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QUASIMÖBIUS MAPS PRESERVE UNIFORM DOMAINS

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Abstract. We show that if a domain Ω in a geodesic metric space is quasimöbius to a uniform domain in some metric space, then Ω is also uniform.

1. Introduction

In this paper we study the behavior of uniform domains under quasisymmetric and quasimöbius maps between metric spaces.

Let (X, d) be a metric space. A subset of X is called a *domain* if it is open and connected. Let $0 < \lambda \le 1$ and $c \ge 1$. We say a domain $\Omega \subset X$ with $\partial\Omega \ne \emptyset$ is (λ, c)-*quasiconvex*, if for any $x \in \Omega$, and any two points $y_1, y_2 \in B(x, \lambda d(x, \partial\Omega))$, there is a path γ in Ω from y_1 to y_2 with length $\ell(\gamma) \le c\, d(y_1, y_2)$.

We say (X, d) is c-*quasiconvex* if for any two points $x, y \in X$, there is a path γ joining x and y with length $\ell(\gamma) \le c\, d(x, y)$. A metric space (X, d) is quasiconvex if it is c-quasiconvex for some $c \ge 1$. Notice that if (X, d) is c-quasiconvex, then every domain in X is (λ, c)-quasiconvex for all $0 < \lambda \le 1$.

Recall that a metric space is *proper* if all its closed balls are compact. The main result of the paper is as follows:

Theorem 1.1. *Let (X_i, d_i) $(i = 1, 2)$ be proper metric spaces, and $\Omega_i \subset X_i$ domains with $\partial\Omega_i \ne \emptyset$. Suppose (Ω_1, d_1) is c_1-uniform, $\Omega_2 \subset (X_2, d_2)$ is (λ, c_2)-quasiconvex for some $0 < \lambda \le 1$ and $c_2 \ge 1$, and there is an η-quasimöbius homeomorphism $(\Omega_1, d_1) \to (\Omega_2, d_2)$. Then (Ω_2, d_2) is c-uniform for some constant c.*

It is reasonable to assume quasiconvexity in Theorem 1.1, at least one needs that the domains are rectifiably connected, as shown by the following example (the author thanks David Herron for pointing out this example). Let $\Delta \subset \mathbf{R}^2$ be the open unit disk in the plane with the Euclidean metric d. The domain Δ is clearly uniform. For any $0 < \varepsilon < 1$, the identity map $(\Delta, d) \to (\Delta, d^\varepsilon)$ is η-quasimöbius with $\eta(t) = t^\varepsilon$. But (Δ, d^ε) is not uniform since there are no rectifiable curves in (Δ, d^ε) except the constant curves.

Theorem 1.1 is not quantitative in the sense that when (Ω_2, d_2) is bounded one can not control the constant c in terms of c_1, c_2, λ and η alone. See Section 5 for an example. It is unclear whether one can make Theorem 1.1 quantitative under the stronger assumption that (Ω_2, d_2) is quasiconvex. See Section 5 for more detail about

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this question. On the other hand, when (Ω_2, d_2) is unbounded or when (X_2, d_2) is quasiconvex and annular convex, there is the following quantitative result. Recall that a metric space is *c-annular convex* for some $c \geq 2$ if for any $x \in X$, each $r > 0$, and any $y, z \in B(x, 2r)\backslash B(x, r)$, there is a path γ from y to z satisfying $\ell(\gamma) \leq c\, d(y, z)$ and $\gamma \cap B(x, r/c) = \emptyset$.

Theorem 1.2. *Let (X_i, d_i) $(i = 1, 2)$ be proper metric spaces, and $\Omega_i \subset X_i$ domains with $\partial\Omega_i \neq \emptyset$. Suppose (Ω_1, d_1) is c_1-uniform, and there is an η-quasimöbius homeomorphism $h\colon (\Omega_1, d_1) \to (\Omega_2, d_2)$.*

(1) *If (Ω_2, d_2) is unbounded and (λ, c_2)-quasiconvex, then (Ω_2, d_2) is c-uniform with $c = c(\eta, c_1, c_2, \lambda)$.*
(2) *If (X_2, d_2) is c_2-quasiconvex and c_2-annular convex, then (Ω_2, d_2) is c-uniform with $c = c(\eta, c_1, c_2)$.*

Theorem 1.2 (2) was first proved in [HSX] by using a characterization of uniform domains in terms of Gromov hyperbolic spaces and the quasiconformal structure on the Gromov boundary. In this paper we give a more elementary proof.

For quasisymmetric maps, we have the following:

Theorem 1.3. *Let (X_i, d_i) $(i = 1, 2)$ be proper metric spaces, $\Omega_i \subset X_i$ domains with $\partial\Omega_i \neq \emptyset$, and $g\colon (\Omega_1, d_1) \to (\Omega_2, d_2)$ an η-quasisymmetric map. Suppose (Ω_1, d_1) is c_1-uniform and (Ω_2, d_2) is (λ, c_2)-quasiconvex. Then (Ω_2, d_2) is c-uniform with $c = c(\eta, c_1, c_2, \lambda)$.*

In the case when the metric spaces are geodesic spaces Theorem 1.3 essentially has been proved by Väisälä (see the proofs of Lemma 10.21 and Theorem 10.22 in [V]), who stated it only for Banach spaces. Our proof is a slight modification of Väisälä's. We include this theorem here since we need it for the proofs of Theorem 1.1 and Theorem 1.2.

The invariance of uniform domains under quasimöbius maps was (implicitly) obtained by Gehring and Martio ([GM]) for Euclidean domains, and has been established by Väisälä for domains in Banach spaces ([V2], [V]).

Theorem 1.1 is proved by using Theorem 1.3 and a construction of Bonk-Kleiner. For any unbounded proper metric space (X, d) and $p \in X$, Bonk and Kleiner constructed a metric $\hat{d}_p$ on the one point compactification $X \cup \{\infty\}$ of X, such that the identity map $(X, d) \to (X, \hat{d}_p)$ is quasimöbius (see [BK] or Section 2). Furthermore, for any domain $\Omega \subset X$ with $\partial\Omega \neq \emptyset$, Ω is uniform with respect to d if and only if Ω is uniform with respect to $\hat{d}_p$ (see Theorem 2.3).

Since a quasimöbius map between bounded metric spaces is quasisymmetric, Theorem 1.1 follows from Theorem 1.3 when both (Ω_1, d_1) and (Ω_2, d_2) are bounded. In general, when we consider the metric $d_i' := \hat{d}_{ip_i}$ $(p_i \in X_i)$ on X_i $(i = 1, 2)$, a quasimöbius map $g\colon (\Omega_1, d_1) \to (\Omega_2, d_2)$ becomes a quasisymmetric map $g\colon (\Omega_1, d_1') \to (\Omega_2, d_2')$ as (X_i, d_i') are bounded for $i = 1, 2$. Theorem 1.3 implies that (Ω_2, d_2') is uniform, and hence (Ω_2, d_2) is also uniform by the preceding paragraph.

Remark. The referee asked whether the results in the paper are still valid if the definition of uniformity is based on diameter or distance (see [V3] for the precise definitions). Theorems 1.1, 1.2, 1.3 all hold for these notions of uniformity under weaker assumptions: one no longer needs to assume the quasiconvexity condition on the domains. The counterexample in Section 5 remains valid as well.

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2. Preliminaries

In this Section we recall some basic definitions and facts, see [V] and [BHX] for more details.

Let (X, d) be a complete metric space, and $\Omega \subset X$ a domain. The metric boundary of Ω is $\partial\Omega = \overline{\Omega}\backslash\Omega$. In this paper we always assume $\partial\Omega \neq \emptyset$. For $x \in \Omega$, we denote $d(x) = d(x, \partial\Omega)$. We say Ω is *rectifiably connected* if for any $x, y \in \Omega$ there is a path in Ω from x to y with finite length. For a rectifiably connected domain Ω, the *quasihyperbolic metric* k on Ω is defined as follows: for $x, y \in \Omega$,

$$k(x, y) := \inf \int_\gamma \frac{1}{d(z)}\, ds(z),$$

where γ runs over all rectifiable curves in Ω joining x and y. Here ds denotes the arc length element along γ. For $x, y \in \Omega$, we set

$$r_\Omega(x, y) = \frac{d(x, y)}{d(x) \wedge d(y)} \quad \text{and} \quad j_\Omega(x, y) = \log\left(1 + r_\Omega(x, y)\right),$$

where $a \wedge b$ denotes $\min\{a, b\}$ for real numbers a, b.

The length metric on Ω is defined as follows: for $x, y \in \Omega$, $l_\Omega(x, y)$ is the infimum of length of paths in Ω from x to y.

Proposition 2.1. (Proposition 2.8 of [BHK]) *Suppose (Ω, d) is locally compact and rectifiably connected. If the identity map* $\mathrm{id}\colon (\Omega, d) \to (\Omega, l_\Omega)$ *is a homeomorphism, then* $\mathrm{id}\colon (\Omega, d) \to (\Omega, k)$ *is also a homeomorphism, and (Ω, k) is a proper geodesic space.*

We observe that if (Ω, d) is (λ, c)-quasiconvex for some $0 < \lambda \leq 1$ and $c \geq 1$, then the identity map $(\Omega, d) \to (\Omega, l_\Omega)$ is locally bilipschitz and hence is a homeomorphism. However, $\mathrm{id}\colon (\Omega, d) \to (\Omega, l_\Omega)$ is not always a homeomorphism (one can easily construct an example using topologist's sine curve).

Lemma 2.2. (Theorem 3.7 (1) of [V]) *The following holds for all $x, y \in \Omega$,*

$$k(x, y) \geq j_\Omega(x, y) \geq \log\frac{d(x)}{d(y)}.$$

Let $c \geq 1$. A path $\gamma\colon [0, 1] \to \Omega$ is called a *c-uniform curve* if:

(1) $\ell(\gamma) \leq c\, d(\gamma(0), \gamma(1))$;
(2) $c\, d(\gamma(t)) \geq \ell(\gamma|[0, t]) \wedge \ell(\gamma|[t, 1])$ for all $t \in [0, 1]$.

The domain $\Omega \subset (X,d)$ is called a *c-uniform domain* in (X,d) if every two points $x, y \in \Omega$ can be joined by a c-uniform curve.

Let $\eta\colon [0,\infty) \to [0,\infty)$ be a homeomorphism. A homeomorphism between metric spaces $f\colon (X,d_X) \to (Y,d_Y)$ is *η-quasisymmetric* if for all pairwise distinct points $x, y, z \in X$, we have

$$\frac{d_Y(f(x),f(y))}{d_Y(f(x),f(z))} \le \eta\left(\frac{d_X(x,y)}{d_X(x,z)}\right).$$

A homeomorphism $f\colon (X,d_X) \to (Y,d_Y)$ is quasisymmetric if it is η-quasisymmetric for some η. The inverse of a quasisymmetric map is quasisymmetric, and the composition of two quasisymmetric maps is also quasisymmetric.

Let $Q = (x_1, x_2, x_3, x_4)$ be a quadruple of pairwise distinct points in (X,d). The *cross ratio* of Q with respect to the metric d is:

$$cr(Q,d) = \frac{d(x_1,x_3)d(x_2,x_4)}{d(x_1,x_4)d(x_2,x_3)}.$$

Let $\eta\colon [0,\infty) \to [0,\infty)$ be a homeomorphism. A homeomorphism between metric spaces $f\colon (X,d_X) \to (Y,d_Y)$ is an *η-quasimöbius* map if

$$cr(f(Q),d_Y) \le \eta(cr(Q,d_X))$$

for all quadruples Q of distinct points in X, where $f(Q) = (f(x_1), f(x_2), f(x_3), f(x_4))$. A homeomorphism $f\colon (X,d_X) \to (Y,d_Y)$ is quasimöbius if it is η-quasimöbius for some η. The inverse of a quasimöbius map is quasimöbius, and the composition of two quasimöbius maps is also quasimöbius.

Quasisymmetric maps are quasimöbius, and quasimöbius maps between bounded metric spaces are quasisymmetric.

Let (X,d) be an unbounded metric space and $p \in X$. Set $S_p(X) = X \cup \{\infty\}$, where ∞ is a point not in X. Let

$$s_p(x,y) = \frac{d(x,y)}{(1+d(x,p))(1+d(y,p))}$$

for $x, y \in X$, $s_p(x,\infty) = s_p(\infty,x) = \frac{1}{1+d(x,p)}$ for $x \in X$ and $s_p(\infty,\infty) = 0$. For $x, y \in S_p(X)$, define

$$\hat{d}_p(x,y) := \inf \sum_{i=0}^{k-1} s_p(x_i, x_{i+1}),$$

where the infimum is taken over all finite sequences of points $x_0, \dots, x_k \in S_p(X)$ with $x_0 = x$ and $x_k = y$. Then $\hat{d}_p$ is a metric on $S_p(X)$ and

$$\frac{1}{4} s_p(x,y) \le \hat{d}_p(x,y) \le s_p(x,y) \quad \text{for } x, y \in S_p(X).$$

Furthermore the identity map $\mathrm{id}\colon (X,d) \to (X,\hat{d}_p)$ is an η-quasimöbius homeomorphism with $\eta(t) = 16t$. If (X,d) is c-quasiconvex and c-annular convex, then

$(S_p(X), \hat{d}_p)$ is c'-quasiconvex and c'-annular convex, where c' depends only on c. See [BHX] for a proof of the above statements.

Theorem 2.3. ([BHX]) *Let (X, d) be an unbounded proper metric space, $\Omega \subset X$ a domain with $\partial\Omega \neq \emptyset$ and $p \in X$. Then $\Omega \subset (X, d)$ is uniform if and only if $\Omega \subset (S_p(X), \hat{d}_p)$ is uniform. Furthermore,*

(1) *if (Ω, d) is unbounded, $p \in \partial\Omega$ and $\Omega \subset (X, d)$ is c-uniform, then $\Omega \subset (S_p(X), \hat{d}_p)$ is c'-uniform with c' depending only on c;*
(2) *if (Ω, d) is unbounded and $\Omega \subset (S_p(X), \hat{d}_p)$ is c-uniform, then $\Omega \subset (X, d)$ is c'-uniform with c' depending only on c.*

Let (X, d) be a metric space and $p \in X$. Set $I_p(X) = X \backslash \{p\}$ if X is bounded and $I_p(X) = (X \backslash \{p\}) \cup \{\infty\}$ if X is unbounded, where ∞ is a point not in X. We shall define a metric d_p on $I_p(X)$.

Let

$$f_p(x, y) = \frac{d(x, y)}{d(x, p)d(y, p)}$$

for $x, y \in X \backslash \{p\}$, $f_p(x, \infty) = f_p(\infty, x) = \frac{1}{d(x,p)}$ for $x \in X \backslash \{p\}$ and $f_p(\infty, \infty) = 0$.

For $x, y \in I_p(X)$, we define

$$d_p(x, y) := \inf \sum_{i=0}^{k-1} f_p(x_i, x_{i+1}),$$

where the infimum is taken over all finite sequences of points $x_0, \cdots, x_k \in I_p(X)$ with $x_0 = x$ and $x_k = y$. Then the following holds for all $x, y \in I_p(X)$:

$$\frac{1}{4} f_p(x, y) \leq d_p(x, y) \leq f_p(x, y). \tag{2.1}$$

It follows that d_p is a metric on $I_p(X)$ and the identity map $\mathrm{id} : (X \backslash \{p\}, d) \to (X \backslash \{p\}, d_p)$ is an η-quasimöbius homeomorphism with $\eta(t) = 16t$. If (X, d) is c-quasiconvex and c-annular convex, then the space $(I_p(X), d_p)$ is c'-quasiconvex and c'-annular convex, where c' depends only on c. See [BHX] for a proof of the above statements.

Theorem 2.4. ([BHX]) *Let (X, d) be a proper metric space, $\Omega \subset X$ a domain and $p \in \partial\Omega$. Assume $\partial\Omega$ contains at least two points if (Ω, d) is bounded. Then $\Omega \subset (X, d)$ is uniform if and only if $\Omega \subset (I_p(X), d_p)$ is uniform. Furthermore,*

(1) *if $\Omega \subset (X, d)$ is c-uniform, then $\Omega \subset (I_p(X), d_p)$ is c'-uniform, where c' depends only on c;*
(2) *if X is c-quasiconvex and c-annular convex, and $\Omega \subset (I_p(X), d_p)$ is c_1-uniform, then $\Omega \subset (X, d)$ is c'-uniform with $c' = c'(c, c_1)$.*

The above two constructions, $S_p(X)$ and $I_p(X)$, are in a sense inverse to each other, as shown by the following result.

Lemma 2.5. ([BHX]) *Let (X,d) be an unbounded metric space and $p \in X$. Set $Y = S_p(X) = X \cup \{\infty\}$ and denote by d' the metric $(\hat{d_p})_\infty$ on $I_\infty(Y) = X$. Then the identity map $id\colon (X,d) \to (I_\infty(Y), d') = (X, d')$ is 16-bilipschitz.*

3. Equivalent definitions of uniform domains

In this Section we prove the equivalence of several different definitions of uniform domains. We need this in Section 4. The results in this Section can be proved by slightly modifying either the proofs of Gehring–Osgood (Section 2 of [GO]) or those of Väisälä (Section 10 of [V]). Both Väisälä's and Gehring–Osgood's proofs use the fact that the metric spaces are geodesic, and this is the only place that needs to be modified. We adopt Väisälä's proofs. We include them here mainly for completeness.

Let (X,d) be a proper metric space, and $\Omega \subset X$ a rectifiably connected domain with $\partial\Omega \neq \emptyset$. Recall that we always have $k(x,y) \geq j_\Omega(x,y)$ for all $x,y \in \Omega$. Let $c \geq 1$. We say $\Omega \subset X$ is a *QH c-uniform domain* if $k(x,y) \leq c\, j_\Omega(x,y)$ for all $x,y \in \Omega$.

We first recall two lemmas from [V].

Lemma 3.1. (Lemma 10.7 in [V]) *Let $\Omega \subset X$ be a QH c-uniform domain, $r > 0$, and let γ be a quasihyperbolic geodesic in Ω such that $d(z) \leq r$ for all $z \in \gamma$. Then $\ell(\gamma) \leq M_1(c)\, r$, where $M_1(c)$ is a constant depending only on c.*

Lemma 3.2. (Lemma 10.8 in [V]) *For each $c \geq 1$, there is a number $q = q(c) \in (0,1)$ with the following property: Let $\Omega \subset X$ be a QH c-uniform domain, $\gamma \subset \Omega$ a quasihyperbolic geodesic with endpoints a_0, a_1, and let $x \in \gamma$ be a point with $d(x) \leq q\, d(a_0)$. Then for $\gamma_x = \gamma[x,a_1]$ we have $\ell(\gamma_x) \leq M_2(c)\, d(x)$, where $M_2(c)$ is a constant depending only on c.*

Lemma 3.3. *Let $\Omega \subset X$ be a (λ_0, c_0)-quasiconvex domain for some $0 < \lambda_0 \leq 1$ and $c_0 \geq 1$. Let $x, y \in \Omega$. If $\frac{d(x,y)}{d(x)} \leq \frac{\lambda_0}{2c_0}$, then $k(x,y) \leq 2c_0 \frac{d(x,y)}{d(x)} \leq \lambda_0$. In particular, if $r_\Omega(x,y) \leq \frac{\lambda_0}{2c_0}$, then $k(x,y) \leq 2c_0 r_\Omega(x,y) \leq \lambda_0$.*

Proof. Suppose that $d(x,y) \leq \frac{\lambda_0}{2c_0} d(x)$. Since $\Omega \subset X$ is (λ_0, c_0)-quasiconvex, there is a path γ from x to y with $\ell(\gamma) \leq c_0 d(x,y) \leq \frac{\lambda_0}{2} d(x) \leq d(x)/2$. It follows that $d(z) \geq d(x)/2$ for all $z \in \gamma$. Now

$$k(x,y) \leq \int_\gamma \frac{1}{d(z)}\, ds(z) \leq \frac{2}{d(x)} \ell(\gamma) \leq \frac{2}{d(x)} c_0\, d(x,y) \leq \lambda_0. \qquad \square$$

Theorem 3.4. *Let (X,d) be a proper metric space, and let $\Omega \subset X$ be a (λ_0, c_0)-quasiconvex domain for some $0 < \lambda_0 \leq 1$ and $c_0 \geq 1$. If $\Omega \subset X$ is QH c_1-uniform, then $\Omega \subset X$ is c_2-uniform with $c_2 = c_2(\lambda_0, c_0, c_1)$.*

Proof. By Proposition 2.1 and the remark after it, (Ω, k) is a geodesic space. Fix $a_0, a_1 \in \Omega$ and let γ be a quasihyperbolic geodesic from a_0 to a_1. We shall show that γ is a c_2-uniform curve with c_2 depending only on λ_0, c_0 and c_1.

(1) We first prove $\ell(\gamma[a_0,x]) \wedge \ell(\gamma[x,a_1]) \leq c\,d(x)$ for all $x \in \gamma$ and some $c = c(c_1)$. Let $x_0 \in \gamma$ be a point with maximal $d(x_0)$. By symmetry, it suffices to find an estimate of the form

$$\ell(\gamma[a_0,x]) \leq c\,d(x)$$

for all $x \in \gamma[a_0,x_0]$. Let $q = q(c_1) \in (0,1)$ be the number given by Lemma 3.2. If $d(x) \leq q\,d(x_0)$, then Lemma 3.2 implies $\ell(\gamma[a_0,x]) \leq M_2(c_1)\,d(x)$. If $d(x) \geq q\,d(x_0)$, we apply Lemma 3.1 with $r = d(x_0)$ and obtain $\ell(\gamma[a_0,x]) \leq \frac{M_1(c_1)}{q}\,d(x)$.

(2) We next prove $\ell(\gamma) \leq c_2\,d(a_0,a_1)$ for some $c_2 = c_2(\lambda_0,c_0,c_1)$. We may assume that $d(a_0) \leq d(a_1)$. Set $t = d(a_0,a_1)$ and $r = d(a_0)$. We consider two cases.

Case (a). $r \leq 2c_0t/\lambda_0$.

We may assume $\ell(\gamma) \geq 2t$. Choose points $b_0, b_1 \in \gamma$ such that $\ell(\gamma[a_0,b_0]) = \ell(\gamma[a_1,b_1]) = t$. By part (1) we have $t \leq c\,d(b_i)$ for $i = 0,1$. We obtain:

$$r_\Omega(b_0,b_1) \leq \frac{d(b_0,a_0) + d(a_0,a_1) + d(a_1,b_1)}{d(b_0) \wedge d(b_1)} \leq \frac{3t}{t/c} = 3c.$$

Since Ω is QH c_1-uniform, this implies that $k(b_0,b_1) \leq c_1 j_\Omega(b_0,b_1) \leq c_1\log(1+3c) = c_3$. For each $x \in \gamma[b_0,b_1]$ we get $k(x,b_0) \leq k(b_0,b_1) \leq c_3$. Since $d(b_0) \leq d(a_0) + d(a_0,b_0) = r + t \leq (1+2c_0/\lambda_0)t$, Lemma 2.2 yields $d(x) \leq d(b_0)e^{c_3} \leq (1+2c_0/\lambda_0)te^{c_3} = c_4t$. Integrating along $\gamma[b_0,b_1]$ gives $k(b_0,b_1) \geq \ell(\gamma[b_0,b_1])/c_4t$. Since $k(b_0,b_1) \leq c_3$, we have $\ell(\gamma[b_0,b_1]) \leq c_3c_4t$ and $\ell(\gamma) \leq t + c_3c_4t + t = (2+c_3c_4)t$.

Case (b). $r \geq 2c_0t/\lambda_0$.

Since $\Omega \subset X$ is (λ_0,c_0)-quasiconvex and $\frac{d(a_0,a_1)}{d(a_0)} = t/r \leq \frac{\lambda_0}{2c_0}$, Lemma 3.3 implies $k(a_0,a_1) \leq 2c_0\frac{t}{r}$. Set $b = \ell(\gamma)$ and let $\gamma'\colon [0,b] \to \gamma$ be the arclength parametrization of γ with $\gamma'(0) = a_0$. Since $d(\gamma'(s)) \leq d(a_0) + d(a_0,\gamma'(s)) \leq r + s$ for all $0 \leq s \leq b$, we get $k(a_0,a_1) = \ell_k(\gamma) \geq \int_0^b \frac{ds}{r+s} = \log(1+b/r)$. Now we have $\log(1+b/r) \leq k(a_0,a_1) \leq 2c_0\frac{t}{r}$. Setting $u = r/t \geq 2c_0/\lambda_0 \geq 2$ we thus have $1 + \frac{b}{tu} \leq e^{2c_0/u}$. It follows that $b/t \leq c_6$, where $c_6 = \max\{u(e^{2c_0/u} - 1) : u \geq 2\}$. Notice that the function $f(u) := u(e^{2c_0/u} - 1)$ is decreasing for $u \geq 2$ and hence $c_6 = 2(e^{c_0} - 1)$. $\square$

Remark 3.5. Buckley and Herron ([BH]) have recently proved a result that is stronger than Theorem 3.4.

Theorem 3.6. *Let (X,d) be a proper metric space, and $\Omega \subset X$ a (λ_0,c_0)-quasiconvex domain for some $0 < \lambda_0 \leq 1$, $c_0 \geq 1$. Then the following conditions are quantitatively equivalent:*

(1) *Ω is c-uniform;*
(2) *Ω is QH c-uniform;*
(3) *$k(x,y) \leq c\,j_\Omega(x,y) + c'$ for all $x,y \in \Omega$, where c and c' are constants.*

The phrase "quantitatively equivalent" should be understood as follows. For example, "(3) $\Rightarrow$ (1) quantitatively" means if $k(x,y) \leq c\,j_\Omega(x,y) + c'$ for all $x,y \in \Omega$, then Ω is c''-uniform with c'' depending only on c, c', λ_0 and c_0.

Proof. We show that (3) $\Rightarrow$ (2) $\Rightarrow$ (1) $\Rightarrow$ (3). The implication (2) $\Rightarrow$ (1) is simply Theorem 3.4. Assume that (3) holds, let $a, b \in \Omega$, and set $r = r_\Omega(a,b)$. If $r \le \frac{\lambda_0}{2c_0}$, then $r \log 2 \le \log(1+r)$, and Lemma 3.3 implies that $k(a,b) \le 2c_0 r \le (2c_0/\log 2) j_\Omega(a,b)$. If $r \ge \frac{\lambda_0}{2c_0}$, then $j_\Omega(a,b) \ge \log(1+\frac{\lambda_0}{2c_0})$. Hence

$$\frac{k(a,b)}{j_\Omega(a,b)} \le c + \frac{c'}{\log(1+\frac{\lambda_0}{2c_0})},$$

and we obtain (2). It remains to prove (1) $\Rightarrow$ (3).

Assume that (1) is true, and let $a, b \in \Omega$. Let $\gamma \subset \Omega$ be a c-uniform arc joining a and b. Let $x_0 \in \gamma$ be the point bisecting the length ℓ_0 of γ. We may assume that $r_\Omega(a,b) \ge \frac{\lambda_0}{2c_0}$, since otherwise Lemma 3.3 gives $k(a,b) \le \lambda_0$ and we have $k(a,b) \le 0 \cdot j_\Omega(a,b) + \lambda_0$. Setting $m = d(a) \wedge d(b)$ we have $\frac{m\lambda_0}{2c_0} \le d(a,b) \le \ell_0$. Choose points $a_1, b_1 \in \gamma$ with $\ell(\gamma[a,a_1]) = \ell(\gamma[b,b_1]) = \frac{m\lambda_0}{4c_0}$. Notice that $r_\Omega(a,a_1), r_\Omega(b,b_1) \le \frac{\lambda_0}{2c_0}$. Hence Lemma 3.3 yields $k(a_1,a) \le \lambda_0$ and $k(b_1,b) \le \lambda_0$. Setting $\beta = \gamma[a_1,x_0]$ we obtain by the uniformity condition

$$k(a_1,x_0) \le \ell_k(\beta) \le c\int_\beta \frac{ds(x)}{\ell(\gamma[a,x])} = c\int_{\frac{m\lambda_0}{4c_0}}^{\frac{\ell_0}{2}} \frac{ds}{s} = c\log\frac{2c_0\ell_0}{m\lambda_0}.$$

The same estimate also holds for $k(b_1,x_0)$. Since $\ell_0 \le c\,d(a,b)$, we have

$$\begin{aligned} k(a,b) &\le k(a,a_1) + k(a_1,x_0) + k(x_0,b_1) + k(b_1,b) \\ &\le 2\lambda_0 + 2c\log\frac{2c_0\ell_0}{m\lambda_0} \le 2\lambda_0 + 2c\log\frac{2c_0cd(a,b)}{m\lambda_0} \\ &= 2\lambda_0 + 2c\log\frac{2c_0c}{\lambda_0} + 2c\log\frac{d(a,b)}{m} \\ &= c' + 2c\log r_\Omega(a,b) \le c' + 2cj_\Omega(a,b), \end{aligned}$$

where $c' = 2\lambda_0 + 2c\log\frac{2c_0c}{\lambda_0}$. □

4. Proof of the main results

In this Section we prove the theorems stated in the Introduction.

We first recall two results of Väisälä.

Lemma 4.1. (Lemma 2.3 of [V]) *Suppose that X is a-quasiconvex, $q > 0$, $b \ge 0$, and that $f\colon X \to Y$ is a map with $d(f(x), f(y)) \le b$ whenever $d(x,y) \le q$. Then $d(f(x), f(y)) \le (ab/q)d(x,y) + b$ for all $x, y \in X$.*

Theorem 4.2. (Theorem 6.12 of [V]) *Suppose that X, Y are metric spaces, $A \subset X$, $f\colon A \to Y$ is η-quasisymmetric, and that $\overline{f(A)}$ is complete. Then f extends to an η-quasisymmetric map $g\colon \overline{A} \to Y$.*

Let $L > 0$ and $A \ge 0$. A map $f\colon X \to Y$ between two metric spaces is an (L, A) *quasi-isometry* if the following two conditions are satisfied:

(1) $d(x_1,x_2)/L - A \le d(f(x_1), f(x_2)) \le L\, d(x_1,x_2) + A$ holds for all $x_1, x_2 \in X$;
(2) For each $y \in Y$, there is some $x \in X$ with $d(f(x), y) \le A$.

Lemma 4.3. *Let (X_i, d_i) $(i = 1, 2)$ be proper metric spaces, $\Omega_i \subset X_i$ rectifiably connected domains with $\partial\Omega_i \ne \emptyset$, and $g\colon \Omega_1 \to \Omega_2$ an η-quasisymmetric map. Suppose (Ω_i, d_i) are (λ_i, c_i)-quasiconvex with $0 < \lambda_i \le 1$ and $c_i \ge 1$. Let k_i be the quasihyperbolic metric on $\Omega_i \subset (X_i, d_i)$. Then the map $g\colon (\Omega_1, k_1) \to (\Omega_2, k_2)$ is an (L, A) quasi-isometry with L and A depending only on λ_1, λ_2, c_1, c_2 and η.*

Proof. By symmetry we only need to show that there exist constants L and A depending only on η, λ_2 and c_2 such that $k_2(g(x), g(y)) \le L\, k_1(x, y) + A$ for all $x, y \in \Omega_1$. Proposition 2.1 and the remark after it imply that (Ω_1, k_1) is a geodesic space. By Lemma 4.1 it suffices to find a constant q depending only on η, λ_2 and c_2 such that $k_2(g(x), g(y)) \le \lambda_2$ whenever $k_1(x, y) \le q$. Let $q = \log[1 + \eta^{-1}(\frac{\lambda_2}{2c_2})]$. Then $\eta(e^q - 1) = \frac{\lambda_2}{2c_2}$. Notice that q depends only on η, λ_2 and c_2. We next show that q has the required property.

Since X_i are proper for $i = 1, 2$ and $g\colon (\Omega_1, d_1) \to (\Omega_2, d_2)$ is η-quasisymmetric, Theorem 4.2 implies that g extends to an η-quasisymmetric homeomorphism $(\overline{\Omega}_1, d_1) \to (\overline{\Omega}_2, d_2)$, which is still denoted by g. Let $x, y \in \Omega_1$ with $k_1(x, y) \le q$. Then

$$q \ge k_1(x, y) \ge \log\left(1 + \frac{d_1(x, y)}{d_1(x) \wedge d_1(y)}\right) \ge \log\left(1 + \frac{d_1(x, y)}{d_1(x)}\right),$$

where $d_i(z) = d_i(z, \partial\Omega_i)$ for $z \in \Omega_i$. It follows that $\frac{d_1(x,y)}{d_1(x)} \le e^q - 1$. Let $z \in \partial\Omega_1$ with $d_2(g(x)) = d_2(g(x), g(z))$. Since g is η-quasisymmetric, we have

$$\frac{d_2(g(x), g(y))}{d_2(g(x))} = \frac{d_2(g(x), g(y))}{d_2(g(x), g(z))} \le \eta\left(\frac{d_1(x, y)}{d_1(x, z)}\right) \le \eta\left(\frac{d_1(x, y)}{d_1(x)}\right) \le \eta(e^q - 1) = \frac{\lambda_2}{2c_2}.$$

Since (Ω_2, d_2) is (λ_2, c_2)-quasiconvex, Lemma 3.3 implies $k_2(g(x), g(y)) \le \lambda_2$. □

We also need the following result (Theorem 6.14 of [V]).

Theorem 4.4. *Suppose that X is a connected metric space and that $f\colon X \to Y$ is η-quasisymmetric. Then f is η_1-quasisymmetric for a function of the form $\eta_1(t) = C(t^\alpha \vee t^{\frac{1}{\alpha}})$, where $C > 0$ and $\alpha \in (0, 1]$ depend only on η.*

Lemma 4.5. *Let (X_i, d_i) $(i = 1, 2)$ be proper metric spaces, $\Omega_i \subset X_i$ domains with $\partial\Omega_i \ne \emptyset$, and $g\colon \Omega_1 \to \Omega_2$ an η-quasisymmetric map. Then there are constants $a > 0$ and $b > 0$ depending only on η such that the following hold:*

$$j_{\Omega_2}(g(x_1), g(y_1)) \le a\, j_{\Omega_1}(x_1, y_1) + b \quad \text{for all } x_1, y_1 \in \Omega_1$$

and

$$j_{\Omega_1}(g^{-1}(x_2), g^{-1}(y_2)) \le a\, j_{\Omega_2}(x_2, y_2) + b \quad \text{for all } x_2, y_2 \in \Omega_2.$$

Proof. By symmetry it suffices to prove that there are constants a and b depending only on η such that $j_{\Omega_1}(g^{-1}(x_2), g^{-1}(y_2)) \le a\, j_{\Omega_2}(x_2, y_2) + b$ for all $x_2, y_2 \in \Omega_2$. To do so, we shall find constants $c > 1$ and $d \ge 1$ depending only on η such that $1 + r_{\Omega_1}(g^{-1}(x_2), g^{-1}(y_2)) \le c\,(1 + r_{\Omega_2}(x_2, y_2))^d$ for all $x_2, y_2 \in \Omega_2$.

By Theorem 4.2, g extends to an η-quasisymmetric homeomorphism $\bar{g}\colon (\overline{\Omega}_1, d_1) \to (\overline{\Omega}_2, d_2)$. Then (see Theorem 6.3 of [V]) $\bar{g}^{-1}\colon (\overline{\Omega}_2, d_2) \to (\overline{\Omega}_1, d_1)$ is η'-quasisymmetric with $\eta'(t) = \eta^{-1}(t^{-1})^{-1}$. By Theorem 4.4, the map $\bar{g}^{-1}$ is η_1-quasisymmetric with $\eta_1(t) = C(t^\alpha \vee t^{\frac{1}{\alpha}})$, where $C > 0$ and $\alpha \in (0,1]$ depend only on η. Set $c = 1 + C$ and $d = \frac{1}{\alpha}$.

Fix $x_2, y_2 \in \Omega_2$ and set $r_1 = r_{\Omega_1}(g^{-1}(x_2), g^{-1}(y_2))$, $r_2 = r_{\Omega_2}(x_2, y_2)$. We may assume $d_1(g^{-1}(x_2)) \le d_1(g^{-1}(y_2))$. Pick $w \in \partial\Omega_1$ with $d_1(g^{-1}(x_2)) = d_1(g^{-1}(x_2), w)$. Since $\bar{g}(w) \in \partial\Omega_2$, we have $d_2(x_2, \bar{g}(w)) \ge d_2(x_2) \ge d_2(x_2) \wedge d_2(y_2)$. Now

$$\begin{aligned} r_1 &= \frac{d_1(g^{-1}(x_2), g^{-1}(y_2))}{d_1(g^{-1}(x_2)) \wedge d_1(g^{-1}(y_2))} = \frac{d_1(g^{-1}(x_2), g^{-1}(y_2))}{d_1(g^{-1}(x_2), w)} \\ &\le \eta_1\left(\frac{d_2(x_2, y_2)}{d_2(x_2, \bar{g}(w))}\right) \le \eta_1\left(\frac{d_2(x_2, y_2)}{d_2(x_2) \wedge d_2(y_2)}\right) = \eta_1(r_2), \end{aligned}$$

that is, $r_1 \le \eta_1(r_2)$.

If $r_2 \le 1$, then $r_1 \le \eta_1(r_2) \le \eta_1(1) = C$, hence $1 + r_1 \le 1 + C = c \le c(1 + r_2)^d$. If $r_2 \ge 1$, then $r_1 \le \eta_1(r_2) = C r_2^d$. It follows that $1 + r_1 \le 1 + C r_2^d \le c(1 + r_2^d) \le c(1 + r_2)^d$. □

Proof of Theorem 1.3. The assumptions imply that (Ω_i, d_i) $(i = 1, 2)$ are rectifiably connected, (Ω_1, d_1) is $(1/2, c_1)$-quasiconvex and (Ω_2, d_2) is (λ, c_2)-quasiconvex. Fix $x_2, y_2 \in \Omega_2$. By Lemma 4.3, $k_2(x_2, y_2) \le L\, k_1(g^{-1}(x_2), g^{-1}(y_2)) + A$, where $L = L(\eta, \lambda, c_1, c_2)$ and $A = A(\eta, \lambda, c_1, c_2)$. Since (Ω_1, d_1) is c_1-uniform, Theorem 3.6 implies

$$k_1(g^{-1}(x_2), g^{-1}(y_2)) \le c'\, j_{\Omega_1}(g^{-1}(x_2), g^{-1}(y_2)),$$

where $c' = c'(c_1)$. On the other hand, by Lemma 4.5,

$$j_{\Omega_1}(g^{-1}(x_2), g^{-1}(y_2)) \le a\, j_{\Omega_2}(x_2, y_2) + b,$$

where $a = a(\eta)$, $b = b(\eta)$. Combining the above inequalities we have $k_2(x_2, y_2) \le a'\, j_{\Omega_2}(x_2, y_2) + b'$ for all $x_2, y_2 \in \Omega_2$, where a', b' depend only on η, λ, c_1 and c_2. Since (Ω_2, d_2) is (λ, c_2)-quasiconvex, Theorem 3.6 implies that (Ω_2, d_2) is c-uniform with $c = c(\eta, \lambda, c_1, c_2)$. □

In the proof of the following lemma, we shall implicitly use the inequality $\frac{1}{4} s_p(x, y) \le \hat{d}_p(x, y) \le s_p(x, y)$ (see Section 2).

Lemma 4.6. *Let (X, d) be an unbounded proper metric space, $\Omega \subset X$ an unbounded domain, $p \in \partial\Omega$, and $0 < \lambda \le 1$, $c \ge 1$. If $\Omega \subset (X, d)$ is (λ, c)-quasiconvex, then $\Omega \subset (X, \hat{d}_p)$ is (λ', c')-quasiconvex with $\lambda' = \lambda'(\lambda, c)$ and $c' = c'(\lambda, c)$.*

Proof. Set $\lambda' = \frac{\lambda}{10000c^2}$ and $c' = 64c$. Fix $x \in \Omega$. Let $\hat{d}_p(x) = \hat{d}_p(x, \hat{\partial}\Omega)$ denote the $\hat{d}_p$-distance from x to $\hat{\partial}\Omega$, where $\hat{\partial}\Omega = \partial\Omega \cup \{\infty\}$ is the boundary of Ω in $(X \cup \{\infty\}, \hat{d}_p)$. For $r > 0$, let $\hat{B}_p(x, r) := \{y \in S_p(X) : \hat{d}_p(x, y) < r\}$.

Claim: $\hat{B}_p(x, \lambda' \hat{d}_p(x)) \subset B(x, \frac{\lambda}{10c} d(x))$.

We first assume the claim and complete the proof of the lemma. Let $y_1, y_2 \in \hat{B}_p(x, \lambda' \hat{d}_p(x))$. Since $\Omega \subset (X, d)$ is (λ, c)-quasiconvex, the claim implies that there is a path γ from y_1 to y_2 such that $\ell(\gamma) \le c\, d(y_1, y_2)$. The claim further implies $d(y, x) \le 3d(x)/10$ for all $y \in \gamma$. Since $p \in \partial\Omega$, we have $d(x, p) \ge d(x)$. It follows that for any $y \in \gamma$ we have $d(x,p)/2 \le d(y,p) \le 2d(x,p)$. Hence $(1 + d(x,p))/2 \le 1 + d(y,p) \le 2(1 + d(x,p))$. Because

$$\frac{d(z_1, z_2)}{4(1 + d(z_1, p))(1 + d(z_2, p))} \le \hat{d}_p(z_1, z_2) \le \frac{d(z_1, z_2)}{(1 + d(z_1, p))(1 + d(z_2, p))}$$

for all $z_1, z_2 \in X$, we conclude that

$$\hat{\ell}_p(\gamma) \le \frac{4\ell(\gamma)}{(1 + d(x,p))^2} \quad \text{and} \quad \hat{d}_p(y_1, y_2) \ge \frac{d(y_1, y_2)}{16(1 + d(x,p))^2},$$

where $\hat{\ell}_p(\gamma)$ denotes the $\hat{d}_p$-length of γ. Together with $\ell(\gamma) \le c\, d(y_1, y_2)$, the above two inequalities imply

$$\hat{\ell}_p(\gamma) \le 64c\hat{d}_p(y_1, y_2) = c' \hat{d}_p(y_1, y_2).$$

We have shown that Ω is (λ', c')-quasiconvex in $(X, \hat{d}_p)$.

Next we prove the claim. Let $y \in \hat{B}_p(x, \lambda' \hat{d}_p(x))$. We need to prove $d(x, y) < \frac{\lambda}{10c} d(x)$. There is some $w \in \partial\Omega$ with $d(x) = d(x, w)$ and some $z \in \hat{\partial}\Omega$ with $\hat{d}_p(x) = \hat{d}_p(x, z)$. We consider two cases depending on whether $z = \infty$ or not.

Case (1). $z = \infty$. Then $\hat{d}_p(x) = \hat{d}_p(x, \infty) \le \frac{1}{1 + d(x,p)}$. The fact that $y \in \hat{B}_p(x, \lambda' \hat{d}_p(x))$ now implies $d(x, y) \le 4\lambda'(1 + d(y, p))$. Since $w \in \partial\Omega$, we have $\hat{d}_p(x, w) \ge \hat{d}_p(x, \infty)$, which implies that $d(x, w) \ge (1 + d(w, p))/4$. If $d(y, p) \le 1$, then

$$\begin{aligned} d(x,y) &\le 4\lambda'(1 + d(y,p)) \le 8\lambda' = \frac{8\lambda}{10000c^2} < \frac{\lambda}{40c} \\ &\le \frac{\lambda}{10c} \frac{1 + d(w,p)}{4} \le \frac{\lambda}{10c} d(x, w) = \frac{\lambda}{10c} d(x), \end{aligned}$$

and we are done. Now assume $d(y, p) \ge 1$. In this case we shall prove $d(x, y) < \frac{\lambda}{10c} d(x)$ by contradiction. So we suppose $d(x, y) \ge \frac{\lambda}{10c} d(x)$. We have $d(x, y) \le 4\lambda'(1 + d(y, p)) \le 8\lambda' d(y, p)$. Hence $d(y, p)/2 \le d(x, p) \le 2\, d(y, p)$ and

$$d(x, w) = d(x) \le d(x, y) \frac{10c}{\lambda} \le 8\lambda' d(y, p) \frac{10c}{\lambda} \le \frac{d(y, p)}{100} \le \frac{d(x, p)}{50}.$$

It follows that $d(w, p) \ge d(x, p)/2$, and therefore

$$d(x, w) \ge (1 + d(w, p))/4 \ge d(w, p)/4 \ge d(x, p)/8,$$

contradicting $d(x, w) \le d(x, p)/50$.

Case (2). $z \in \partial\Omega$. The inequalities $\hat{d}_p(x,y) \le \lambda' \hat{d}_p(x,z)$, $\hat{d}_p(x,z) \le \hat{d}_p(x,w)$ and $\hat{d}_p(x,z) \le \hat{d}_p(x,\infty)$ yield the inequalities

$$\frac{d(x,y)}{4(1+d(y,p))} \le \lambda' \frac{d(x,z)}{1+d(z,p)}, \quad \frac{d(x,z)}{(1+d(z,p))} \le \frac{4d(x,w)}{1+d(w,p)} \quad \text{and} \quad \frac{d(x,z)}{(1+d(z,p))} \le 4$$

respectively. If $d(y,p) \le 1$, then

$$\begin{aligned} d(x,y) &\le 4(1+d(y,p)) \cdot \lambda' \frac{d(x,z)}{1+d(z,p)} \le 8\lambda' \frac{4d(x,w)}{1+d(w,p)} \\ &\le 32\lambda' d(x,w) < \frac{\lambda}{10c} d(x,w) = \frac{\lambda}{10c} d(x). \end{aligned}$$

On the other hand, if $d(y,p) \ge 1$, then

$$d(x,y) \le 4(1+d(y,p)) \cdot \lambda' \frac{d(x,z)}{1+d(z,p)} \le 8d(y,p) \cdot \lambda' \cdot 4 = 32\lambda' d(y,p) \le d(y,p)/10.$$

If $d(x,w) \ge \frac{d(y,p)}{10c}$, then $d(x,y) \le 32\lambda' d(y,p) \le 32\lambda' 10c d(x,w) < \frac{\lambda}{10c} d(x,w)$, and we are done. If $d(x,w) \le \frac{d(y,p)}{10c}$, then $d(w,y) \le d(w,x) + d(x,y) \le d(y,p)/5$ and hence $d(w,p) \ge d(y,p)/2$. It follows that

$$\begin{aligned} d(x,y) &\le 4(1+d(y,p)) \cdot \lambda' \frac{d(x,z)}{1+d(z,p)} \le 8d(y,p) \cdot \lambda' \cdot \frac{4d(x,w)}{1+d(w,p)} \\ &\le 32\lambda' d(y,p) \frac{d(x,w)}{d(w,p)} \le 32\lambda' \cdot 2d(x,w) = 64\lambda' d(x,w) \\ &< \frac{\lambda}{10c} d(x,w) = \frac{\lambda}{10c} d(x). \quad \square \end{aligned}$$

Proof of Theorem 1.1. Let $i = 1$ or 2. If (Ω_i, d_i) is bounded, let $d'_i = d_i$; and if (Ω_i, d_i) is unbounded, pick $p_i \in \partial\Omega_i$ and set $d'_i := \hat{d}_{ip_i}$. Recall that the identity map $(X_i, d_i) \to (X_i, d'_i)$ is quasimöbius. As (Ω_i, d'_i) is bounded, a quasimöbius map $g\colon (\Omega_1, d_1) \to (\Omega_2, d_2)$ becomes a quasisymmetric map $g\colon (\Omega_1, d'_1) \to (\Omega_2, d'_2)$. Since (Ω_1, d_1) is uniform, Theorem 2.3 implies that (Ω_1, d'_1) is uniform. On the other hand, (Ω_2, d_2) is (λ, c_2)-quasiconvex. By Lemma 4.6, (Ω_2, d'_2) is (λ', c')-quasiconvex for some $0 < \lambda' \le 1$ and $c' \ge 1$. Now it follows from Theorem 1.3 that (Ω_2, d'_2) is uniform. Hence (Ω_2, d_2) is also uniform by Theorem 2.3. $\square$

Proof of Theorem 1.2 (2). Let $i = 1$ or 2. If Ω_i is bounded, set $X'_i = X_i$ and $d'_i = d_i$; if Ω_i is unbounded, then fix any base point $p_i \in \partial\Omega_i$ and set $X'_i = S_{p_i}(X_i)$ and $d'_i = \hat{d}_{ip_i}$. Denote by $\partial\Omega'_i$ the boundary of Ω_i in (X'_i, d'_i) and $\overline{\Omega}'_i$ the closure of Ω_i in (X'_i, d'_i). Let $f_i\colon (\Omega_i, d_i) \to (\Omega_i, d'_i)$ be the identity map and set

$$h' := f_2 \circ h \circ f_1^{-1}\colon (\Omega_1, d'_1) \to (\Omega_2, d'_2).$$

Let $\eta_0(t) = 16t$. Then h' is an $\eta' := \eta_0 \circ \eta \circ \eta_0$-quasimöbius homeomorphism between bounded metric spaces, and hence is a quasisymmetric map. By Theorem 4.2, the map h' extends continously to a homeomorphism $(\overline{\Omega}'_1, d'_1) \to (\overline{\Omega}'_2, d'_2)$, which is still denoted by h'. In particular, there exist $a_1 \in \partial\Omega'_1$, $a_2 \in \partial\Omega'_2$ such that for any

$\{x_i\} \subset \Omega_1$ with $x_i \to a_1$ we have $h'(x_i) \to a_2$. If $\partial\Omega'_1$ is a single point, then $\partial\Omega'_2$ and $\partial\Omega_2$ are also single points. Since (X_2, d_2) is c_2-quasiconvex and c_2-annular convex, the fact that $\partial\Omega_2$ is a single point implies that (Ω_2, d_2) is $6c_2^2$-uniform (see Lemma 9.4 of [HSX]). From now on, we assume $\partial\Omega'_1$ contains at least two points.

Now we fix $a_1 \in \partial\Omega'_1, a_2 \in \partial\Omega'_2$ such that for any $\{x_i\} \subset \Omega_1$ with $x_i \to a_1$ we have $h'(x_i) \to a_2$. Let $X''_i = I_{a_i}(X'_i) = X'_i\backslash\{a_i\}$, $d''_i = (d'_i)_{a_i}$. Denote by $\partial\Omega''_i$ the boundary of Ω_i in (X''_i, d''_i) and $\overline{\Omega}''_i$ the closure of Ω_i in (X''_i, d''_i). Note that $\partial\Omega''_i = \partial\Omega'_i\backslash\{a_i\}$ and $\overline{\Omega}''_i = \overline{\Omega}'_i\backslash\{a_i\}$ as sets. Let $g_i\colon (X'_i\backslash\{a_i\}, d'_i) \to (X'_i\backslash\{a_i\}, d''_i)$ be the identity map and set

$$h'' := g_2 \circ h' \circ g_1^{-1} : (\overline{\Omega}''_1, d''_1) \to (\overline{\Omega}''_2, d''_2).$$

Since g_i is η_0-quasimöbius, h'' is η''-quasimöbius, where $\eta'' := \eta_0 \circ \eta' \circ \eta_0$. The choice of a_1 and a_2 implies that for any $\{x_i\} \subset \Omega_1$ with $d''_1(x_i, x_1) \to \infty$ we have $d''_2(h''(x_i), h''(x_1)) \to \infty$. It follows that h'' is an η''-quasisymmetric homeomorphism.

Since (Ω_1, d_1) is c_1-uniform, Theorem 2.3 (1) implies that (Ω_1, d'_1) is c'_1-uniform with $c'_1 = c'_1(c_1)$. Since $\partial\Omega'_1$ contains at least two points and $a_1 \in \partial\Omega'_1$, it follows from Theorem 2.4 (1) that (Ω_1, d''_1) is c''_1-uniform with $c''_1 = c''_1(c'_1) = c''_1(c_1)$. On the other hand, since (X_2, d_2) is c_2-quasiconvex and c_2-annular convex, it follows from Section 2 that (X''_2, d''_2) is c''_2-quasiconvex with $c''_2 = c''_2(c_2)$. Therefore, $\Omega_2 \subset (X''_2, d''_2)$ is $(1/(3c''_2), c''_2)$-quasiconvex.

Now Theorem 1.3 applied to h'' implies that (Ω_2, d''_2) is c'-uniform with $c' = c'(1/(3c''_2), c''_2, c''_1, \eta'') = c'(c_1, c_2, \eta)$. Now the result follows from Theorem 2.4 (2) and Theorem 2.3 (2). □

Proof of Theorem 1.2 (1). The proof is similar to that of Theorem 1.2 (2), and we only indicate what should be modified. By the assumption of Theorem 1.2 (1) (Ω_2, d_2) is unbounded, hence $\infty \in \partial\Omega'_2$. We choose $a_2 = \infty$ and $a_1 = (h')^{-1}(\infty)$. The proof of Theorem 1.2 (2) shows that (Ω_1, d''_1) is c''_1-uniform with $c''_1 = c''_1(c_1)$ and h'' is an η''-quasisymmetric homeomorphism with $\eta'' = \eta''(\eta)$.

Since $a_2 = \infty$, Lemma 2.5 implies that the identity map $(X_2, d_2) \to (X_2, d''_2)$ is 16-bilipschitz. Now the fact that (Ω_2, d_2) is (λ, c_2)-quasiconvex implies that (Ω_2, d''_2) is (λ'', c''_2)-quasiconvex with $\lambda'' = \lambda/256$ and $c''_2 = 256c_2$.

Now Theorem 1.3 applied to h'' implies that (Ω_2, d''_2) is c'-uniform with $c' = c'(\lambda'', c''_2, c''_1, \eta'') = c'(\lambda, c_1, c_2, \eta)$. Now the result follows from Lemma 2.5. □

5. Example and open questions

In this Section we give an example that shows Theorem 1.1 can not be made quantitative, and present two related questions.

Let $\mathbf{R}^n$ be the n-dimensional Euclidean space and $\tau\colon \mathbf{R}^n\backslash\{p\} \to \mathbf{R}^n\backslash\{p\}$, $\tau(x) = x/|x|^2$ the inversion about the unit sphere centered at the origin p. Let d denote the Euclidean metric. We can define a new metric d' on $\mathbf{R}^n\backslash\{p\}$ by pulling back the Euclidean metric via τ: $d'(x,y) = d(\tau(x), \tau(y))$. One checks that

$$d'(x,y) = \frac{d(x,y)}{d(x,p)d(y,p)}.$$

Let $A \subset \mathbf{R}^n$ be a subset containing p, and consider the metric spaces (A, d) and $(A\backslash\{p\}, d_p)$. Notice that $f_p(x,y) = d'(x,y)$ for all $x, y \in A\backslash\{p\}$. Since d' is a metric on $\mathbf{R}^n\backslash\{p\}$, f_p is a metric on $A\backslash\{p\}$. Now the definition of d_p and the triangle inequality show that $d_p = f_p$ on $A\backslash\{p\}$. It follows that for any $x, y \in A\backslash\{p\}$, we have $d_p(x,y) = f_p(x,y) = d'(x,y) = d(\tau(x), \tau(y))$; that is, $\tau\colon (A\backslash\{p\}, d_p) \to (\tau(A\backslash\{p\}), d)$ is an isometry.

Now consider $\mathbf{R}^2$. We identity $\mathbf{R}^2$ with $\mathbf{C}$ and use complex number notations. For $0 < u < \pi/2$, let $X = \{\frac{1}{2}(i + e^{i\theta}) : -\pi/2 \le \theta \le 3\pi/2 - u\}$ and $\Omega = X\backslash\{p, q\}$, where $q = \frac{1}{2}(i + e^{i(3\pi/2-u)})$. One checks that $\Omega \subset (X, d)$ is a $(1/2, \pi)$-quasiconvex domain. By Section 2, the identity map $(X\backslash\{p\}, d) \to (X\backslash\{p\}, d_p)$ is η-quasimöbius with $\eta(t) = 16t$. By the preceding paragraph, $(X\backslash\{p\}, d_p)$ is isometric to $(\tau(X\backslash\{p\}), d)$. Set $u' = \frac{\cos(3\pi/2-u)}{1+\sin(3\pi/2-u)}$. We notice that $\tau(X\backslash\{p\}) = \{x + i : u' \le x < \infty\}$ is a ray and $\tau(\Omega) = \{x + i : u' < x < \infty\}$. It is now clear that $\Omega \subset (X\backslash\{p\}, d_p)$ is 1-uniform. On the other hand, by considering two points in Ω close to p and q, we see that the uniformity constant of $\Omega \subset (X, d)$ is in the order of $1/u$, which tends to infinity as $u \to 0$. This example shows that Theorem 1.1 can not be made quantitative.

In view of the above example and the Theorems in this paper, it is natural to ask the following question:

Question 5.1. Let (X_i, d_i) *($i = 1, 2$)* be proper metric spaces, and $\Omega_i \subset X_i$ domains with $\partial\Omega_i \ne \emptyset$. Suppose (Ω_1, d_1) is c_1-uniform, (Ω_2, d_2) is bounded and c_2-quasiconvex, and there is an η-quasimöbius homeomorphism $(\Omega_1, d_1) \to (\Omega_2, d_2)$. By Theorem 1.1, (Ω_2, d_2) is c-uniform for some constant c. Is it possible to obtain an upper bound for c in terms only of c_1, c_2 and η?

A special case of Question 5.1 is the following:

Question 5.2. Let (X, d) be a proper metric space, $\Omega \subset X$ a bounded domain and $p \in \partial\Omega$. Assume $\partial\Omega$ contains at least two points, (Ω, d_p) is c_1-uniform and (Ω, d) is c_2-quasiconvex. By Theorem 2.4, (Ω, d) is c-uniform for some constant c. Is it possible to obtain an upper bound for c in terms only of c_1 and c_2?

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