_{\substack{I' \in \mathscr{D}^0 \\ I' \subset I}} \int\limits_{y \in v+rI'} \partial_d(\gamma^0 \psi^{v,r}(I')) \cdot dy \cdot \operatorname{diam}(I)\, dr \cdot \operatorname{diam}(I)\, dv$$

$$= \sum_{I \in \mathscr{D}^0} \operatorname{diam}(I)^{-3} \int\limits_{v=0}^1 \int\limits_{r=0}^1 \sum_{\substack{I' \in \mathscr{D}^0 \\ I' \subset I}} \partial_d(\gamma^0 \psi^{v,r}(I')) \cdot r\mathscr{H}^1(I') \cdot \operatorname{diam}(I)\, dr \cdot \operatorname{diam}(I)\, dv$$

$$= \int\limits_{v=0}^1 \int\limits_{r=0}^1 \sum_{I \in \mathscr{D}^0} \frac{1}{\operatorname{diam}(I)} \sum_{\substack{I' \in \mathscr{D}^0 \\ I' \subset I}} \partial_d(\gamma^0 \psi^{v,r}(I')) \cdot r\mathscr{H}^1(I')\, dr\, dv$$

$$= \int\limits_{v=0}^1 \int\limits_{r=0}^1 \sum_{I' \in \mathscr{D}^0} \sum_{I \supset I'} \frac{\mathscr{H}^1(I')}{\operatorname{diam}(I)} \partial_d(\gamma^0 \psi^{v,r}(I')) \cdot r\, dr\, dv$$

$$\lesssim \int\limits_{v=0}^1 \int\limits_{r=0}^1 \sum_{I' \in \mathscr{D}^0} \partial_d(\gamma^0 \psi^{v,r}(I')) \cdot r\, dr\, dv$$
$$\lesssim \ell(\Gamma).$$

Similarly,

$$\sum_{I \in \mathscr{D}^1} \operatorname{diam}(I)^{-3} \int\limits_{a(I)}^{b(I)} \int\limits_x^{b(I)} \int\limits_y^{b(I)} \partial_1 \circ \gamma^1(x, y, z)\, dz\, dy\, dx \lesssim \ell(\Gamma)$$

Hence

$$\sum_{\substack{\tau_B \\ B \in \mathscr{G}_1}} \tilde{\beta}^2(\tau_B) \operatorname{diam}(\tau_B) \lesssim \ell(\Gamma).$$

Lemma 2.10. *We have inequality (2.5) for $i = 1$.*

Proof.

$$(2.7) \qquad \sum_{B \in \mathscr{G}_1} \beta(B)^2 \operatorname{diam}(B) \lesssim \sum_{B \in \mathscr{G}_1} \tilde{\beta}(\tau_B)^2 \operatorname{diam}(\tau_B) \lesssim \ell(\Gamma). \qquad \square$$

2.3. Almost flat arcs. In this subsection we prove inequality (2.5) for $i = 2$.

This subsection will have two parts. We first show that for every ball $B \in \mathscr{G}_2$ there exist two special arcs, $\eta_1(B) \in \Lambda^1(B)$ and $\eta_2(B) \in \Lambda^2(B)$. These arcs will have properties useful for the second part of this subsection, where we construct a bounded weight which will in turn give us the desired result.

Part I

Lemma 2.11. *Let $B \in \mathscr{G}_2$. Let $\xi \in \Lambda^2(B)$. If for every arc $\tau_i \in \Lambda^1(B)$ we have*

$$(2.8) \qquad \ell(\tau_i)^{-1} \int_{\tau_i} \operatorname{dist}(\cdot, \xi) \le \varepsilon_4 \beta_2^2(B) \operatorname{diam}(B)$$

then for every triple of arcs $\tau_i, \tau_j, \tau_k \in \Lambda^1(B)$ we have

$$\operatorname{diam}(B)^{-3} \int_{\tau_i} \int_{\tau_j} \int_{\tau_k} \partial(\{x, y, z\}) \, d\ell(z) \, d\ell(y) \, d\ell(x) \le C_2(\varepsilon_2^2 + \varepsilon_4) \beta_2^2(B) \operatorname{diam}(B)$$

where C_2 is a constant which depends only on the 1-Ahlfors-regularity constant of Γ.

Proof. Let $(\gamma(x_1), \gamma(x_2), \gamma(x_3)) \in \Gamma^3$ be an ordered triple. Let S_3 be the permutation group on $\{1, 2, 3\}$. We define for $\sigma \in S_3$

$$\partial_\sigma(\gamma(x_1), \gamma(x_2), \gamma(x_3)) := \partial_1(\gamma(x_{\sigma(1)}), \gamma(x_{\sigma(2)}), \gamma(x_{\sigma(3)})).$$

We will let σ depend on a triple $\bar{x} = (x_1, x_2, x_3)$ and we will denote this by $\sigma_{\bar{x}}$.

Recall that $\partial(\{\cdot\})$ is a continuous function. We denote by $\mathbf{D}_{\tau, n}$ the collection of 2^n points in the domain of τ, evenly spaced according to arc-length. Let $N_0 = N_0(B)$ be chosen large enough so that for all $\tau_i, \tau_j, \tau_k \in (\Lambda^1(B) \cup \{\xi\})$ (possibly non-different) and $n_1, n_2, n_3 \ge N_0$

$$(2.9) \qquad \begin{aligned} \operatorname{diam}(B)^{-3} &\int_{\tau_i} \int_{\tau_j} \int_{\tau_k} \partial(\{x, y, z\}) \, d\ell(z) \, d\ell(y) \, d\ell(x) \\ &\sim 2^{-n_1 - n_2 - n_3} \sum_{x \in \mathbf{D}_{\tau_i, n_1}} \sum_{y \in \mathbf{D}_{\tau_j, n_2}} \sum_{z \in \mathbf{D}_{\tau_k, n_3}} \partial(\{\gamma(x), \gamma(y), \gamma(z)\}), \end{aligned}$$

and for all $n \geq N_0$

$$(2.10) \qquad \ell(\tau_i)^{-1} \int_{\tau_i} \mathrm{dist}(\cdot, \xi) \sim 2^{-n} \sum_{x \in \mathbf{D}_{\tau_i, n}} \mathrm{dist}(\gamma(x), \xi)$$

Let $\tau_1, \tau_2, \tau_3 \in \Lambda^1(B)$. Write

$$\mathbf{D}_{\tau_1, N_0} = \{O_1, O_2, \ldots\},$$

where

$$\mathrm{dist}(\gamma(O_i), \xi) \leq \mathrm{dist}(\gamma(O_{i+1}), \xi).$$

Now let us assume for a moment that $\mathrm{dist}(\tau_1, \xi) > 0$. Let N_1 be chosen such that

$$2^{-N_1} < \mathrm{dist}(\gamma(O_1), \xi).$$

Take $N = \max\{N_1, N_0\}$. We define a function f with domain $\mathbf{D}_{\tau_1, N_0}$ taking values of probability measures on $\mathbf{D}_{\xi, N}$ as follows. We go over the O_i's as ordered by i. Let F_i be the set

$$F_i = \{x' \in \mathbf{D}_{\xi, N} : \mathrm{dist}(\gamma(x'), \gamma(O_i)) \leq 2\,\mathrm{dist}(\gamma(O_i), \xi)\},$$

which is non-empty by our choice of N_1. Define $f(O_1)$ as the uniform probability measure on F_1. Given $f(O_1), \ldots, f(O_{k-1})$, define $f(O_k)$ as the probability measure on F_k, so that the measure

$$(2.11) \qquad \sum_{i \leq k} f(O_i)|_{F_k}$$

is as close as possible (in sup norm!) to $2^N k$ times the uniform distribution on F_k (this is our way of ensuring that (2.11) is as uniform as possible). We have for all $x \in \mathbf{D}_{\tau_1, N_0}$ and $x' \in \mathrm{supp}(f(x))$,

$$\mathrm{dist}(\gamma(x), \gamma(x')) \leq 2\,\mathrm{dist}(\gamma(x), \xi).$$

We also have for any $x' \in \mathbf{D}_{\xi, N}$

$$(2.12) \qquad 2^{-N_0} \sum_{x \in \mathbf{D}_{\tau_1, N_0}} f(x)\{x'\} \leq C 2^{-N}$$

where C is a constant which depends only on the 1-Ahlfors-regularity constant of Γ. To see inequality (2.12), assume the contrary. Let O_k be the last element such that $f(O_k)\{x'\}$ was positive. Then by construction of $f(O_k)$, we have that for all $x'' \in F_k$

$$\sum_{i \leq k} f(O_i)\{x''\} \geq \sum_{i \leq k} f(O_i)\{x'\} \geq C 2^{-N+N_0}.$$

Summing over F_k we get a total mass of

$$\sum_{x'' \in F_k} \sum_{i \leq k} f(O_i)\{x''\} \geq \sharp F_k \cdot C 2^{-N+N_0} \geq C 2^{N_0} \frac{\mathrm{dist}(\gamma(O_k), \xi)}{\ell(\xi)}.$$

All this mass, however, came from O_i's such that

$$\text{dist}(\gamma(O_i), \gamma(O_k)) \leq 2\,\text{dist}(\gamma(O_i), \xi) + \text{diam}(F_k) + 2\,\text{dist}(\gamma(O_k), \xi)$$
$$\leq 10\,\text{dist}(\gamma(O_k), \xi)$$

and so by enlarging C we get a contradiction to 1-Ahlfors-regularity. This gives inequality (2.12).

We similarly define f on $\mathbf{D}_{\tau_2, N_0}$ and $\mathbf{D}_{\tau_3, N_0}$. Now,

$$\text{diam}(B)^{-3} \int_{\tau_1} \int_{\tau_2} \int_{\tau_3} \partial(\{x, y, z\})\, d\ell(z)\, d\ell(y)\, d\ell(x)$$

$$\sim 2^{-N_0 - N_0 - N_0} \sum_{x \in \mathbf{D}_{\tau_1, N_0}} \sum_{y \in \mathbf{D}_{\tau_2, N_0}} \sum_{z \in \mathbf{D}_{\tau_3, N_0}} \partial(\{\gamma(x), \gamma(y), \gamma(z)\})$$

$$\lesssim 2^{-N_0}\Big(\sum_{x \in \mathbf{D}_{\tau_1, N_0}} \text{dist}(\gamma(x), \xi) + \sum_{y \in \mathbf{D}_{\tau_2, N_0}} \text{dist}(\gamma(y), \xi) + \sum_{z \in \mathbf{D}_{\tau_3, N_0}} \text{dist}(\gamma(z), \xi)\Big)$$

$$+ 2^{-N_0 - N_0 - N_0} \sum_{x \in \mathbf{D}_{\tau_1, N_0}} \sum_{y \in \mathbf{D}_{\tau_2, N_0}} \sum_{z \in \mathbf{D}_{\tau_3, N_0}} \sum_{x' \in \text{supp} f(x)} \sum_{y' \in \text{supp} f(y)}$$

$$\sum_{z' \in \text{supp} f(z)} f(x)\{x'\} \cdot (f(y)\{y'\} \cdot f(z)\{z'\} \cdot \partial(\{\gamma(x'), \gamma(y'), \gamma(z')\})$$

$$\lesssim 2^{-N_0}\Big(\sum_{x \in \mathbf{D}_{\tau_1, N_0}} \text{dist}(\gamma(x), \xi) + \sum_{y \in \mathbf{D}_{\tau_2, N_0}} \text{dist}(\gamma(y), \xi) + \sum_{z \in \mathbf{D}_{\tau_3, N_0}} \text{dist}(\gamma(z), \xi)\Big)$$

$$+ C^3 2^{-N - N - N} \sum_{x' \in \mathbf{D}_{\xi, N}} \sum_{y' \in \mathbf{D}_{\xi, N}} \sum_{z' \in \mathbf{D}_{\xi, N}} \partial_{\sigma_{(x', y', z')}}(\gamma(x'), \gamma(y'), \gamma(z')).$$

We have yet to specify the function σ and have total freedom in choosing its values in S_3. Choose $\sigma_{(x', y', z')}$ such that $\sigma_{(x', y', z')}(x', y', z')$ has increasing order when ordered by ξ. From inequalities (2.8), (2.9), and (2.10) we now get the lemma.

The case $\text{dist}(\tau_1, \xi) = 0$ can either be assumed not to happen by using remark 2.4 or by computing the above integrals (sums) as limits of the corresponding integrals (sums) in the loops γ_n from Remark 2.4. $\qquad\square$

Let $\xi_2(B) \in \Lambda^2(B)$ be an arc containing the center of B. We upper bound the size of ε_2 and fix ε_4 in the proof of the following lemma.

Lemma 2.12. *Let $B \in \mathscr{G}_2$. We have an arc $\xi_1(B) \in \Lambda^1(B)$ such that*

$$\bar{d}(B) := \ell(\xi_1(B))^{-1} \int_{\xi_1(B)} \text{dist}(\cdot, \xi_2(B)) \geq \varepsilon_4 \beta_2^2(B)\,\text{diam}(B)$$

Proof. If the contrary is true then by reducing ε_4 and ε_2 we get a contradiction from the previous lemma and Ahlfors-regularity (the latter bounds the number of triples). $\qquad\square$

We define $\hat{\beta}(B)$ by

$$\hat{\beta}^2(B)\ell(\xi_1(B)) := \bar{d}(B) = \ell(\xi_1(B))^{-1}\int_{\xi_1(B)}\mathrm{dist}(\cdot,\xi_2(B))$$

Remark 2.13.

$$1 \geq \hat{\beta}(B) \gtrsim \sqrt{\varepsilon_4}\beta_2(B),$$

with constant depending only on the 1-Ahlfors-regularity constant of Γ.

Part II

Lemma 2.14. *Let $R > 0$ be given. There is a $P_1 = P_1(R)$ such that one can write a disjoint union*

$$\mathscr{G} = \mathscr{G}^1 \cup \cdots \cup \mathscr{G}^{P_1}$$

where for each $1 \leq p_1 \leq P_1$ and $B_1, B_2 \in \mathscr{G}^{p_1}$ with $\mathrm{radius}(B_1) = \mathrm{radius}(B_2)$, we have

$$\mathrm{dist}(B_1, B_2) \geq R \cdot \mathrm{radius}(B_1).$$

Proof. By 1-Ahlfors-regularity we have for each $B_0 \in \mathscr{G}$

$$\sharp\{B \in \mathscr{G} : (R+1) \cdot B \cap (R+1) \cdot B_0 \neq \emptyset, \ \mathrm{radius}(B) = \mathrm{radius}(B_0)\} \leq C_1$$

where C_1 is some constant depending only on the 1-Ahlfors-regularity constant and the choice of A and R. We create the desired disjoint union by going over the balls in order. We set $P_1 = C_1$. By the pigeon-hole principle a ball B can be placed in at least one collection $\mathscr{G}^{p_1}$ such that the result of the lemma will not be contradicted. $\qquad\square$

The choice of R will be a consequence of Lemma 2.16. Fix $1 \leq p_1 \leq P_1(R)$. Let $M > 0$ be any positive integer. Consider $\Delta_M^{p_1} \subset \mathscr{G}_2 \cap \mathscr{G}^{p_1}$ defined by

$$\Delta_M^{p_1} := \{B \in \mathscr{G}_2 \cap \mathscr{G}^{p_1} : 2^{-M} \leq \frac{1}{2}\hat{\beta}^2(B) < 2^{-M+1}\}.$$

Write $\Delta_M^{p_1} = \Delta_M^{p_1,1} \cup \ldots \cup \Delta_M^{p_1,KM}$ where

$$\Delta_M^{p_1,p_2} := \{B \in \Delta_M^{p_1} : \mathrm{radius}(B) = A2^{-nKM+p_2}, n \in \mathbf{Z}\}, \quad 1 \leq p_2 \leq KM.$$

Fix $M > 0$ and $1 \leq p_2 \leq KM$ (K will be fixed later). Fix $\Delta \subset \Delta_M^{p_1,p_2}$ a finite subset. Take $B \in \Delta$.

We define $Q(B) \subset (1 + 4 \cdot 2^{-KM})2B$ as follows. Set

$$U_0 := 2B$$

$$U_{n+1} := U_n \cup \bigcup\{2B' : B' \in \Delta, \ 2B' \cap U_n \neq \emptyset, \ \mathrm{radius}(B) \geq \mathrm{radius}(B')\}$$

$$Q(B) := \bigcup_n U_n.$$

Proposition 2.15. $Q(B) \subset (1 + 4 \cdot 2^{-KM})2B$.

We first consider the following lemma.

Lemma 2.16. *Consider any metric space. Assume $R = R > 0$ is sufficiently large. Let $0 < \delta < \frac{1}{3}$. Let $\{B_i\}_1^n$ be a sequence of balls in this metric space so that $B_k \cap B_{k+1} \neq \emptyset$, $\mathrm{radius}(B_i) \in \{\delta^k : k \text{ integer}\}$, and $\mathrm{dist}(B_{i_1}, B_{i_2}) \geq R \cdot d$ for balls B_{i_1}, B_{i_2} satisfying $\mathrm{radius}(B_{i_1}) = \mathrm{radius}(B_{i_2}) = d$. Then for any $x \in \cup B_i$,*

$$\mathrm{dist}(x, \mathrm{center}(B_{k_0})) \leq \mathrm{radius}(B_{k_0})(1 + 2\delta + \ldots + 2^k \delta^k + \ldots),$$

where k_0 is chosen so that

$$\mathrm{radius}(B_{k_0}) = \max_{i=1}^n \mathrm{radius}(B_i).$$

Proof. This follows by induction on n.

– For $n = 1$ this is clear as $\mathrm{dist}(x, \mathrm{center}(B_1)) \leq \mathrm{radius}(B_1)$.

– For $n = N + 1$: Consider the sequence $B_1, B_2, \ldots, B_{k_0-1}$. Let k_1 be so that $\mathrm{radius}(B_{k_1}) = \max_{i=1}^{k_0-1} \mathrm{radius}(B_i)$. By induction, for any $y \in \cup_{i=1}^{k_0-1} B_i$

$$\mathrm{dist}(y, \mathrm{center}(B_{k_1})) \leq \mathrm{radius}(B_{k_1})(1 + 2\delta + \ldots + 2^k \delta^k + \ldots).$$

Hence, if R is large enough, $\mathrm{radius}(B_{k_1}) \neq \mathrm{radius}(B_{k_0})$, and hence $\mathrm{radius}(B_{k_1}) < \mathrm{radius}(B_{k_0})$ which gives

$$\mathrm{dist}(y, \mathrm{center}(B_{k_1})) \leq \delta \, \mathrm{radius}(B_{k_0})(1 + 2\delta + \ldots + 2^k \delta^k + \ldots).$$

Similarly for the sequence $B_{k_0+1}, B_{k_0+2}, \ldots, B_n$. We conclude

$$\begin{aligned}
\mathrm{dist}(x, \mathrm{center}(B_{k_0})) &\leq \mathrm{radius}(B_{k_0}) + 2\delta \, \mathrm{radius}(B_{k_0})(1 + 2\delta + \ldots + 2^k \delta^k + \ldots) \\
&= \mathrm{radius}(B_{k_0})(1 + 2\delta(1 + 2\delta + \ldots + 2^k \delta^k + \ldots)) \\
&= \mathrm{radius}(B_{k_0})(1 + 2\delta + \ldots + 2^k \delta^k + \ldots).
\end{aligned}$$

This concludes the induction. $\qquad\square$

We now prove Proposition 2.15.

Proof. Recall that the number of balls in Δ is finite. We denote by $\delta = 2^{-MK}$. If $K > 2$ we have $\delta < \frac{1}{3}$. Let $x \in Q$. Then there is a sequnce of balls $\{B_i\}_1^n$ such that $x \in B_n$, $\frac{1}{2}B_i \in \Delta$, $B_k \cap B_{k+1} \neq \emptyset$ with $\mathrm{radius}(B_i) \leq \mathrm{radius}(2B)$ and $B_1 = 2B$. Using Lemma 2.16 we get the desired estimate as

$$(1 + 2\delta + \ldots + 2^k \delta^k + \ldots) \leq 1 + 4 \cdot 2^{-KM}. \qquad \square$$

The family $\{Q(B) : B \in \Delta\}$ has the property that if Q_1 and Q_2 are in it, then if $Q_1 \cap Q_2 \neq \emptyset$ we have $Q_1 \subset Q_2$ or $Q_2 \subset Q_1$.

We write

$$(2.13) \qquad\qquad Q = \left(\bigcup_i Q^i \right) \cup R_Q$$

where Q^i is maximal such that $Q^i = Q(B^i)$, $B^i \in \Delta$ and $Q^i \subsetneq Q$. We choose R_Q so that all the unions in equation (2.13) are disjoint.

Let $B \in \mathscr{G}$ be a ball. for $\tau \in \Lambda(B)$ we denote by τ^Q the extension of τ to an arc in $\Lambda(Q(B))$. We set

$$(2.14) \qquad \Lambda^Q(B) := \{\tau^Q : \tau \in \Lambda(B)\}.$$

Remark 2.17. We have (using regularity) that if $B \in \mathscr{G}_2$ then for all $\tau \in \Lambda^Q(B)$

$$\tilde{\beta}(\tau) \lesssim \varepsilon_2 \beta_2(B) \lesssim \varepsilon_2 \sqrt{\varepsilon_4^{-1}} \hat{\beta}(B).$$

We also denote by $\xi_2(Q)$ a connected component of $\xi_2(B) \cap Q$ which contains the center of B. We will denote by $J_1(Q)$ and $J_2(Q)$ the index sets

$$J_1(Q) = \{i : Q^i \cap \xi_1(B) \neq \emptyset\}$$
$$J_2(Q) = \{i : Q^i \cap \xi_2(Q) \neq \emptyset\}.$$

Remark 2.18. By enlarging K if necessary, if $x \in \xi_1(B)$ such that $\mathrm{dist}(x, \xi_2(Q)) \geq \frac{1}{4} 2^{-M} \mathrm{diam}(2B)$ and $x \in Q^j$, then $j \in J_1 \setminus J_2$.

Proposition 2.19. *Let $B \in \Delta$ and $Q = Q(B)$. Then*

$$\ell(R_Q) + \sum_j \mathrm{diam}(Q^j) \geq \ell(R_Q) + \sum_{j \in J_1 \cup J_2} \mathrm{diam}(Q^j) \geq (1 + c'' \hat{\beta}(B)) \mathrm{diam}(Q)$$

for some constant $c'' > 0$ depending only on the 1-Ahlfors-regularity of Γ.

Before we can prove this proposition we need two lemmas.

Lemma 2.20. *There is a constant $c > 0$, independent of ε_2, K, and M, so that for any $Q = Q(B)$ and $\xi_1 = \xi_1(B)$, $\xi_2 = \xi_2(Q)$ we have*

$$\ell(R_Q \cap \xi_1) + \sum_{j \in J_1 \setminus J_2} \mathrm{diam}(Q^j) \geq c 2^{\frac{-M}{2}} \mathrm{diam}(Q).$$

Proof. Let $\bar{d} = \bar{d}(B)$. Assume for a moment

$$(2.15) \qquad \ell(\{x \in \xi_1 : \mathrm{dist}(x, \xi_2(B)) \geq \frac{\bar{d}}{2}\}) \leq \hat{\beta}(B)\ell(\xi_1).$$

Then

$$(1 - \hat{\beta}(B))\ell(\xi_1)\frac{\bar{d}}{2} + \hat{\beta}(B)\ell(\xi_1)d_\infty \geq \bar{d}\ell(\xi_1),$$

where

$$d_\infty = \sup_{x \in \xi_1} \mathrm{dist}(x, \xi_2(B)).$$

Hence

$$d_\infty \hat{\beta}(B) \geq \bar{d} - (1 - \hat{\beta}(B))\frac{\bar{d}}{2} = \frac{\bar{d}}{2} + \hat{\beta}(B)\frac{\bar{d}}{2}$$

or

$$d_\infty \geq \hat{\beta}(B)^{-1}\frac{\bar{d}}{2} + \frac{\bar{d}}{2}$$

and hence (since ξ_1 is connected and we are assuming (2.15)), the diameter of the largest-diameter connected component of

$$\{x \in \xi_1 : \operatorname{dist}(x, \xi_2(B)) \geq \frac{\bar{d}}{2}\}$$

is at least

$$\hat{\beta}(B)^{-1} \frac{\bar{d}}{2} = \frac{1}{2}\hat{\beta}(B)\ell(\xi_1).$$

Either way (with or without assumption (2.15)) we have

$$\ell(\{x \in \xi_1 : \operatorname{dist}(x, \xi_2(Q)) \geq \frac{\bar{d}}{2}\}) \geq \ell(\{x \in \xi_1 : \operatorname{dist}(x, \xi_2(B)) \geq \frac{\bar{d}}{2}\}) \geq \frac{1}{2}\hat{\beta}(B)\ell(\xi_1)$$

where the first inequality follows from $\xi_2(Q) \subset \xi_2(B)$. By remark 2.18 and the definitions of $\hat{\beta}$ and ξ_1,

$$\ell(R_Q \cap \xi_1) + \sum_{j \in J_1 \smallsetminus J_2} \operatorname{diam}(Q^j) \gtrsim \ell(\{x \in \xi_1 : \operatorname{dist}(x, \xi_2(Q)) \geq \frac{\bar{d}}{2}\})$$

$$\geq \frac{1}{2}\hat{\beta}(B)\ell(\xi_1) \gtrsim \hat{\beta}(B)\operatorname{diam}(Q) \gtrsim 2^{\frac{-M}{2}}\operatorname{diam}(Q).$$

An important thing to note is that all the similarity constants are independent of ε_2, K, and M since these are rough lower bounds. This gives

$$\ell(R_Q \cap \xi_1) + \sum_{j \in J_1 \smallsetminus J_2} \operatorname{diam}(Q^j) \geq c2^{\frac{-M}{2}}\operatorname{diam}(Q)$$

with c independent of ε_2, K, and M.   □

Lemma 2.21. *There is a constant $\varepsilon_3 > 0$ (independent of M), which we can make arbitrarily small by reducing ε_2 and increasing K, such that for any $Q = Q(B)$ and $\xi_2 = \xi_2(Q)$ we have*

$$\ell(R_Q \cap \xi_2) + \sum_{j \in J_2} \operatorname{diam}(Q^j) \geq (1 - \varepsilon_3 2^{\frac{-M}{2}})\operatorname{diam}(Q).$$

Proof. Throughout the proof we assume ε_2 is sufficiently small. Recall that by construction we have

$$\operatorname{diam}(Q^j) \leq (1 + 4 \cdot 2^{-KM})2^{-KM}\operatorname{diam}(Q)$$

and

$$\operatorname{diam}(Q) \leq (1 + 4 \cdot 2^{-MK})\operatorname{diam}(2B).$$

Let $\xi_{2.0} \in \Lambda(2B)$ be a subarc of $\xi_2(Q)$ containing the center of B. Let O be the center of B, and O_1, O_2 the entry and exit points of $\xi_{2.0}$ from $2B$. Assume without loss of generality that $O_1 < O < O_2$ as ordered by ξ_2. Consider a ball $\operatorname{Ball}(O_1, r)$, with $r \leq \operatorname{radius}(B)$. Let $O_1^r \in \xi_2$ be the (unique) point such that $\operatorname{dist}(O_1^r, O_1) = r$, $O_1 < O_1^r < O$, and any other such point X satisfies $X < O_1^r$. Symmetrically, let

$O_2^r \in \xi_2$ be the (unique) point s.t. $\text{dist}(O_2^r, O_2) = r$, $O < O_2^r < O_2$, and any other such point X satisfies $X > O_2^r$.

The constants r_0 and C_{r_0} will be fixed below, independently of ε_2 and M. Suppose for a moment that there is no pair $r_1, r_2 \in [0, r_0 \operatorname{diam}(Q)]$ such that

$$(2.16) \qquad \partial_1(O_1^{r_1}, O, O_2^{r_2}) < C_{r_0} \varepsilon_2 \sqrt{\varepsilon_4^{-1}} 2^{\frac{-M}{2}} \operatorname{diam}(Q).$$

Then

$$\varepsilon_2^2 \beta^2(B) \operatorname{diam}(B)$$
$$\gtrsim \tilde{\beta}^2(\xi_2) \operatorname{diam}(\xi_2)$$
$$\gtrsim \operatorname{diam}(B)^{-3} (C_{r_0} \varepsilon_2 \sqrt{\varepsilon_4^{-1}} 2^{\frac{-M}{2}} \operatorname{diam}(Q))^2 \cdot r_0 \operatorname{diam}(Q) \cdot r_0 \operatorname{diam}(Q)$$
$$\sim C_{r_0}^2 r_0^2 \varepsilon_2^2 \varepsilon_4^{-1} 2^{-M} \operatorname{diam}(Q)$$
$$\gtrsim C_{r_0}^2 r_0^2 \varepsilon_2^2 \beta^2(B) \operatorname{diam}(B).$$

Thus by setting C_{r_0} large enough with respect to r_0 we get a contradiction. So we let $r_1, r_2 \in [0, r_0 \operatorname{diam}(Q)]$ be a pair such that (2.16) holds. This implies

$$\text{dist}(O_1^{r_1}, O_2^{r_2})$$
$$\geq \text{dist}(O_1^{r_1}, O) + \text{dist}(O_2^{r_2}, O) - C\varepsilon_2 \sqrt{\varepsilon_4^{-1}} 2^{\frac{-M}{2}} \operatorname{diam}(Q)$$
$$\geq \text{radius}(2B) - r_1 + \text{radius}(2B) - r_2 - C\varepsilon_2 \sqrt{\varepsilon_4^{-1}} 2^{\frac{-M}{2}} \operatorname{diam}(Q)$$
$$\geq \operatorname{diam}(Q) - r_1 - r_2 - 4 \cdot 2^{-MK} \operatorname{diam}(Q) - C\varepsilon_2 \sqrt{\varepsilon_4^{-1}} 2^{\frac{-M}{2}} \operatorname{diam}(Q).$$

If $r_1 \leq 2 \cdot 2^{-MK} \operatorname{diam}(Q)$ define $\xi_{2.1} = \emptyset$. If $r_2 \leq 2 \cdot 2^{-MK} \operatorname{diam}(Q)$ define $\xi_{2.2} = \emptyset$. Otherwise, we define $\xi_{2.1}$ or $\xi_{2.2}$ as follows.

For points $X, Y \in \xi_2$, we will denote by $X \rightsquigarrow Y$ the subarc of ξ_2 connecting X and Y. Assume $r_i > 2 \cdot 2^{-MK} \operatorname{diam}(Q)$. Let

$$B_i = \left(1 - 2 \cdot 2^{-MK} \frac{\operatorname{diam}(Q)}{r_i} \right) \text{Ball}(O_i, r_i).$$

By the definition of $O_i^{r_i}$ we have that

$$\text{dist}\left(O_1^{r_1} \rightsquigarrow O_2^{r_2}, \ B_i \right) \geq 2 \cdot 2^{-MK} \operatorname{diam}(Q).$$

By reducing r_0 the balls $\text{Ball}(O_1, r_1)$ and $\text{Ball}(O_2, r_2)$ have distance at least $2 \cdot 2^{-MK} \operatorname{diam}(Q)$ from each other. Define $\xi_{2.i}$ to be the largest-diameter (connected) subarc of $(O_i \rightsquigarrow O_i^{r_i}) \cap B_i$.

In either case, $\operatorname{diam}(\xi_{2.i}) \geq r_i - 2 \cdot 2^{-MK} \operatorname{diam}(Q)$.

Denote by $\xi_{2.3}$ the subarc $O_1^{r_1} \rightsquigarrow O_2^{r_2}$. By the above we have that no Q^j intersects 2 of these subarcs, and that (by increasing K for the last inequality)

$$\text{diam}(\xi_{2.1}) + \text{diam}(\xi_{2.2}) + \text{diam}(\xi_{2.3})$$

$$\geq r_1 - 2 \cdot 2^{-MK} \text{diam}(Q) + r_2 - 2 \cdot 2^{-MK} \text{diam}(Q)$$

$$+ \text{ diam}(Q) - r_1 - r_2 - 4 \cdot 2^{-MK} \text{diam}(Q) - C\varepsilon_2 \sqrt{\varepsilon_4^{-1}} 2^{\frac{-M}{2}} \text{diam}(Q)$$

$$\geq \text{diam}(Q) - 8 \cdot 2^{-MK} \text{diam}(Q) - C\varepsilon_2 \sqrt{\varepsilon_4^{-1}} 2^{\frac{-M}{2}} \text{diam}(Q)$$

$$\geq (1 - \varepsilon_3 2^{\frac{-M}{2}}) \text{diam}(Q).$$

Furthermore, since $\text{diam}(\xi_{2.i} \cap Q^j) \leq \text{diam}(Q^j)$, we have

$$\ell(R_Q \cap \xi_2) + \sum_{j \in J_2} \text{diam}(Q^j) \geq (1 - \varepsilon_3 2^{\frac{-M}{2}}) \text{diam}(Q). \qquad \square$$

We now get Proposition 2.19:

Proof.

$$\ell(R_Q) + \sum_j \text{diam}(Q^j)$$

$$\geq \ell(R_Q \cap \xi_2) + \sum_{j \in J_2} \text{diam}(Q^j) + \ell(R_Q \cap \xi_1) + \sum_{j \in J_1 \smallsetminus J_2} \text{diam}(Q^j)$$

$$\geq (1 - \varepsilon_3 2^{\frac{-M}{2}}) \text{diam}(Q) + c 2^{\frac{-M}{2}} \text{diam}(Q).$$

As we may get ε_3 arbitrarily small, we have obtained the proposition. $\qquad \square$

Lemma 2.22. *We have*

$$\sum_{B \in \Delta} \hat{\beta}(B)^2 \, \text{diam}(B) \lesssim 2^{\frac{-M}{2}} \mathscr{H}^1(\Gamma).$$

Proof. For $B \in \Delta$ and $Q = Q(B)$, we will construct a weight w_Q that satisfies (i), (ii) and (iii):

(i) $\int_Q w_Q d\ell \geq \text{diam}(Q)$,

(ii) for almost every $x_0 \in \Gamma$, $\sum_{B \in \Delta} w_{Q(B)}(x_0) < C 2^{\frac{M}{2}}$,

(iii) $\text{supp}(w_Q) \subset Q$,

where C is a constant which depends only on the 1-Ahlfors-regularity constant of Γ.

We will construct w_Q as a martingale. We denote by $w_Q(Z) := \int_Z w_Q \, d\ell$. Set

$$w_Q(Q) = \text{diam}(Q).$$

Assume now that $w_Q(Q')$ is defined. We define $w_Q(Q'^i)$ and $w_Q(R_{Q'})$, where

$$Q' = (\bigcup Q'^i) \cup R_{Q'},$$

a decomposition as given by equation (2.13).

Take

$$w_Q(R_{Q'}) = \frac{w_Q(Q')}{s'} \ell(R_{Q'})$$

and

$$w_Q(Q'^i) = \frac{w_Q(Q')}{s'} \operatorname{diam}(Q'^i),$$

where

$$s' = \ell(R_{Q'}) + \sum_i \operatorname{diam}(Q'^i).$$

This will give us w_Q. Note that $s' \lesssim \ell(\Gamma \cap Q')$. Clearly (i) and (iii) are satisfied. To see (ii):

$$\begin{aligned}
\frac{w_Q(Q'^{i*})}{\operatorname{diam}(Q'^{i*})} &= \frac{w_Q(Q')}{s'} \\
&= \frac{w_Q(Q')}{\operatorname{diam}(Q')} \frac{\operatorname{diam}(Q')}{s'} \\
&= \frac{w_Q(Q')}{\operatorname{diam}(Q')} \frac{\operatorname{diam}(Q')}{\ell(R_{Q'}) + \sum_i \operatorname{diam}(Q'^i)} \\
&\leq \frac{w_Q(Q')}{\operatorname{diam}(Q')} \frac{1}{1 + c'' \hat{\beta}(B)}.
\end{aligned}$$

And so,

$$\frac{w_Q(Q'^{i*})}{\operatorname{diam}(Q'^{i*})} \leq q \frac{w_Q(Q')}{\operatorname{diam}(Q')}$$

with $q = \frac{1}{1 + c'' 2^{\frac{-M}{2}}}$.

Now, suppose that $x \in Q_N \subset \ldots \subset Q_1$. we get:

$$\begin{aligned}
\frac{w_{Q_1}(Q_N)}{\operatorname{diam}(Q_N)} &\leq q \frac{w_{Q_1}(Q_{N-1})}{\operatorname{diam}(Q_{N-1})} \\
&\leq \cdots \\
&\leq q^{N-1} \frac{w_{Q_1}(Q_1)}{\operatorname{diam}(Q_1)} = q^{N-1}.
\end{aligned}$$

Hence, we have $w_{Q_1}(x) \lesssim q^{N-1}$. This will give us (ii) as a sum of a geometric series' since

$$\sum q^n = \frac{1}{1-q} \lesssim \frac{1}{2^{\frac{-M}{2}}} = 2^{\frac{M}{2}}.$$

Now,

$$\sum_{B \in \Delta} \hat{\beta}(B)^2 \operatorname{diam}(B) \lesssim 2^{-M} \sum_{B \in \Delta} \operatorname{diam}(B)$$

$$\lesssim 2^{-M} \sum_{B \in \Delta} \int w_{Q(B)}(x) \, d\ell(x)$$

$$= 2^{-M} \int \sum_{B \in \Delta} w_{Q(B)}(x) \, d\ell(x)$$

$$\lesssim 2^{-M} \int 2^{\frac{M}{2}} \, d\ell(x)$$

$$\lesssim 2^{\frac{-M}{2}} \mathscr{H}^1(\Gamma). \quad \square$$

Remark 2.23. By taking an increasing sequence of $\Delta \to \Delta_M^{p_1,p_2}$ we get that Lemma 2.22 holds with Δ is replaced by $\Delta_M^{p_1,p_2}$.

We now get Theorem 1.9 since

$$\sum_{B \in \mathscr{G}_2} \beta_2^2(B)^2 \operatorname{diam}(B) \lesssim \sum_{B \in \mathscr{G}_2} \hat{\beta}(B)^2 \operatorname{diam}(B)$$

$$= \sum_{M=1}^{\infty} \sum_{\substack{B \in \mathscr{G}_2 \\ 2^{-M} \le \frac{1}{2}\hat{\beta}(B)^2 < 2^{-M+1}}} \hat{\beta}(B)^2 \operatorname{diam}(B)$$

$$\lesssim \sum_{M=1}^{\infty} M 2^{\frac{-M}{2}} \mathscr{H}^1(\Gamma)$$

$$\lesssim \mathscr{H}^1(\Gamma).$$

3. Modifications for proofs of Theorems 1.10 and 1.11

In this section we give the needed modifications to obtain Theorems 1.10 and 1.11.

Consider a ball $\operatorname{Ball}(z, R)$ where $R > 0$ and $z \in \Gamma$. Let $\{\Gamma_i\}$ be the connected components of $\Gamma \cap \operatorname{Ball}(z, 10R)$ which intersect $\operatorname{Ball}(z, R)$. If there is only one such component then Theorems 1.8 and 1.9 give Theorems 1.10 and 1.11. Otherwise, all components Γ_i must have diameter at least $9R$, and so by 1-Ahlfors-regularity there are at most P of them, where P depends only on the 1-Ahlfors-regularity constant of Γ. Parameterize each Γ_i by γ_i, as assured by Lemma 2.3.

Informally speaking, the proofs we have of Theorems 1.8 and 1.9 now work word for word, since they only depend on the existence of a parameterization for each connected component. Rather than checking this, we use the following trick.

One may simply connect the end of γ_i to the beginning of γ_{i+1} with an arc-length parameterization. The total added length will be at most $20PR$. Call this new path γ, and its image $\tilde{\Gamma}$. One may apply Theorems 1.8 and 1.9 to get the desired results now.

This completes the proof of Theorems 1.10 and 1.11. $\qquad\square$

4. Appendix

4.1. Proof of Lemma 2.3. We assume $\Gamma \subset \mathscr{M}$ be a compact connected set of finite $\mathscr{H}^1$ measure.

Using the Kuratowski embedding theorem (see [Hei03]), we have an isometric embedding $f \colon \Gamma \to \ell_\infty(\Gamma)$. Let $\Gamma' = f(\Gamma)$.

The following two lemmas have proofs identical to what appears in [Sch].

Lemma 4.1. *Let $C_1, C_2 > 0$ be given. Given a compact set $\Gamma' \subset \ell_\infty(\Gamma)$ the set $E := \{x \in \ell_\infty(\Gamma) : x = tx_1 + (1-t)x_2, x_i \in \Gamma', -C_1 \le t \le C_2\}$ is compact.*

Proof. Suppose $\{x^i\} \subset E$ is a sequence. We can write $x^i = t^i x_1^i + (1 - t^i)x_2^i$ as in the definition of E. By the compactness of Γ' we have i_k such that $x_1^{i_k} \to x_1$. By compactness of Γ' again, $x_2^{i_{k_j}} \to x_2$. By compactness of $[-C_1, C_2]$ we have $t^{i_{k_{j_l}}} \to t$. $x_1, x_2 \in \Gamma', t \in [-C_1, C_2]$. Hence $x^{i_{k_{j_l}}} \to tx_1 + (1-t)x_2 \in E$. $\qquad\square$

Lemma 4.2. *Let $\Gamma' \subset \ell_\infty(\Gamma)$ be a compact connected set of finite length. Then we have a Lipschitz function $\gamma \colon [0,1] \to \ell_\infty(\Gamma)$ such that $\mathrm{Image}(\gamma) = \Gamma'$ and $\|\gamma\|_{\mathrm{Lip}} \le 32\mathscr{H}^1(\Gamma')$.*

Proof. We use a well known result from graph theory (which we call (*)):

If G is a connected graph with finitely many edges, then there is a path that traverses each edge of G exactly twice (once in each direction). This result is easily seen by induction on the number of edges.

For $n \ge 0$, let $X_n = X_n^{\Gamma'}$ (i.e. take $X_n \subset \Gamma'$ a 2^{-n}-net). Note that since Γ' is compact, each X_n is finite. We want to get a connected set E_n. We do this by adding line segments inductively. Set $E_n^0 = X_n$. We get E_n^{i+1} from E_n^i by adding a line segment between points $x_1, x_2 \in X_n$ such that $\mathrm{dist}(x_1, x_2) < 2^{-n+3}$ and they are not yet in the same connected component of E_n^i. If there are no two such points we stop and call the resulting set E_n. Let G_n be the obvious abstract graph associated to E_n. If G_n is not connected then $\mathrm{Vertex}(G_n) = A \cup B$ with $\mathrm{dist}(A, B) \ge 2^{-n+2}$ and A separated from B. By the construction of E_n and X_n we have that $\mathrm{dist}(\mathscr{N}_{2^{-n}}(A), \mathscr{N}_{2^{-n}}(B)) \ge 2^{-n}$ and $\Gamma' \subset \mathscr{N}_{2^{-n}}(A) \cup \mathscr{N}_{2^{-n}}(B)$. This is a contradiction to Γ' being connected. Hence G_n is connected.

Note that $\mathscr{H}^1(E_n) \le \sharp(X_n)2^{-n+3} \le 16\mathscr{H}^1(\Gamma')$, where the final inequality follows from the fact that the balls $\{B(x, 2^{-n-1}) : x \in X_n\}$ are disjoint.

We can thus parameterize E_n by a Lipschitz curve $\gamma_n \colon [0,1] \to \ell_\infty(\Gamma)$. The image of this parameterization is in E as defined in the previous lemma. By Arzela-Ascoli we have a subsequence converging to γ'. We have that $\mathrm{Image}(\gamma') = \Gamma'$ by say

$$\sup_{x \in E_n} \mathrm{dist}(x, \Gamma') + \sup_{y \in \Gamma'} \mathrm{dist}(E_n, y) \le 4 \cdot 2^{-n} + 2^{-n} = 5 \cdot 2^{-n}$$

and a triangle inequality. $\qquad\square$

Now, Consider the mapping $\gamma'' = f^{-1}\gamma'$. The map γ'' gives the first part of Lemma 2.3 with $\mathbf{T}$ replaced by $[0,1]$. To correct this one simply defines $\gamma(t) = \gamma''(2t)$ for $0 \le t \le \frac{1}{2}$ and $\gamma(t) = \gamma''(1 - (2t - 1))$ for $\frac{1}{2} \le t \le 1$. The map γ has $\mathbf{T}$ as its domain and Γ as its image.

Assume now that Γ is also 1-Ahlfors-regular with constant C. Then in the proof above, E_n is also 1-Ahlfors-regular. Hence

$$\frac{R}{C'} \le \mathscr{H}^1(\gamma_n^{-1}(\mathrm{Ball}(x, R))) \le C'R \quad \forall x \in E_n, \ 0 < R \le \mathrm{diam}(E_n)$$

by the result (*). Given R, one may choose n large enough so that this implies the second part of Lemma 2.3.

This completes the proof of Lemma 2.3. □

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