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THE CALCULUS OF CONFORMAL METRICS

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Abstract. Minda and Peschl invented a kind of derivative of maps between Riemann surfaces, which depends on the choice of conformal metric. We give explicit formulas relating the Minda–Peschl derivatives to the Levi–Civita connection, which express the difference between the two in terms of the curvature of the metric. Furthermore, we exhibit a geometric interpretation of the derivatives in terms of a decomposition of the space of symmetric complex differentials. Finally, this decomposition is used to give simple formulas for parallel transport of complex differentials which hold for conformal metrics on a Riemann surface.

1. Introduction

Peschl introduced to function theory the idea of invariant derivatives associated to certain conformal metrics [13]. Later, Minda generalized these derivatives to arbitrary conformal metrics although this was not published [11]. His generalization used an inductive definition of a kind of connection. We will refer to these general derivatives as the Minda–Peschl derivatives.

Their usefulness stems from the fact that theorems regarding classes of holomorphic mappings from a fixed domain into another can sometimes be better phrased in terms of canonical metrics on these domains. Indeed, special cases of the derivatives, especially for the Euclidean, hyperbolic and spherical metrics, have been used many times in function theory for this reason (e.g. [1], [6], [10]). Minda's connection also appears in conformal field theory (cf. [12] Chapter 14 and references therein).

The Minda–Peschl derivatives of course have a geometric interpretation. However, they are not simply derivatives with respect to the familiar Levi–Civita connection of Riemannian geometry, as one might expect. They are rather derivatives with respect to the Levi–Civita connection followed by a projection onto a certain space of differentials. This interpretation was observed independently by Kim and Sugawa [5] and myself [14], [15]. Kim and Sugawa also give closed forms for the higher-order invariant derivatives for arbitrary conformal metrics in terms of Bell polynomials, and showed that they satisfy a Faà di-Bruno formula.

The contribution of the present paper is the following. If the metric has curvature, then the higher-order Minda–Peschl derivatives of a holomorphic function differ from the derivatives with respect to the Levi–Civita connection. We supply 1) an inductive formula relating the Minda–Peschl and the Levi–Civita derivatives

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of a holomorphic function, where the difference is given explicitly in terms of the curvature of the metric (Theorem 4.7), and 2) the closed-form relation between the Levi–Civita and Minda–Peschl derivatives for arbitrary orders of differentiation for a family of constant curvature metrics (Theorem 4.15). This family includes the hyperbolic, Euclidean and spherical metrics.

An essential part of the picture is the space of symmetric tensors and its decomposition into terms according to the number of dz's and $d\bar{z}$'s. This is analogous to the decomposition of antisymmetric tensors (forms) into types based on the number of dz's and $d\bar{z}$'s. In a sense Minda's connection bears the same relation to symmetric tensors as ∂ and $\overline{\partial}$ bear to forms.

Another aim of this paper is to make the Minda–Peschl derivatives accessible to function theorists, by giving a coherent framework for computation. The relation to the Levi–Civita connection is necessary for this, as is the formalism of parallel transport, which allows one to convert the Levi–Civita derivative into an ordinary limit expression. As a consequence of the decomposition of symmetric tensors into spaces of differentials, the parallel transport map between two points along a curve considerably simplifies and can in fact be represented by a single complex number. Although this observation is fairly simple, it does not appear in standard references on Riemannian geometry or complex function theory—neither, for that matter, does Minda's connection—so it seemed appropriate to include it here. Expressions for the parallel transport map for particular metrics which commonly appear in function theory are also provided.

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2. Spaces of differentials

In this section, we define certain spaces of symmetric tensors, and present a decomposition of the symmetric tensors into certain spaces of differentials. This is analogous to the decomposition of complex differential forms on a complex manifold into terms of the form $dz_{i_1} \wedge \cdots \wedge dz_{i_r} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_s}$. The decomposition into symmetric tensors is essential to understanding Minda's connection and higher-order derivatives of a holomorphic function in conformal metrics.

In general, the higher-order derivatives of a map between manifolds M are symmetric tensors. It must be observed that differential forms are inappropriate for representing higher-order derivatives of functions, as can be inferred from the fact that $\partial^2 = 0$ and $\bar{\partial}^2 = 0$. Minda's connections ∇ and $\overline{\nabla}$ (which will be defined in the next section) play the same role for symmetric tensors as ∂ and $\bar{\partial}$ do for antisymmetric tensors.

The necessary algebraic machinery will be carefully developed shortly, but first we briefly give some motivation. Let f be a complex-valued function on an open subset of $\mathbf{C}$. Disregarding for a moment the complex structure and treating f as a real function, we have that its nth derivative at a point p is a symmetric map $T_p\mathbf{R}^2 \times \cdots \times T_p\mathbf{R}^2 \to T_{f(p)}\mathbf{R}^2$, where $T_p\mathbf{R}^2$ denotes the tangent space of $\mathbf{R}^2$ at p. We can identify the tangent space $T_{f(p)}\mathbf{R}^2$ with $T_p\mathbf{R}^2$ using the first derivative of f. Thus one can identify the nth derivatives of f with a local section of $T\mathbf{R}^2 \otimes T^*\mathbf{R}^2 \otimes \cdots \otimes T^*\mathbf{R}^2$ which is symmetric in the covariant components.

If the function f is holomorphic, then the Cauchy-Riemann equations provide relations between the components of the nth derivative tensor, and these relations are conveniently written in terms of the complex algebra. Of course, we usually think of the higher derivatives of a holomorphic function themselves as functions rather than tensors. However, for general conformal metrics it is not possible to make this simplification (this will be clarified in Remark 4.13 ahead). Thus in the following, given a Riemann surface R, we will consider symmetric local sections of $T^{\mathbf{C}}R \otimes T^{\mathbf{C}*}R \otimes \cdots T^{\mathbf{C}*}R$, where $T^{\mathbf{C}}R$ and $T^{\mathbf{C}*}R$ denote the complexified tangent and cotangent bundles respectively.

Remark 2.1. In order to define even the second derivative of a map from R to $\mathbf{R}$ (the Hessian) in a way which is independent of the choice of coordinates, it is necessary to have a metric with which to identify the tangent spaces at different points. For a map between Riemann surfaces, we need a metric both on the domain and image of the mapping.

We now describe the symmetric tensors ([4] Appendix B.) First some notation is necessary. Let R be a Riemann surface, and let $\mathfrak{X}$ denote the space of complex vector fields over R. Let $\mathfrak{D}^k_l$ denote the complex tensors of covariant degree l and contravariant degree k; i.e. $\mathfrak{D}^k_l$ is the space of sections of $\bigotimes^k T^{\mathbf{C}}R \otimes \bigotimes^l T^{\mathbf{C}*}R$. We will mostly be working with complex tensors. When it is necessary to refer to real vector fields or tensors we will write $\mathfrak{X}^{\mathbf{R}}$ and $\mathfrak{D}^{\mathbf{R}}$ respectively.

Given a vector space V, we can define the n-fold symmetric product $\mathrm{Sym}^n(V)$ of V as the quotient of $\bigotimes^n V$ by the relations $v_1 \otimes \cdots \otimes v_n - v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(2)}$ for any permutation σ of $(1, \ldots, n)$. Let $\pi_s\colon \bigotimes^n V \to \mathrm{Sym}^n(V)$ be the quotient map, and define the symmetric product "$\cdot$" by

$$v_1 \cdot v_2 \cdots v_n = \pi_s(v_1 \otimes v_2 \otimes \cdots \otimes v_n).$$

We can identify $\mathrm{Sym}^n(V)$ with the subspace of $\bigotimes^n V$ consisting of elements of the form

$$\frac{1}{n!}\sum_{\sigma} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

where $v_1, \ldots, v_n \in V$ and σ ranges over all the permutations of $(1, \ldots, n)$. With this identification, we have for example that $v_1 \cdot v_2 = (v_1 \otimes v_2 + v_2 \otimes v_1)/2$.

Example 2.1. One often sees the expression $\rho^2|dz|^2$ for a conformal metric, by which is meant the symmetric product

$$\rho(z)^2|dz|^2 = \frac{1}{2}\rho(z)^2\left(dz \otimes d\bar{z} + d\bar{z} \otimes dz\right) = \rho(z)^2\left(dx \otimes dx + dy \otimes dy\right).$$

Note that $\rho(z)^2 dz \otimes d\bar{z}$ is not symmetric and so is not a Riemannian metric.

We then have the following symmetric tensor spaces.

Definition 2.2. Let $\mathfrak{S}^k_l$ denote the complex symmetric tensors of contravariant degree k and covariant degree l; i.e. the space of sections of $\mathrm{Sym}^k(\mathrm{T}^{\mathbf{C}}R) \otimes \mathrm{Sym}^l(\mathrm{T}^{\mathbf{C}*}R)$.

We require a decomposition of the symmetric product of a direct sum: for vector spaces V and W, there is a canonical identification [4]

$$\mathrm{Sym}^n(V \oplus W) \cong \sum_{k=0}^{n} \mathrm{Sym}^k(V) \otimes \mathrm{Sym}^{n-k}(W).$$

Regarding the right hand side as a subset of $\mathrm{Sym}^n(V \oplus W) \subset \bigotimes^n(V \oplus W)$ and carrying the product "$\cdot$" into this space as above, we can write

$$\mathrm{Sym}^n(V \oplus W) = \sum_{k=0}^{n} \mathrm{Sym}^k(V) \cdot \mathrm{Sym}^{n-k}(W).$$

In particular, applying this to the decompositions of the complexified tangent and cotangent spaces $\mathrm{T}^{\mathbf{C}}_p\mathrm{R} = \mathrm{T}^{(1,0)}_p\mathrm{R} \oplus \mathrm{T}^{(0,1)}_p\mathrm{R}$ and $\mathrm{T}^{\mathbf{C}*}_p\mathrm{R} = \mathrm{T}^{(1,0)*}_p\mathrm{R} \oplus \mathrm{T}^{(0,1)*}_p\mathrm{R}$ for all $p \in R$, we have the decomposition of tensor bundles

$$\begin{aligned} \mathrm{Sym}^n(\mathrm{T}^{\mathbf{C}}\mathrm{R}) &= \sum_{k=0}^{n} \mathrm{Sym}^k(\mathrm{T}^{(1,0)}\mathrm{R}) \cdot \mathrm{Sym}^{n-k}(T^{(0,1)}\mathrm{R}) \\ \mathrm{Sym}^n(\mathrm{T}^{\mathbf{C}*}\mathrm{R}) &= \sum_{k=0}^{n} \mathrm{Sym}^k(\mathrm{T}^{(1,0)*}\mathrm{R}) \cdot \mathrm{Sym}^{n-k}(\mathrm{T}^{(0,1)*}\mathrm{R}) \end{aligned}$$

Definition 2.3. Let $\mathscr{D}^{r,s}_{m,n}$ be the space of complex differentials of covariant type (m,n) and contravariant type (r,s); i.e. the space of sections of $\mathrm{Sym}^r(\mathrm{T}^{(1,0)}\mathrm{R}) \cdot \mathrm{Sym}^s(\mathrm{T}^{(0,1)}\mathrm{R}) \otimes \mathrm{Sym}^m(\mathrm{T}^{(1,0)*}\mathrm{R}) \cdot \mathrm{Sym}^n(\mathrm{T}^{(0,1)*}\mathrm{R})$.

We have thus shown that

$$\mathfrak{S}^k_l = \bigoplus_{r+s=k, m+n=l} \mathscr{D}^{r,s}_{m,n}. \tag{2.1}$$

Remark 2.4. In terms of a locally biholomorphic parameter z, $\mathscr{D}^{r,s}_{m,n}$ consists of tensors of the form

$$h(z)\frac{\partial^r}{\partial z^r} \cdot \frac{\partial^s}{\partial \bar{z}^s} \otimes dz^m \cdot d\bar{z}^n.$$

For example, a Beltrami differential on R is an element of $\mathscr{D}^{1,0}_{0,1}$, and a conformal metric is an element of $\mathscr{D}^{0,0}_{1,1}$ (see Example 2.1).

In the rest of the paper, the symmetric product will often be written without a dot; a product of differentials should always be taken to mean the symmetric product unless otherwise noted. Furthermore we will abbreviate $dz \cdot dz \cdots dz$ by

dz^n, and similarly $\partial^r/\partial z^r$ represents a product of r factors of $\partial/\partial z$. Thus for example $dz \cdot d\bar{z} \cdot dz = dz^2 d\bar{z}$.

Definition 2.5. There is a natural multiplication $\mathscr{D}^{r_1,s_1}_{m_1,n_1} \times \mathscr{D}^{r_2,s_2}_{m_2,n_2} \to \mathscr{D}^{r_1+r_2,s_1+s_2}_{m_1+m_2,n_1+n_2}$, which in a local coordinate z is given by

$$\left(h_1(z)\frac{\partial^{r_1}}{\partial z^{r_1}} \cdot \frac{\partial^{s_1}}{\partial \bar{z}^{s_1}} \otimes dz^{m_1} \cdot d\bar{z}^{n_1}\right) \cdot \left(h_2(z)\frac{\partial^{r_2}}{\partial z^{r_2}} \cdot \frac{\partial^{s_2}}{\partial \bar{z}^{s_2}} \otimes dz^{m_2} \cdot d\bar{z}^{n_2}\right)$$
$$= h_1(z)h_2(z)\frac{\partial^{r_1+r_2}}{\partial z^{r_1+r_2}} \cdot \frac{\partial^{s_1+s_2}}{\partial \bar{z}^{s_1+s_2}} \otimes dz^{m_1+m_2} \cdot d\bar{z}^{n_1+n_2}.$$

It is easily checked that this definition is independent of the choice of local coordinate.

Finally, the following remark clarifies the relation between two different local-coordinate representations of tangent vectors.

Remark 2.6. (Representations of tangent vectors in local coordinates) There are two possible representations of tangent vectors in local coordinates, which we clarify here for later use. Let $z = x + iy$ be a locally biholomorphic parameter near a point $p \in R$. We may identify the tangent space $T^{(1,0)}_p R$ at a point p with $\mathbf{C}$ via

$$I_1 \colon T^{(1,0)}_p R \to \mathbf{C}$$
$$(a+ib)\frac{\partial}{\partial z} \mapsto a + ib.$$

On the other hand the real tangent space $T_p R$ can be identified with $\mathbf{C}$ using the map

$$I_2 \colon T_p R \to \mathbf{C}$$
$$a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} \mapsto a + ib.$$

These two representations are compatible under the standard isomorphism $\Phi : T^{(1,0)}_p R \to T_p R$ given by $\Phi(Z) = 2\mathrm{Re}(Z)$ [8] since $I_2 \circ \Phi = I_1$. Note that while I_1 and I_2 depend on the choice of local parameter, Φ does not.

3. The connections

3.1. The Levi–Civita connection. In order that this paper be self-contained, and to set notation, we give a brief outline of necessary facts concerning the Levi–Civita connection. For a detailed presentation see [7].

Let g be a smooth Riemannian metric compatible with the complex structure of R; i.e. if J is the almost complex structure, then $g_p(\mathrm{J}_p X_p, \mathrm{J}_p Y_p) = g_p(X_p, Y_p)$ for any point $p \in \mathrm{R}$ and vectors $X_p, Y_p \in \mathrm{T}_p\mathrm{R}$. Equivalently, if z is a local biholomorphic parameter, then g can be written in the form $\rho(z)^2|dz|^2$ for some smooth non-vanishing real-valued function $\rho(z)$. Metrics which are compatible with the complex structure will be called conformal.

Associated to g is the Levi–Civita connection ∇^L, which is the unique mapping from pairs of vector fields to vector fields $(X,Y) \mapsto \nabla^L_X Y$ which satisfies, for any vector fields X, Y and Z and smooth function f, 1) $\nabla^L_{fX} Y = f\nabla^L_X Y$, 2) $\nabla^L_X fY = (Xf)Y - f\nabla^L_X Y$, 3) $\nabla^L_X Y - \nabla^L_Y X = XY - YX$ (∇^L is torsion-free) and 4) $X(g(Y,Z)) - g(\nabla^L_X Y, Z) - g(Y, \nabla^L_X Z) = 0$ (compatibility with the metric). If the metric is compatible with the almost complex structure J then it can be shown that for any vector fields X and Y

$$\nabla^L_X \mathrm{J} Y = \mathrm{J}\nabla^L_X Y. \tag{3.1}$$

The Levi–Civita connection has an extension to real tensor fields $\nabla^L \colon \mathfrak{X}^{\mathbf{R}} \times \mathfrak{D}^{\mathbf{R}}{}^k_l \to \mathfrak{D}^{\mathbf{R}}{}^k_l$, which satisfies the following properties. 1) $\nabla^L_X f = Xf$ for any smooth function f on R, 2) $\nabla^L_X(\alpha \otimes \beta) = (\nabla^L_X \alpha) \otimes \beta + \alpha \otimes (\nabla^L_X \beta)$. This extension is unique. It has the further property that if C is a contraction of any two indices in $\mathfrak{D}^{\mathbf{R}}{}^k_l$ then $\nabla^L_X C(\alpha) = C(\nabla^L_X \alpha)$ ([7] Proposition 2.7).

For a conformal metric g on a Riemann surface R, there are relations which result in a simple form for the Levi–Civita connection. To exploit them we complex linearly extend ∇^L:

$$\nabla^L_{X_1 + iY_1}(X_2 + iY_2) = \nabla^L_{X_1} X_2 - \nabla^L_{Y_1} Y_2 + i\nabla^L_{X_1} Y_2 + i\nabla^L_{Y_1} X_2$$

for vector fields X_i and Y_i, $i = 1, 2$. We give some formulas for ∇^L in local coordinates. For a locally biholomorphic parameter z in which g has the form $g = \rho^2 |dz|^2$ let the "Christoffel symbol" Γ^ρ be defined by

$$\nabla^L_{\frac{\partial}{\partial z}} \frac{\partial}{\partial z} = \Gamma^\rho \frac{\partial}{\partial z}. \tag{3.2}$$

It can be shown ([8] IX.5) that for a conformal metric

$$\Gamma^\rho = 2\frac{\partial}{\partial z} \log \rho. \tag{3.3}$$

Under a change of parameter $z = f(w)$, so that

$$g = \rho(z)^2 |dz|^2 = \rho \circ f(w)^2 |f'(w)|^2 |dw|^2,$$

the Christoffel symbol transforms according to

$$\Gamma^{\rho \circ f |f'|} = \Gamma^\rho \circ f \cdot f' + \frac{f''}{f'}. \tag{3.4}$$

We also have the following relations:

$$\begin{aligned} \nabla^L_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} &= -\nabla^L_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} = 2\,\mathrm{Re}(\nabla^L_{\frac{\partial}{\partial z}} \frac{\partial}{\partial z}), \\ \nabla^L_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} &= \nabla^L_{\frac{\partial}{\partial y}} \frac{\partial}{\partial x} = -2\,\mathrm{Im}(\nabla^L_{\frac{\partial}{\partial z}} \frac{\partial}{\partial z}). \end{aligned} \tag{3.5}$$

This follows directly from the equation (3.1) and the fact that the connection is torsion free. Thus, the Levi–Civita connection is entirely specified in local coordinates by Γ^ρ. We summarize this in the following proposition.

Proposition 3.1. *If ρ^2 and ∇^L are as above then*

$$\nabla^L_{\frac{\partial}{\partial z}} \frac{\partial}{\partial z} = \Gamma^\rho \frac{\partial}{\partial z}, \quad \nabla^L_{\frac{\partial}{\partial \bar z}} \frac{\partial}{\partial \bar z} = \overline{\Gamma}^\rho \frac{\partial}{\partial \bar z}, \quad \nabla^L_{\frac{\partial}{\partial \bar z}} \frac{\partial}{\partial z} = \nabla^L_{\frac{\partial}{\partial z}} \frac{\partial}{\partial \bar z} = 0$$

and

$$\nabla^L_{\frac{\partial}{\partial z}} dz = -\Gamma^\rho dz, \quad \nabla^L_{\frac{\partial}{\partial \bar z}} d\bar z = -\overline{\Gamma}^\rho d\bar z, \quad \nabla^L_{\frac{\partial}{\partial \bar z}} dz = \nabla^L_{\frac{\partial}{\partial z}} d\bar z = 0.$$

Proof. The first set of equations can be calculated from $\nabla^L_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x}$, $\nabla^L_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y}$ and equation (3.5). The second set follows from the first and the fact that ∇^L commutes with contractions and satisfies a Leibniz rule: we have that

$$\nabla^L_{\frac{\partial}{\partial z}} \left(dz \otimes \frac{\partial}{\partial z} \right) = \nabla^L_{\frac{\partial}{\partial z}} dz \otimes \frac{\partial}{\partial z} + dz \otimes \nabla^L_{\frac{\partial}{\partial z}} \frac{\partial}{\partial z} = \nabla^L_{\frac{\partial}{\partial z}} dz \otimes \frac{\partial}{\partial z} + dz \otimes \Gamma^\rho \frac{\partial}{\partial z}.$$

Contracting both sides and noting $\nabla^L(1) = 0$ shows that

$$\nabla^L_{\frac{\partial}{\partial z}} dz \left(\frac{\partial}{\partial z} \right) = -\Gamma^\rho.$$

Applying a similar argument we can deduce that

$$\nabla^L_{\frac{\partial}{\partial z}} dz \left(\frac{\partial}{\partial \bar z} \right) = 0$$

so $\nabla^L dz = -\Gamma^\rho dz$. The remaining formulas follow similarly. ([8] IX.5). □

3.2. Restriction of the Levi–Civita connection to symmetric tensors. When restricted to symmetric tensors, the Levi–Civita connection ∇^L can be regarded as raising the covariant degree in the following sense. If $\mathfrak{S}^{\mathbf{R}}{}^k_l$ denotes the sections of $\mathrm{Sym}^k(\mathrm{TR}) \otimes \mathrm{Sym}^l(\mathrm{T^*R})$, then the Levi–Civita connection can be viewed as a map $\nabla^L \colon \mathfrak{S}^{\mathbf{R}}{}^k_l \to \mathfrak{S}^{\mathbf{R}}{}^k_{l+1}$ in the following way: if α is a section of $\mathrm{Sym}^k(\mathrm{TR})$ and β is a section of $\mathrm{Sym}^l(\mathrm{T^*R})$, we define

$$\nabla^L(\alpha \otimes \beta) \equiv (\nabla^L_{\frac{\partial}{\partial x}} \alpha) \otimes \beta \cdot dx + (\nabla^L_{\frac{\partial}{\partial y}} \alpha) \otimes \beta \cdot dy + \alpha \otimes (\nabla^L_{\frac{\partial}{\partial x}} \beta) \cdot dx + \alpha \otimes (\nabla^L_{\frac{\partial}{\partial y}} \beta) \cdot dy.$$

This defines ∇^L on arbitrary sections by linearity. Note that this definition is independent of the choice of coordinate $z = x + iy$. This definition also extends to complex symmetric tensors $\nabla^L : \mathfrak{S}^k_l \to \mathfrak{S}^k_{l+1}$.

Remark 3.2. (Necessity of using symmetric tensors) There is no unique way to regard ∇^L as a map from $\mathfrak{D}^{\mathbf{R}}{}^k_l$ to $\mathfrak{D}^{\mathbf{R}}{}^k_{l+1}$ simply because there is an arbitrary choice in ordering the covariant factors. For example, if α is a one form in local coordinates (x, y), one has two choices for the extension: either

$$\nabla^L(\alpha) =^? \left(\nabla_{\frac{\partial}{\partial x}} \alpha \right) \otimes dx + \left(\nabla_{\frac{\partial}{\partial y}} \alpha \right) \otimes dy$$

or

$$\nabla^L(\alpha) =^? dx \otimes \left(\nabla_{\frac{\partial}{\partial x}} \alpha \right) + dy \otimes \left(\nabla_{\frac{\partial}{\partial y}} \alpha \right).$$

The same observation holds for the complexified tensor spaces.

As we would like to describe derivatives of maps between Riemann surfaces with possibly different metrics, we need to allow the possibility that one metric be used to differentiate the contravariant components and another be used to differentiate the covariant components.

Definition 3.3. Let R be a Riemann surface equipped with two metrics g_1 and g_2. Let ∇^{L,g_i}, $i=1,2$ be the unique complex linear extension of the Levi–Civita connections of g_i to complex tensors of arbitrary type; that is $\nabla^{L,g_i}:\mathfrak{X}\times\mathfrak{D}\to\mathfrak{D}$. For symmetric tensor fields define $\nabla^{L,g_1,g_2}:\mathfrak{S}^k_l\to\mathfrak{S}^k_{l+1}$ as follows. If α and β are sections of $\mathrm{Sym}^k(\mathrm{T}^{\mathbf{C}}\mathrm{R})$ and $\mathrm{Sym}^l(\mathrm{T}^{\mathbf{C}*}\mathrm{R})$ respectively, then

$$\nabla^{L,g_1,g_2}(\alpha\otimes\beta)\equiv(\nabla^{L,g_2}_{\frac{\partial}{\partial x}}\alpha)\otimes\beta\cdot dx+(\nabla^{L,g_2}_{\frac{\partial}{\partial y}}\alpha)\otimes\beta\cdot dy+\alpha\otimes(\nabla^{L,g_1}_{\frac{\partial}{\partial x}}\beta)\cdot dx+\alpha\otimes(\nabla^{L,g_1}_{\frac{\partial}{\partial y}}\beta)\cdot dy.$$

This extends linearly to arbitrary elements of $\mathfrak{S}^k_l$.

In order to avoid unwieldy superscripts, we will simply denote this extension by ∇^L, where the underlying metrics g_1 and g_2 are assumed to be specified.

Remark 3.4. An alternative to symmetrizing would be to specify that the new covariant factor always appears on the right (as in [14]). This leads to more complicated formulas for the relation between ∇^L and the Minda–Peschl derivatives.

3.3. The connections ∇ and $\overline{\nabla}$. Next, we define two connections, which when applied to a holomorphic function result in the Minda–Peschl derivatives.

Definition 3.5. Let $g_1=\rho_1^2|dz|^2$ and $g_2=\rho_2^2|dz|^2$ be two conformal metrics in terms of a local parameter z, which will be associated with contravariant and covariant tensors respectively. The connection $\nabla^{g_1,g_2}\colon\mathscr{D}^{r,s}_{m,n}\to\mathscr{D}^{r,s}_{m+1,n}$ is defined in local coordinates by

$$\nabla^{g_1,g_2}\left(h(z)\frac{\partial^r}{\partial z^r}\frac{\partial^s}{\partial\bar z^s}\otimes dz^m d\bar z^n\right)=\left(\frac{\partial h}{\partial z}+\left(r\Gamma^{\rho_2}-m\Gamma^{\rho_1}\right)h\right)\frac{\partial^r}{\partial z^r}\frac{\partial^s}{\partial\bar z^s}\otimes dz^{m+1}d\bar z^n,$$

and $\overline{\nabla}^{g_1,g_2}\colon\mathscr{D}^{r,s}_{m,n}\to\mathscr{D}^{r,s}_{m,n+1}$ by

$$\overline{\nabla}^{g_1,g_2}\left(h(z)\frac{\partial^r}{\partial z^r}\frac{\partial^s}{\partial\bar z^s}\otimes dz^m d\bar z^n\right)=\left(\frac{\partial h}{\partial\bar z}+\left(s\overline{\Gamma}^{\rho_2}-n\overline{\Gamma}^{\rho_1}\right)h\right)\frac{\partial^r}{\partial z^r}\frac{\partial^s}{\partial\bar z^s}\otimes dz^m d\bar z^{n+1},$$

It follows from equation (3.4) that ∇^{g_1,g_2} and $\overline{\nabla}^{g_1,g_2}$ do not depend on the choice of local parameter z.

Actually, in order to describe derivatives only the case that $r=1$ and $s=1$ are necessary, but the definition here has been extended to allow for an efficient statement of the Leibniz rule ahead.

As with ∇^L, the superscripts g_1 and g_2 will be dropped to simplify the notation, except where the choice of metrics is not clear from context.

Some remarks are in order regarding the origin of this definition. ∇ and $\overline{\nabla}$ were first invented in the function-theoretic context by Minda [11] without reference to the underlying space of differentials, but not published. In the special case that $g_1=g_2$, they also appear in the physics literature as ‘raising and lowering operators’ (see [12]

Chpt. 14). In this case one can eliminate all of the $\bar{z}$ components by taking traces with respect to the metric, and the difference $r-n$ is referred to as the "helicity".

Using Definition 3.5 it can be immediately computed that ∇ and $\overline{\nabla}$ satisfy a Leibniz rule with respect to the multiplication of Definition 2.5.

Proposition 3.6. *If $\alpha \in \mathscr{D}^{r_1,s_1}_{m_1,n_1}$ and $\beta \in \mathscr{D}^{r_2,s_2}_{m_2,n_2}$, then*

$$\nabla(\alpha \cdot \beta) = (\nabla\alpha)\cdot\beta + \alpha\cdot(\nabla\beta)$$
$$\overline{\nabla}(\alpha \cdot \beta) = (\overline{\nabla}\alpha)\cdot\beta + \alpha\cdot(\overline{\nabla}\beta)$$

The Levi–Civita connection ∇^L, ∇, and $\overline{\nabla}$ are related in the following way.

Proposition 3.7. *Let ∇^L be the complex linear extension of the Levi–Civita connection restricted to the symmetric tensors $\mathfrak{S}$, corresponding to a pair of metrics $g_i = \rho_i^2|dz|^2$, $i = 1, 2$ as described in Definition 3.3. Then*

$$\nabla^L = \nabla + \overline{\nabla}.$$

Proof. Since the symmetric tensor fields decomposes into the direct sum of the spaces $\mathscr{D}^{r,s}_{m,n}$, it suffices to check this for $\alpha \in \mathscr{D}^{r,s}_{m,n}$. For such tensors we have

$$\begin{aligned}\nabla^L\alpha &= \left(\nabla^L_{\frac{\partial}{\partial x}}\alpha\right)\cdot dx + \left(\nabla^L_{\frac{\partial}{\partial y}}\alpha\right)\cdot dy\\ &= \frac{1}{2}\left(\nabla^L_{\frac{\partial}{\partial z}+\frac{\partial}{\partial \bar{z}}}\alpha\right)\cdot(dz+d\bar{z}) + \frac{1}{2}\left(\nabla^L_{i\frac{\partial}{\partial z}-i\frac{\partial}{\partial \bar{z}}}\alpha\right)\cdot(id\bar{z}-idz)\\ &= \left(\nabla^L_{\frac{\partial}{\partial z}}\alpha\right)\cdot dz + \left(\nabla^L_{\frac{\partial}{\partial \bar{z}}}\alpha\right)\cdot d\bar{z}.\end{aligned}$$

The Leibniz rule for the Levi–Civita connection under tensor product extends by linearity to the symmetric product, so that

$$\begin{aligned}\nabla^L_{\frac{\partial}{\partial z}}\left(h(z)\frac{\partial^r}{\partial z^r}\frac{\partial^s}{\partial \bar{z}^s}\otimes dz^m d\bar{z}^n\right) &= \nabla^L_{\frac{\partial}{\partial z}}(h(z))\frac{\partial^r}{\partial z^r}\frac{\partial^s}{\partial \bar{z}^s}\otimes dz^m d\bar{z}^n\\ &\quad + h(z)\nabla^L_{\frac{\partial}{\partial z}}\left(\frac{\partial^r}{\partial z^r}\right)\frac{\partial^s}{\partial \bar{z}^s}\otimes dz^m d\bar{z}^n\\ &\quad + h(z)\frac{\partial^r}{\partial z^r}\frac{\partial^s}{\partial \bar{z}^s}\otimes\nabla^L_{\frac{\partial}{\partial z}}(dz^m)\,d\bar{z}^n\\ &= \left(\frac{\partial h}{\partial z}(z) + (r\Gamma^{\rho_2} - s\Gamma^{\rho_1})\,h(z)\right)\frac{\partial^r}{\partial z^r}\frac{\partial^s}{\partial \bar{z}^s}\otimes dz^m d\bar{z}^n\end{aligned}$$

by Proposition 3.1. Thus for any $\alpha \in \mathscr{D}^{r,s}_{m,n}$

$$\left(\nabla^L_{\frac{\partial}{\partial z}}\alpha\right)\cdot dz = \nabla\alpha.$$

Similarly

$$\left(\nabla^L_{\frac{\partial}{\partial \bar{z}}}\alpha\right)\cdot d\bar{z} = \overline{\nabla}\alpha. \qquad \square$$

Remark 3.8. Alternatively Proposition 3.7 could be used as the coordinate-free definition of ∇ and $\overline{\nabla}$, and Definition 3.5 can be derived from it.

4. Covariant derivatives of a holomorphic function

In this section we derive the relation between the Minda–Peschl derivatives $D_n^{\rho,\sigma} f$ and the Levi–Civita connection. In particular, we show that for metrics with non-zero curvature, the two derivatives differ by terms involving the curvature. We give two explicit results: first, an inductive formula for the difference between the Minda–Peschl and Levi–Civita derivatives, and second, we give closed formula for all n for certain special metrics of constant curvature.

4.1. Preliminaries. In this section we define the Minda–Peschl derivatives in the form that they appear in the literature, and prove a key invariance property which can be used to give an alternate definition for specific metrics.

In the following, on a Riemann surface R_1 let $E \in \mathscr{D}^{1,0}_{1,0}$ denote the complex tensor which is given in terms of a local biholomorphic parameter z by

$$E = \frac{\partial}{\partial z} \otimes dz.$$

Note that under a change of parameter, this expression stays the same; i.e. if $w = g(z)$ then

$$\frac{\partial}{\partial z} \otimes dz = \frac{\partial}{\partial w} \otimes dw$$

and thus E is a well-defined tensor. Under the standard identification of $T_p^{\mathbf{C}} R_1 \otimes T_p^{\mathbf{C}*} R_1$ with complex linear maps from $T_p^{\mathbf{C}} R_1 \to T_p^{\mathbf{C}} R_1$, E can be interpreted as the projection of the identity map onto $\mathscr{D}^{1,0}_{1,0}$. That is, if $z = x + iy$,

$$\frac{\partial}{\partial x} \otimes dx + \frac{\partial}{\partial y} \otimes dy = \frac{\partial}{\partial z} \otimes dz + \frac{\partial}{\partial \bar{z}} \otimes d\bar{z}.$$

Definition 4.1. Let R_1 and R_2 be Riemann surfaces endowed with conformal metrics g_1 and g_2 respectively. Let $\tilde{f} : R_1 \to R_2$ be a locally one-to-one holomorphic map. Define higher derivatives of $\tilde{f}$ inductively with respect to $\nabla^{g_1, \tilde{f}^* g_2}$ as follows:

$$\nabla_1^{g_1, \tilde{f}^* g_2} \tilde{f} = E$$

$$\text{and} \qquad \nabla_{n+1}^{g_1, \tilde{f}^* g_2} \tilde{f} = \nabla^{g_1, \tilde{f}^* g_2} \left(\nabla_n^{g_1, \tilde{f}^* g_2} \tilde{f} \right),$$

Let z and w be local biholomorphic parameters on R_1 and R_2 respectively, and $w = f(z)$, $\rho(z)^2 |dz|^2$ and $\sigma(w)^2 |dw|^2$ be the expressions for $\tilde{f}$, g_1 and g_2 respectively, in terms of these parameters. Define the Minda–Peschl derivatives $D_n^{\rho,\sigma} f$ by

$$\nabla_n^{g_1, \tilde{f}^* g_2} \tilde{f} = \frac{\rho(z)^n D_n^{\rho,\sigma} f(z)}{\sigma \circ f(z) f'(z)} \frac{\partial}{\partial z} \otimes dz^n.$$

Remark 4.2. The fact that the first derivative is a projection of the identity map, is a consequence of our choice of expressing the derivatives as tensors on R_1, rather than as maps $T^{\mathbf{C}} R_1 \times \cdots \times T^{\mathbf{C}} R_1 \to T^{\mathbf{C}} R_2$.

Proposition 3.7 establishes the geometric interpretation of Minda's invariant derivatives as follows. Successive derivatives of a function by ∇^L are symmetric tensors in $\mathfrak{S}^1_n$, which by equation (2.1) decompose into a sum of terms in $\mathscr{D}^{1,0}_{k,l}$ for $k+l=n$. Thus by Proposition 3.7

$$\nabla^{g_1,\tilde{f}^*g_2}_n \tilde{f} = \frac{\rho(z)^n D^{\rho,\sigma}_n f(z)}{\sigma\circ f(z) f'(z)} \frac{\partial}{\partial z} \otimes dz^n$$

is the component of $\nabla^{g_1,\tilde{f}^*g_2}_n f$ in $\mathscr{D}^{1,0}_{n,0}$. Equivalently, if one evaluates the nth derivative of $\nabla^L_n \tilde{f}$ on the multi-vector $(\partial/\partial z,\ldots,\partial/\partial z)$, all the terms vanish except for the term in $\mathscr{D}^{1,0}_{n,0}$. It follows from Definition 3.5, Definition 4.1 and equation (3.4) that $D^{\rho,\sigma}_n f$ satisfy the recursive relation

$$\rho\, D^{\rho,\sigma}_{n+1} f = \frac{\partial}{\partial z}\left(D^{\rho,\sigma}_n f\right) + \frac{1}{2}\left(\Gamma^\sigma\circ f\cdot f' - n\Gamma^\rho\right) D^{\rho,\sigma}_n f. \tag{4.1}$$

If $\rho=1$ and $\sigma=1$ in the parameters z and w, then it is easy to check that

$$D^{\rho,\sigma}_n f(z) = f^{(n)}(z) \tag{4.2}$$

for all n.

Remark 4.3. (Dependence of $D^{\rho,\sigma}_n f$ on local parameters) Since ∇ does not depend on the choice of a local parameter, neither does $\nabla^{g_1,\tilde{f}^*g_2}_n \tilde{f}$. However, the Minda–Peschl derivatives do depend on the choice of parameters, in the following way. Let $\zeta=g(z)$ and $\xi=h(w)$ be locally biholomorphic changes of parameters on R_1 and R_2 respectively. We then have corresponding changes in f, ρ and σ: $\hat{f}(\zeta)=h\circ f\circ g^{-1}(\zeta)$, $\hat{\rho}(g(z))|g'(z)|=\rho(z)$ and $\hat{\sigma}(h(w))|h'(w)|=\sigma(w)$. Since $\nabla^{g_1,\tilde{f}^*g_2}_n \tilde{f}$ is independent of coordinates we have that

$$\frac{\rho(z)^n D^{\rho,\sigma}_n f(z)}{\sigma\circ f(z) f'(z)} \frac{\partial}{\partial z}\otimes dz^n = \frac{\hat{\rho}(\zeta)^n D^{\hat{\rho},\hat{\sigma}} \hat{f}(\zeta)}{\hat{\sigma}\circ \hat{f}(\zeta)\hat{f}'(\zeta)} \frac{\partial}{\partial \zeta}\otimes d\zeta^n$$

so

$$D^{\hat{\rho},\hat{\sigma}}_n \hat{f}(\zeta) = \frac{h'(w)}{|h'(w)|}\frac{|g'(z)|^n}{g'(z)^n} D^{\rho,\sigma}_n f(z).$$

For particular metrics, the invariant derivatives appear in the literature with a rather different definition [6] [10] [13]. We now give this definition and demonstrate its equivalence with the one above, for these special metrics. An alternate approach appears in [5].

First we define certain special conformal metrics.

Definition 4.4. Let

$$\mathbf{D}_k = \begin{cases} \{z : |z| < 1/\sqrt{|k|}\,\}, & k<0,\\ \mathbf{C}, & k=0,\\ \overline{\mathbf{C}}, & k>0,\end{cases}$$

and

$$\lambda_k = \frac{1}{1+k|z|^2}.$$

A calculation using the well-known formula for the Gaussian curvature [9]

$$K = -\Delta \log \lambda_k / \lambda_k^2 \tag{4.3}$$

shows that the curvature of λ_k is $4k$. In the case that $k < 0$, these are the only complete constant curvature metrics on $\mathbf{D}_k$ up to scale. In the case that $k > 0$, there are more; however up to scale the metrics above are the only ones for which 0 and ∞ are antipodal (in the sense that every geodesic connecting 0 and ∞ has the same length).

The maps

$$T(z) = e^{i\theta} \frac{z+a}{1-k\bar{a}z}$$

satisfy the identity

$$1 + k|T(z)|^2 = (1+k|z|^2)|T'(z)|,$$

which shows that they are isometries of λ_k. These are in fact all of the orientation-preserving isometries [11], and they preserve the domain $\mathbf{D}_k$.

Given any two non-antipodal points z and w in $\mathbf{D}_k$, they are connected by a shortest geodesic; using an isometry above we can move this geodesic to a radial line segment starting at 0, without changing the length. A simple computation of the length of this segment results in the following expression for the distance between z and w.

$$d_{\lambda_k}(w,z) = \begin{cases} \frac{1}{\sqrt{-k}} \operatorname{arctanh}\left(\sqrt{-k}\left|\frac{w-z}{1+k\bar{w}z}\right|\right), & k < 0, \\ |w-z|, & k = 0, \\ \frac{1}{\sqrt{k}} \arctan\left(\sqrt{k}\left|\frac{w-z}{1+k\bar{w}z}\right|\right), & k > 0. \end{cases}$$

We have the following invariance property of the higher-order derivatives. Recall that for a locally holomorphic map g, $g^*(\rho(z)|dz|^2) = \rho \circ g(z)|g'(z)|^2|dz|^2$ denotes the pullback $\rho(z)^2|dz|^2$; we will stretch the notation slightly and say $g^*(\rho) = \rho \circ g|g'|$. In general, the pull-back of a tensor of type $\mathscr{D}^{r,s}_{m,n}$ is given in local coordinates by

$$g^*\left(h(z)\frac{\partial^r}{\partial z^r}\frac{\partial^s}{\partial \bar{z}^s} \otimes dz^m d\bar{z}^n\right) = h(g(z))g'(z)^{m-r}\overline{g'(z)}^{n-s}\frac{\partial^r}{\partial z^r}\frac{\partial^s}{\partial \bar{z}^s} \otimes dz^m d\bar{z}^n.$$

Proposition 4.5. *Let $\tilde{f} : \mathrm{R}_1 \to \mathrm{R}_2$ be a conformal map between Riemann surfaces R_1 and R_2, equipped with conformal metrics g_1 and g_2 respectively. Let $\tilde{T}$ be a holomorphic local isometry of g_1 and $\tilde{S}$ be a holomorphic local isometry of g_2. Then*

$$\nabla_n^{\tilde{T}^* g_1, \tilde{S}\circ\tilde{f}\circ\tilde{T}^* g_2}(\tilde{S}\circ\tilde{f}\circ\tilde{T}) = \tilde{T}^*(\nabla_n^{g_1,\tilde{f}^* g_2}\tilde{f}).$$

If $\tilde{T}$ is given by $T(z)$ in terms of a local parameter z on R_1, and the coefficient of an element of $\mathscr{D}^{1,0}_{n,0}$ is denoted by

$$\{g(z)\frac{\partial}{\partial z} \otimes dz^n\} = g(z),$$

then we have that

$$\{\nabla_n^{\tilde{T}^*g_1,\tilde{S}\circ\tilde{f}\circ\tilde{T}^*g_2}(\tilde{S}\circ\tilde{f}\circ\tilde{T})\} = \{\nabla_n^{g_1,\tilde{f}^*g_2}\tilde{f}\}\circ T\cdot T'^{n-1}.$$

Proof. We prove the formula in local coordinates by induction. The case $n=1$ is immediate. Next, note that by equation (3.3) we have

$$\Gamma^{T^*\rho} = \Gamma^{\rho}\circ T\cdot T' + \frac{T''}{T'} \tag{4.4}$$

and

$$\Gamma^{f^*\sigma} = \Gamma^{\sigma}\circ f\cdot f' + \frac{f''}{f'}$$

which implies that

$$\Gamma^{S\circ f\circ T^*\sigma} = \Gamma^{f^*\sigma}\circ T\cdot T' + \frac{T''}{T'}. \tag{4.5}$$

using the fact that $\Gamma^{S^*\sigma}=\Gamma^{\sigma}$. Assuming that the result holds for the nth derivative, applying Definition 3.5 and equations (4.4) and (4.5) we have

$$\begin{aligned}
\{\nabla_{n+1}^{\tilde{T}^*g_1,\,\tilde{S}\circ\tilde{f}\circ\tilde{T}^*g_2}(\tilde{S}\circ\tilde{f}\circ\tilde{T})\} &= \frac{\partial}{\partial z}\{\nabla_n^{\tilde{T}^*g_1,\,\tilde{S}\circ\tilde{f}\circ\tilde{T}^*g_2}(\tilde{S}\circ\tilde{f}\circ\tilde{T})\}\\
&\quad + \left(\Gamma^{S\circ f\circ T^*\sigma} - n\Gamma^{T^*\rho}\right)\{\nabla_n^{\tilde{T}^*g_1,\,\tilde{S}\circ\tilde{f}\circ\tilde{T}^*g_2}\tilde{S}\circ\tilde{f}\circ\tilde{T}\}\\
&= \frac{\partial}{\partial z}\left(\{\nabla_n^{g_1,\tilde{f}^*g_2}\tilde{f}\}\circ T\cdot T'^{n-1}\right)\\
&\quad + \left(\Gamma^{f^*\sigma}\circ T\cdot T' - n\Gamma^{\rho}\circ T\cdot T' - (n-1)\frac{T''}{T'}\right)\\
&\quad \cdot\{\nabla_n^{g_1,\,\tilde{f}^*g_2}\tilde{f}\}\circ T\cdot T'^{n-1}\\
&= \{\nabla_{n+1}^{g_1,\,\tilde{f}^*g_2}\tilde{f}\}\circ T\cdot T'^{n}. \qquad \square
\end{aligned}$$

Note that in the statement of the Theorem, one may replace $\tilde{T}^*g_1$ by g_1 since $\tilde{T}$ is an isometry of g_1.

If g_1 and g_2 are given by ρ and σ in local coordinates, then when written in terms of $D_n^{\rho,\sigma}f$ Proposition 4.5 takes the form

$$D_n^{\rho,\sigma}(S\circ f\circ T) = \frac{S'\circ f\circ T}{|S'\circ f\circ T|}(D_n^{\rho,\sigma}f)\circ T\frac{T'}{|T'|}. \tag{4.6}$$

This follows from Proposition 4.5, Definition 4.1 and the fact that for isometries we have $\rho\circ T\,|T'|=\rho$ and $\sigma\circ S\,|S'|=\sigma$.

This leads us to the definition of the invariant derivatives due to Peschl, extended slightly.

Proposition 4.6. (Peschl's definition of invariant derivatives) *Let $k,k'\in\mathbf{R}$. Given a locally one-to-one holomorphic function $f\colon\mathbf{D}_k\to\mathbf{D}_{k'}$, and $a\in\mathbf{D}_k$, let*

$T(z) = (z+a)/(1-k\bar{a}z)$ and $S(w) = (w - f(a))/(1 + k'\overline{f(a)}w)$. Then

$$D_n^{\lambda_k,\lambda_{k'}} f(a) = \left.\frac{\partial^n}{\partial z^n}\right|_{z=0} (S \circ f \circ T).$$

Proof. We have that $T(0) = a$ and $S(f(a)) = 0$, and also that

$$\frac{T'(0)}{|T'(0)|} = 1 \quad \text{and} \quad \frac{S'(f(a))}{|S'(f(a))|} = 1.$$

Thus by equation (4.6) we have that $D_n^{\lambda_k,\lambda_{k'}} f(a) = D_n^{\lambda_k,\lambda_{k'}} (S \circ f \circ T)(0)$.

It is easily checked that

$$\frac{\partial^n \Gamma}{\partial z^n}(0) = 0$$

for Γ^{λ_k} and $\Gamma^{\lambda_{k'}}$ for all $n \geq 0$, and thus $D_n^{\lambda_k,\lambda_{k'}}(S \circ f \circ T)(0) = D_n^{1,1}(S \circ f \circ T) = (S \circ f \circ T)^{(n)}(0)$ by equations (4.1) and (4.2). □

4.2. Relation between $\nabla_n^L f$ and $\nabla_n f$. We now give a general formula for the relation between the Levi–Civita derivative and the covariant derivatives of a holomorphic function. The difference is related to the curvature of the metrics on the domain and image. If g_1 and g_2 are Riemannian metrics on R_1 and R_2 given by $\rho(z)^2|dz|^2$ and $\sigma(w)^2|dw|^2$ in local parameters, define the following quantities. In terms of local parameters let

$$\Psi_1 \equiv \Gamma_{\bar{z}}^{\rho}\, dz \cdot d\bar{z} = -\frac{1}{2}\rho^2 K_\rho\, dz \cdot d\bar{z},$$
$$\Phi_1 \equiv \Gamma_{\bar{z}}^{f^*\sigma}\, dz \cdot d\bar{z} = -\frac{1}{2} f^*\sigma^2 K_{f^*\sigma}\, dz \cdot d\bar{z},$$

where K_ρ and $K_{f^*\sigma}$ are the Gaussian curvatures of ρ and $f^*\sigma$ respectively (equation (4.3)). Furthermore define Ψ_n and Φ_n inductively by

$$\Psi_{n+1} = \nabla \Psi_n,$$
$$\Phi_{n+1} = \nabla \Phi_n.$$

Note that when applied to Φ_n and Ψ_n, ∇ only depends on the metric $g_1 = \rho(z)^2|dz|^2$ since they are tensors of type $\mathscr{D}_{n,1}^{0,0}$.

Theorem 4.7. *Let $f\colon R_1 \to R_2$ be a holomorphic map between Riemann surfaces, which are endowed with conformal metrics g_1 and g_2. If ∇ and $\overline{\nabla}$ are the connections associated to these metrics, then*

$$\overline{\nabla}\nabla_n f = \sum_{k=1}^{n-1} \left[\binom{n-1}{k-1} \Phi_{n-k} - \binom{n}{k-1} \Psi_{n-k} \right] \nabla_k f.$$

Proof. Let $g_1 = \rho(z)^2|dz|^2$ and $g_2 = \sigma(w)^2|dw|^2$ in local parameters z and w on R_1 and R_2 respectively. As in Proposition 4.5 denote the coefficient of $\nabla_n f$ by $\{\nabla_n f\}$, and similarly the coefficients of Ψ_n and Φ_n by $\{\Psi_n\}$ and $\{\Phi_n\}$. By definition,

$$\{\nabla_{n+1} f\} = \{\nabla_n f\}_z + \left(\Gamma^{f^*\sigma} - n\Gamma^{\rho}\right)\{\nabla_n f\}.$$

We claim that

$$\{\nabla_n f\}_{\bar z} = \sum_{k=1}^{n-1}\left[\binom{n-1}{k-1}\{\Phi_{n-k}\} - \binom{n}{k-1}\{\Psi_{n-k}\}\right]\{\nabla_k f\}.$$

This is easily checked for $n = 2$. Assume it holds for n:

$$\begin{aligned}
\{\nabla_{n+1} f\}_{\bar z} &= \{\nabla_n f\}_{\bar z z} + \left(\Gamma_{\bar z}^{f^*\sigma} - n\Gamma_{\bar z}^{\rho}\right)\{\nabla_n f\} + \left(\Gamma^{f^*\sigma} - n\Gamma^{\rho}\right)\{\nabla_n f\}_{\bar z}\\
&= \sum_{k=1}^{n-1}\left[\binom{n-1}{k-1}\{\Phi_{n-k}\}_z - \binom{n}{k-1}\{\Psi_{n-k}\}_z\right]\{\nabla_k f\}\\
&\quad + \sum_{k=1}^{n-1}\left[\binom{n-1}{k-1}\{\Phi_{n-k}\} - \binom{n}{k-1}\{\Psi_{n-k}\}\right]\{\nabla_k f\}_z\\
&\quad + \left(\{\Phi_1\} - n\{\Psi_1\}\right)\{\nabla_n f\}\\
&\quad + \left(\Gamma^{f^*\sigma} - n\Gamma^{\rho}\right)\sum_{k=1}^{n-1}\left[\binom{n-1}{k-1}\{\Phi_{n-k}\} - \binom{n}{k-1}\{\Psi_{n-k}\}\right]\{\nabla_k f\}.
\end{aligned}$$

Using Definition 3.5 and shifting the index in the first sum, we have

$$\begin{aligned}
\{\nabla_{n+1} f\}_{\bar z} &= \left(\{\Phi_1\} - n\{\Psi_1\}\right)\{\nabla_n f\}\\
&\quad + \sum_{k=1}^{n-1}\left[\binom{n-1}{k-1}\{\Phi_{n-k}\} - \binom{n}{k-1}\{\Psi_{n-k}\}\right]\\
&\quad \cdot \left(\{\nabla_k f\}_z + \left(\Gamma^{f^*\sigma} - k\Gamma^{\rho}\right)\{\nabla_k f\}\right)\\
&\quad + \sum_{k=1}^{n-1}\Bigg(\left[\binom{n-1}{k-1}\{\Phi_{n-k}\}_z - \binom{n}{k-1}\{\Psi_{n-k}\}_z\right]\\
&\quad - (n-k)\,\Gamma^{\rho}\left[\binom{n-1}{k-1}\{\Phi_{n-k}\} - \binom{n}{k-1}\{\Psi_{n-k}\}\right]\Bigg)\{\nabla_k f\}\\
&= \left(\{\Phi_1\} - n\{\Psi_1\}\right)\{\nabla_n f\}\\
&\quad + \sum_{k=2}^{n}\left[\binom{n-1}{k-2}\{\Phi_{n-k+1}\} - \binom{n}{k-2}\{\Psi_{n-k+1}\}\right]\{\nabla_k f\}\\
&\quad + \sum_{k=1}^{n-1}\left[\binom{n-1}{k-1}\{\Phi_{n-k+1}\} - \binom{n}{k-1}\{\Psi_{n-k+1}\}\right]\{\nabla_k f\}.
\end{aligned}$$

Now using Pascal's identity we have that

$$\begin{aligned}\{\nabla_{n+1}f\}_{\bar z} = \sum_{k=2}^{n}\Bigg[&\left(\binom{n-1}{k-2}+\binom{n-1}{k-1}\right)\{\Phi_{n-k+1}\}\\ &-\left(\binom{n}{k-2}+\binom{n}{k-1}\right)\{\Psi_{n-k+1}\}\Bigg]\{\nabla_k f\}\\ &+\left[\binom{n-1}{0}\{\Phi_n\}-\binom{n}{0}\{\Psi_n\}\right]\{\nabla_1 f\}\\ =\sum_{k=1}^{n}&\left[\binom{n}{k-1}\{\Phi_{n-k+1}\}-\binom{n+1}{k-1}\{\Psi_{n-k+1}\}\right]\{\nabla_k f\}. \qquad \square\end{aligned}$$

It then follows immediately from Proposition 3.7 that

Theorem 4.8. *Given the conditions of Theorem 4.7, if ∇^L is the Levi–Civita connection, then*

$$\nabla^L\nabla_n f = \nabla_{n+1}f + \sum_{k=1}^{n-1}\left[\binom{n-1}{k-1}\Phi_{n-k} - \binom{n}{k-1}\Psi_{n-k}\right]\nabla_k f.$$

Remark 4.9. Since $\nabla_n f \in \mathscr{D}^{1,0}_{n,0}$ we have that

$$\overline{\nabla}\nabla_n f = \frac{\partial}{\partial \bar z}\left(\frac{\rho^n D_n^{\rho,\sigma} f}{\sigma\circ f f'}\right)\frac{\partial}{\partial z}\otimes d\bar z\cdot dz^n.$$

The $\overline{\nabla}$ term takes a simple form for metrics of constant curvature.

Corollary 4.10. *Given the conditions of Theorem 4.7, suppose further that $g_1 = \rho(z)^2|dz|^2$ is a metric of constant curvature K, and $g_2 = \sigma(w)|dw|^2 = |dw|^2$. Then,*

$$\overline{\nabla}\nabla_n f = \frac{n(n-1)}{4}\rho^2\, K\, \nabla_{n-1} f\, dz\, d\bar z.$$

In terms of $D_n f$ we have

$$\frac{\partial}{\partial \bar z}(\rho^n D_n^{\rho,1} f\, dz^n) = \frac{n(n-1)}{4}\rho^{n+1}\, K\, D_{n-1}^{\rho,1} f.$$

Proof. Apply the previous theorem using $\Psi_1 = -\rho^2 K/2$ and $\Phi_1 = 0$, noting that $\Psi_n = 0$ for all $n \geq 2$. The second claim follows from Remark 4.9 and the fact that f' is holomorphic. $\square$

Now that the main results are established, we highlight four important points regarding the geometric interpretation of the Minda–Peschl derivatives.

Remark 4.11. ($\nabla_n f$ is the projection of $\nabla^L_n f$ onto $\mathscr{D}^{1,0}_{n,0}$) It follows from Theorem 4.8 that

$$\nabla^L_n f = \nabla_n f + \text{terms with } d\bar z;$$

that is $\nabla_n f$ is the projection of $\nabla^L_n f$ onto the $\mathscr{D}^{1,0}_{n,0}$ component in the decomposition (2.1).

Remark 4.12. ($\nabla^L_n f$ and $\nabla_n f$ differ because of curvature) Theorem 4.8 shows that curvature accounts for the difference between Minda–Peschl derivatives and the Levi–Civita derivatives of a holomorphic function. In the case that the metric has zero curvature, we have that these two derivatives are the same.

Remark 4.13. (Identifying the nth derivative with a function) In the case of zero curvature we have a special situation: the successive derivatives of a function lie entirely in $\mathscr{D}^{1,0}_{n,0}$. Since this space is one complex dimensional, it is possible to identify the derivative tensor of any order with its complex coefficient in some local coordinate. For example the nth derivative of a function defined on a subset of the complex plane in the Euclidean metric is

$$f^{(n)}(z)\frac{\partial}{\partial z} \otimes dz^n$$

which can be identified with $f^{(n)}(z)$ in the familiar way.

This handy coincidence does not occur for arbitrary conformal metrics since the nth derivative tensor contains terms with factors of $d\bar{z}$. Since the space of nth derivative tensors with respect to ∇^L is not one-complex dimensional, we cannot identify the derivative tensor $\nabla^L_n f$ with a function.

On the other hand, the tensors $\nabla_n f$ all lie in the one complex dimensional space $\mathscr{D}^{1,0}_{n,0}$ and thus *can* be identified with a complex function. So in this sense, for arbitrary conformal metrics the Minda–Peschl derivatives can be regarded as the "next best thing" to the ordinary derivative $f^{(n)}$.

Remark 4.14. ($\nabla^L_n f$ can be expressed in terms of $\nabla_n f$) Theorem 4.8 shows that the nth derivative $\nabla^L_n f$ can be written entirely in terms of $\nabla_k f$ for $k = 1, \cdots, n$ and the derivatives of the curvature with respect to ∇ and $\overline{\nabla}$.

The following example shows that the second term in Theorem 4.8 appears in an elementary context.

Example 4.1. (The Marty relation) Consider the problem of maximizing $\mathrm{Re}(a_n)$ over functions $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ which are univalent on $\mathbf{D}_1$. Choosing $\sigma = 1$ and $\rho = \lambda_1$ (the hyperbolic metric on $\mathbf{D}_1$), by the invariance property (4.6) this is equivalent to the problem of finding

$$\sup \mathrm{Re}\left(\frac{D_n f(z)}{D_1 f(z)}\right) \tag{4.7}$$

for any fixed z, where the supremum is taken over holomorphic one-to-one maps of the disc. This supremum is independent of z by (4.6).

Now for any function h and curve γ, we have that

$$\frac{d}{dt}\mathrm{Re}\left(\frac{h\circ\gamma}{\rho^{n-1}\circ\gamma}\right) = \frac{1}{\rho^{n-1}\circ\gamma}\mathrm{Re}\left(\left(h_z\circ\gamma - (n-1)\Gamma^\rho\circ\gamma\,\mathrm{Re}(h\circ\gamma)\right)\dot{\gamma} + h_{\bar{z}}\circ\gamma\,\overline{\dot{\gamma}}\right).$$

Assume that the function f takes on the supremum above at z. Since the upper bound in (4.7) is independent of z, setting $h = \rho^{n-1} D_n f / D_1 f$ we must have that

$$\left.\frac{d}{dt}\right|_{t=0} \operatorname{Re}\left(\frac{h \circ \gamma}{\rho^{n-1} \circ \gamma}\right) = 0$$

for any curve γ such that $\gamma(0) = z$. Furthermore, by equation (4.6) $D_n(e^{i\theta} f) = e^{i\theta} D_n f$, so h must be real in order that the supremum be obtained. Thus

$$\operatorname{Re}\left((h_z \circ \gamma - (n-1)\Gamma^\rho \circ \gamma \, h \circ \gamma) \cdot \dot{\gamma} + h_{\bar{z}} \circ \gamma \cdot \overline{\dot{\gamma}}\right) = 0. \tag{4.8}$$

The first term $h_z - (n-1)\Gamma^\rho h$ of (4.8) is the coefficient of $\nabla(\nabla_n f / \nabla_1 f)$, which by the Leibniz rule (Proposition 3.6) is

$$\nabla\left(\frac{\nabla_n f}{\nabla_1 f}\right) = \frac{\nabla_{n+1} f}{\nabla_1 f} - \frac{\nabla_2 f}{\nabla_1 f}\frac{\nabla_n f}{\nabla_1 f} = \left(\rho^n \frac{D_{n+1} f}{D_1 f} - \rho^n \frac{D_2 f}{D_1 f}\frac{D_n f}{D_1 f}\right) dz^n.$$

By Corollary 4.10, since the curvature of ρ is -4 we have

$$h_{\bar{z}} = -n(n-1)\rho^n \frac{D_{n-1} f}{D_1 f}.$$

We thus have

$$\operatorname{Re}\left(\left(\frac{D_{n+1} f}{D_1 f} - \frac{D_2 f}{D_1 f}\frac{D_n f}{D_1 f}\right)\dot{\gamma} - n(n-1)\frac{D_{n-1} f}{D_1 f}\overline{\dot{\gamma}}\right) = 0$$

at the point z. Since the argument of $\dot{\gamma}$ is arbitrary

$$\frac{D_{n+1} f}{D_1 f} - \frac{D_2 f}{D_1 f}\frac{D_n f}{D_1 f} - n(n-1)\frac{\overline{D_{n-1} f}}{\overline{D_1 f}} = 0.$$

Setting $z = 0$ and noting that $n! a_n = D_n f(0)/D_1 f(0)$ this is the Marty relation

$$(n+1)a_{n+1} - 2a_2 a_n = (n-1)\overline{a_{n-1}}.$$

Let $\nabla_n^L f = \nabla^L \circ \cdots \circ \nabla^L f$ where ∇^L is applied n times. We now derive a general formula for $\nabla_n^L f$ in terms of $\nabla_k f$ for $k = 1 \cdots, n$ for the special case of the metrics λ_k. In order to derive this relation, we compare the coefficients of two different power series.

Let $T(z) = (z+a)/(1 - k\bar{a}z)$ for some $a \in \mathbf{D}_k$ for $k \neq 0$. Let $D_n f = D_n^{\lambda_k, 1} f$. Expanding $f \circ T$ in a power series at 0 we have

$$f(w) = f(a) + D_1 f(a)\frac{w-a}{1+k\bar{a}w} + \frac{1}{2}D_2 f(a)\left(\frac{w-a}{1+k\bar{a}w}\right)^2 + \cdots. \tag{4.9}$$

by equation (4.6).

Let $a, w \in \mathbf{D}_k$, and let $\gamma(s)$ be the unique geodesic segment connecting these points, parametrized so that $\gamma(0) = 0$ and $\gamma(t_k) = w$ where t_k is the arc-length of the segment. The isometry $T(z) = (z+a)/(1 - k\bar{a}z)$ takes 0 to a and takes a radial line through the origin to the curve γ. Let θ be the argument of the ray joining 0 to $T^{-1}(w)$. Now let X_θ denote the vector field (restricted to γ) consisting of the λ_k-unit tangent vectors to γ.

Now $T^{-1}(\gamma(s))$ lies on the ray through the origin for all s, so

$$T^{-1}(w) = \frac{w-a}{1+k\bar{a}w} = \left|\frac{w-a}{1+k\bar{a}w}\right| e^{i\theta} \tag{4.10}$$
$$= \begin{cases} \frac{1}{\sqrt{-k}}\, e^{i\theta}\mathrm{arctanh}\left(\sqrt{-k}\, t_k\right), & k<0, \\ \frac{1}{\sqrt{k}}\, e^{i\theta}\mathrm{arctan}\left(\sqrt{k}\, t_k\right), & k>0. \end{cases}$$

Note that if w is allowed to vary along the geodesic, θ does not change.

Now let f be a holomorphic function in a neighborhood of a. Expanding f in a power series in t_k along the geodesic γ:

$$f(w) = f(a) + X_\theta f(a) t_k + \frac{1}{2} X_\theta^2 f(a) t_k^2 + \cdots .$$

Now since $\nabla^L_{X_\theta} X_\theta = 0$, we have that $\nabla^L_n f(a) = X_\theta^n f(a)$, since for all n,

$$\nabla^L_{n+1} f(X_\theta, \dots, X_\theta) = X_\theta(\nabla^L_n f(X_\theta, \dots, X_\theta)) - \nabla^L_n f(\nabla^L_{X_\theta} X_\theta, \dots, X_\theta)$$
$$\cdots - \nabla^L_n f(X_\theta, \dots, \nabla^L_{X_\theta} X_\theta).$$

So

$$f(w) = f(a) + \nabla^L_1 f(X_\theta)(a) t_k + \frac{1}{2}\nabla^L_2 f(X_\theta, X_\theta)(a) t_k^2 + \cdots . \tag{4.11}$$

Comparing (4.9) and (4.11), using (4.10), we have proven the following theorem.

Theorem 4.15. *For $k \neq 0$*

$$\frac{1}{n!}\nabla^L_n f^{\lambda_k}(X_\theta, \dots, X_\theta)(a) = \sum_{m=1}^{n} \frac{1}{m!} e^{im\theta} c^k_{nm} D^{\lambda_k}_m f(a). \tag{4.12}$$

where c^k_{nm} is the nth coefficient of the Taylor series in t of $\left(\frac{1}{\sqrt{-k}}\tanh\sqrt{-k}\, t\right)^m$ if $k<0$, and of $\left(\frac{1}{\sqrt{k}}\tan\sqrt{k}\, t\right)^m$ if $k>0$.

In principle this could also be derived directly from Corollary 4.10 and Proposition 3.7.

5. Parallel transport and conformal metrics

Parallel transport is a way of identifying the tangent spaces along a curve. On an arbitrary Riemann surface there is no coordinate-free way to do this without a metric. Briefly, a vector field restricted to a curve γ is said to be parallel if its covariant derivative with respect to the tangent vector to the curve is zero. As this is a differential equation with a unique solution, the result is a linear mapping between tangent spaces along the curve. This definition extends to tensors of any type via the extension of ∇^L to arbitrary tensor fields. A more detailed presentation of parallel transport can be found in [3].

In the case of conformal metrics on a Riemann surface, the parallel transport map takes a particularly simple form. As a consequence of the decomposition (2.1) and the fact that $\mathscr{D}^{r,s}_{m,n}$ is one complex dimensional, it is possible to express the parallel transport map for arbitrary tensors in terms of a single complex number. In the next section, we make this observation explicit. Afterwards, we give formulas for the parallel transport map along geodesics for specific metrics which commonly appear in function theory.

5.1. Parallel transport using the decomposition of symmetric tensors. Let $\gamma(t)$ be a smooth curve in the Riemann surface R for t in some subinterval of $\mathbf{R}$, and denote the derivative vector at $\gamma(t)$ by $\dot\gamma(t) \in T_{\gamma(t)}R$. Define the "parallel transport map" as follows. If $\mathfrak{D}_p$ denotes the complex tensor algebra at $p \in R$, let $\mathbf{P}_{\gamma(t)\gamma(u)} \colon \mathfrak{D}_{\gamma(t)} \to \mathfrak{D}_{\gamma(u)}$ be defined by the requirements that for all $\alpha \in \mathfrak{D}_{\gamma(t)}$

$$\nabla^L_{\dot\gamma(s)} \left(\mathbf{P}_{\gamma(t)\gamma(s)}\alpha\right) = 0$$

for all s in an open interval containing t and u, and $\mathbf{P}_{\gamma(t)\,\gamma(t)}$ is the identity. Note that $\mathbf{P}_{pq}$ depends on the curve γ joining p and q, but not its parametrization. This map is simply the complex linear extension of the standard parallel transport map. In general for a tensor $\alpha \in \mathfrak{D}$ defined on a neighbourhood of the curve γ,

$$\left(\nabla^L_{\dot\gamma(u)}\alpha\right)_{\gamma(u)} = \left.\frac{d}{dt}\right|_{t=u} \mathbf{P}_{\gamma(t)\gamma(u)}\alpha_{\gamma(t)}. \tag{5.1}$$

For conformal metrics on a Riemann surface, the parallel transport map has the following properties.

Proposition 5.1. (1) *The parallel transport map commutes with the almost complex structure* J; *i.e. for* $\alpha \in \mathfrak{D}_{\gamma(t)}$

$$\mathbf{P}_{\gamma(t)\gamma(u)}\mathrm{J}\alpha = \mathrm{J}\mathbf{P}_{\gamma(t)\gamma(u)}\alpha.$$

(2) *Given* $\alpha, \beta \in \mathfrak{D}_{\gamma(t)}$,

$$\mathbf{P}_{\gamma(t)\gamma(u)}(\alpha \otimes \beta) = (\mathbf{P}_{\gamma(t)\gamma(u)}\alpha) \otimes (\mathbf{P}_{\gamma(t)\gamma(u)}\beta).$$

Similarly

$$\mathbf{P}_{\gamma(t)\gamma(u)}(\alpha \cdot \beta) = (\mathbf{P}_{\gamma(t)\gamma(u)}\alpha) \cdot (\mathbf{P}_{\gamma(t)\gamma(u)}\beta).$$

(3) *For* $\alpha \in T^{\mathbf{C}*}_{\gamma(t)}R$ *and* $Z \in T^{\mathbf{C}}_{\gamma(t)}R$,

$$\mathbf{P}_{\gamma(t)\gamma(u)}(\alpha(Z)) = (\mathbf{P}_{\gamma(t)\gamma(u)}\alpha)(\mathbf{P}_{\gamma(t)\gamma(u)}Z).$$

Proof. 1) For any tensor $\alpha \in \mathfrak{D}_{\gamma(t)}$, $\nabla^L_{\dot\gamma(u)}\mathrm{J}\mathbf{P}_{\gamma(t)\gamma(u)}\alpha = \mathrm{J}\nabla^L_{\dot\gamma(u)}\mathbf{P}_{\gamma(t)\gamma(u)}\alpha = 0$. So since $\mathrm{J}\mathbf{P}_{\gamma(t)\gamma(t)}\alpha = \mathrm{J}\alpha$ we must have that $\mathrm{J}\mathbf{P}_{\gamma(t)\gamma(u)}\alpha = \mathbf{P}_{\gamma(t)\gamma(u)}\mathrm{J}\alpha$.

2) We have that $\nabla^L_{\dot\gamma}$ satisfies the Leibniz rule with respect to tensor products. So

$$\nabla^L_{\dot\gamma(u)} \left[(\mathbf{P}_{\gamma(t)\gamma(u)}\alpha) \otimes (\mathbf{P}_{\gamma(t)\gamma(u)}\beta)\right] = 0,$$

from which the first claim follows. The second claim follows from the linearity of $\mathbf{P}_{\gamma(t)\gamma(u)}$ (equivalently, from the linearity of ∇^L).

3) By definition, the parallel transport of a scalar along a curve is constant. So we need only check that $(\mathbf{P}_{\gamma(t)\gamma(u)}\alpha)(\mathbf{P}_{\gamma(t)\gamma(u)}Z)$ is constant. Since $\nabla^L_{\dot\gamma(u)}$ commutes with contractions and satisfies the Leibniz rule, we have that

$$\begin{aligned}\frac{d}{du}[(\mathbf{P}_{\gamma(t)\gamma(u)}\alpha)(\mathbf{P}_{\gamma(t)\gamma(u)}Z)] &= (\nabla_{\dot\gamma(u)}\mathbf{P}_{\gamma(t)\gamma(u)}\alpha)(\mathbf{P}_{\gamma(t)\gamma(u)}Z)\\ &\quad + (\mathbf{P}_{\gamma(t)\gamma(u)}\alpha)(\nabla_{\dot\gamma(u)}\mathbf{P}_{\gamma(t)\gamma(u)}Z) = 0. \qquad \square\end{aligned}$$

For elements of $\mathscr{D}^{r,s}_{n,m}$ at $\gamma(t)$, $\mathbf{P}_{\gamma(t)\gamma(u)}$ can be represented by multiplication by a complex number, since $\mathscr{D}^{r,s}_{m,n}$ has complex dimension one. In fact, part two of Proposition 5.1 implies that $\mathbf{P}_{\gamma(t)\gamma(u)}$ is entirely determined by its action on $T^{(1,0)}_{\gamma(t)}R$. The following proposition summarizes this.

Proposition 5.2. *Let R be a Riemann surface equipped with a conformal metric g, and γ be a smooth curve in R. Let z be a locally biholomorphic parameter and $P_{\gamma(t)\gamma(s)}$ be the complex number representing the parallel transport map from $T^{(1,0)}_{\gamma(t)}$ to $T^{(1,0)}_{\gamma(u)}$; that is*

$$\mathbf{P}_{\gamma(t)\gamma(u)}\left(\frac{\partial}{\partial z}\right)_{\gamma(t)} = \left(P_{\gamma(t)\gamma(u)}\frac{\partial}{\partial z}\right)_{\gamma(u)}.$$

(1) *The parallel transport map along γ preserves the type of a differential:*

$$\mathbf{P}_{\gamma(t)\gamma(u)}\colon \mathscr{D}^{r,s}_{m,n\,\gamma(t)} \to \mathscr{D}^{r,s}_{m,n\,\gamma(u)}.$$

For any element

$$h(z)\frac{\partial^r}{\partial z^r}\cdot\frac{\partial^s}{\partial \bar z^s}\otimes dz^m d\bar z^n \in \mathscr{D}^{r,s}_{m,n}$$

we have that

$$\begin{aligned}\mathbf{P}_{\gamma(t)\gamma(u)}h(z)\frac{\partial^r}{\partial z^r}\cdot\frac{\partial^s}{\partial \bar z^s}\otimes dz^m d\bar z^n &= (P_{\gamma(t)\gamma(u)})^{r-m}(\overline{P_{\gamma(t)\gamma(u)}})^{s-n}h(z)\\ &\quad\cdot\frac{\partial^r}{\partial z^r}\cdot\frac{\partial^s}{\partial \bar z^s}\otimes dz^m d\bar z^n.\end{aligned}$$

(2) *If $g=\rho(z)^2|dz|^2$ in local coordinates, then*

$$|P_{\gamma(t)\gamma(u)}| = \frac{\rho(\gamma(t))}{\rho(\gamma(u))}.$$

(3) *In the special case that γ is a geodesic, if we treat the derivative vector $\dot\gamma$ as a complex number as in Remark 2.6, we have that*

$$P_{\gamma(t)\gamma(u)} = \frac{\dot\gamma(u)}{\dot\gamma(t)}.$$

Proof. 1) Since $\mathscr{D}^{1,0}_{0,0\,\gamma(t)} = T^{(1,0)}_{\gamma(t)}R$ and $\mathscr{D}^{0,1}_{0,0\,\gamma(t)} = T^{(0,1)}_{\gamma(t)}R$ are characterized as the $\pm i$-eigenspaces of J, and similarly for the tangent spaces at $\gamma(u)$, part 1) of Proposition 5.1 shows that the claim is true for $r=1, s=m=n=0$ and $s=1, r=m=n=0$. Similarly the claim is true for $m=1, r=s=n=0$ and

$n = 1, r = s = m = 0$. The general claim then follows from part 2) of Proposition 5.1.

2) $\nabla^L_{X_p} g = 0$ for any vector X_p and point p by the definition of the Levi–Civita connection, and thus g is parallel. In local coordinates, we have by part 1) of this proposition that

$$\rho(\gamma(u))^2|dz|^2 = \mathbf{P}_{\gamma(t)\gamma(u)}\rho(\gamma(t))^2|dz|^2 = \rho(\gamma(t))^2|P_{\gamma(t)\gamma(u)}|^{-2}|dz|^2.$$

3) This follows immediately from Remark 2.6 and the definition of $P_{\gamma(t)\gamma(u)}$. □

We thus have the following expression for the Levi–Civita derivative along a curve γ. Let R be a Riemann surface equipped with a pair of conformal metrics g_1 and g_2, associated to covariant and contravariant tensors respectively as in Definition 3.5. For any smooth curve γ, if $P^i_{\gamma(t)\gamma(u)}$ denotes the complex numbers associated to the parallel transport maps with respect to g_i, $i = 1, 2$, then we have the following.

Proposition 5.3. *We can express the Levi–Civita derivative of an element of* $\mathscr{D}^{r,s}_{m,n}$ *in terms of an ordinary derivative of its coefficient in the following way:*

$$\left[\nabla^L_{\dot\gamma} h(z)\frac{\partial^r}{\partial z^r}\cdot\frac{\partial^s}{\partial \bar z^s}\otimes dz^m d\bar z^n\right]_{\gamma(u)} = \frac{d}{dt}\left[\frac{(P^2_{\gamma(t)\gamma(u)})^r(\overline{P^2_{\gamma(t)\gamma(u)}})^s}{(P^1_{\gamma(t)\gamma(u)})^m(\overline{P^1_{\gamma(t)\gamma(u)}})^n} h(\gamma(t))\right]_{t=u} \cdot\frac{\partial^r}{\partial z^r}\cdot\frac{\partial^s}{\partial \bar z^s}\otimes dz^m d\bar z^n.$$

Note that this result is specific to conformal metrics on Riemann surfaces.

We close the section with a differential equation for $P_{\gamma(t)\gamma(u)}$ in the general case.

Proposition 5.4. *Let R, $g = \rho(z)|dz|^2$ and γ be as above with z a local parameter. Then*

$$\left.\frac{d}{dt}\right|_{t=r} P_{\gamma(t)\gamma(u)} = P_{\gamma(r)\gamma(u)}\Gamma^\rho(\gamma(r))\dot\gamma(r).$$

Here $\dot\gamma$ denotes the derivative of γ represented as a complex number as in Remark 2.6.

Proof. In the local parameter z, consider the vector $\partial/\partial z \in T^{(1,0)}_{\gamma(r)}R$. Define a vector field along the curve γ by $X_{\gamma(t)} = P_{\gamma(r)\gamma(t)}\partial/\partial z$. This vector field is parallel, so

$$0 = \nabla^L_{\dot\gamma}\left(P_{\gamma(r)\gamma(t)}\frac{\partial}{\partial z}\right)\Bigg|_{t=r} = \left(\left.\frac{d}{dt}\right|_{t=r} P_{\gamma(r)\gamma(t)}\right)\frac{\partial}{\partial z} + \nabla^L_{\dot\gamma(r)}\frac{\partial}{\partial z},$$

(where we are treating $\dot\gamma$ in the second term as an element of $T_{\gamma(r)}R$). We have by Proposition 3.1 that if $\dot\gamma = \gamma_1\partial/\partial x + \gamma_2\partial/\partial y$ then

$$\nabla^L_{\dot\gamma(r)}\frac{\partial}{\partial z} = \gamma_1\nabla^L_{\partial/\partial x}\frac{\partial}{\partial z} + \gamma_2\nabla^L_{\partial/\partial y}\frac{\partial}{\partial z} = (\gamma_1 + i\gamma_2)\nabla^L_{\partial/\partial z}\frac{\partial}{\partial z} = \dot\gamma(r)\Gamma^\rho(\gamma(r)).$$

Differentiating both sides of $P_{\gamma(t)\gamma(r)}P_{\gamma(r)\gamma(t)} = 1$ shows that

$$\left.\frac{d}{dt}\right|_{t=r} P_{\gamma(t)\gamma(r)} = -\left.\frac{d}{dt}\right|_{t=r} P_{\gamma(r)\gamma(t)} = \Gamma^\rho(\gamma(r))\dot\gamma(r).$$

The claim then follows from the fact that $P_{\gamma(t)\gamma(u)} = P_{\gamma(r)\gamma(u)}P_{\gamma(t)\gamma(r)}$. $\square$

5.2. The parallel transport map of some specific metrics.

Proposition 5.5. *Let $\Omega \subset \mathbf{C}$ be a planar domain, $z_1, z_2 \in \Omega$, and $f\colon \Omega \to \mathbf{C}$ be a locally univalent map. Let $|f'(z)|^2|dz|^2$ be the conformal metric obtained by pulling back the Euclidean metric $|dw|^2$ under the map $w = f(z)$. Then for any curve connecting z_1 to z_2 in Ω, we have $P_{z_1z_2} = f'(z_1)/f'(z_2)$.*

Proof. Let $a\partial/\partial z$ be a vector in $T^{(1,0)}_{z_1}\Omega$. The push forward under f to $T^{(1,0)}_{f(z_1)}\mathbf{C}$ is $af'(z_1)\partial/\partial w$. Since the Euclidean metric is flat, the parallel transport of this vector along any curve to $f(z_2)$ is also $af'(z_1)\partial/\partial w$; pulling this back to z_2 gives $a(f'(z_1)/f'(z_2))\partial/\partial z$. Alternatively, one may check that $f'(z_1)/f'(z_2)$ satisfies the differential equation of Proposition 5.4. $\square$

If there is a unique geodesic segment connecting a pair of points, then we can define a map from $\mathfrak{D}_{z_1}$ to $\mathfrak{D}_{z_2}$ by parallel transport along this geodesic. We now find the complex number $P_{z_1z_2}$ representing this map for the special metrics λ_k of Definition 4.4. Although for $k > 0$ the geodesic segment connecting a pair of points is not unique, parallel transport along the two possible segments is the same, so the map is well-defined.

Proposition 5.6. *Let $z_1, z_2 \in \mathbf{D}_k$, and let $P_{z_1z_2}$ be the complex number representing parallel transport along a geodesic segment from z_1 to z_2. Then,*

$$P_{z_1z_2} = \frac{(1 + k|z_2|^2)}{(1 + k|z_1|^2)}\frac{(1 + k\bar{z}_1z_2)}{(1 + kz_1\bar{z}_2)}.$$

Proof. First assume that $z_1 = 0$ and $z_2 = x > 0$. In this case, P_{0x} is positive real. To see this, observe that the geodesic connecting 0 and x is the segment $[0, x]$ of the real line, and the tangent vector to this curve is parallel. The claim then follows from Proposition 5.2 part three. By Proposition 5.2 part two we have that $P_{0x} = |P_{0x}| = \lambda_k(0)/\lambda_k(x) = 1 + k|x|^2$.

Now given arbitrary points z_1 and z_2 choose an isometry T so that $T(z_1) = 0$ and $T(z_2) = x$ for some $x > 0$. T has the form

$$T(z) = e^{i\theta}\frac{z - z_1}{1 + k\bar{z}_1z}.$$

for some θ. Since T is an isometry, parallel transport must be given by first pushing forward from z_1 to 0 with T, then to x using P_{0x}, and then pushing forward from x to z_2 using T^{-1}. So

$$\begin{aligned} P_{z_1z_2} &= T'(z_1)(1 + kx^2)T^{-1\prime}(x) = \frac{T'(z_1)}{T'(z_2)}(1 + k|T(z_2)|^2) \\ &= T'(z_1)\frac{|T'(z_2)|}{T'(z_2)}(1 + k|z_2|^2) \end{aligned}$$

using the identity $(1+k|T(z_2)|^2)=|T'(z_2)|(1+k|z_2|^2)$. Since $T'(z_1)=e^{i\theta}(1+k|z_1|^2)^{-1}$ and

$$\frac{T'(z_2)}{|T'(z_2)|}=e^{i\theta}\frac{1+kz_1\bar{z}_2}{1+k\bar{z}_1z_2}$$

the claim follows. □

Remark 5.7. If T is an isometry of λ_k, and $T(z_1)=w_1$, $T(z_2)=w_2$, the following identity holds:

$$\frac{T'(w_1)}{T'(w_2)}\frac{(1+k|w_1|^2)}{(1+k|w_2|^2)}\frac{(1+k\bar{w}_2w_1)}{(1+k\bar{w}_1w_2)}=\frac{(1+k|z_1|^2)}{(1+k|z_2|^2)}\frac{(1+k\bar{z}_2z_1)}{(1+kz_2\bar{z}_1)}.$$

This is equivalent to the fact that parallel transport of a vector along a geodesic commutes with isometries.

Example 5.1. The Koebe function $\kappa(z)=z/(1-z)^2$ has the interesting property that $\nabla_n\kappa/\nabla_1\kappa$ is parallel along $-1<x<1$ for $g_1=\lambda_{-1}^2|dz|^2$ and $g_2=|dz|^2$. To see this, choose $k=-1$ and $z_i=x_i\in(-1,1)$, $i=1,2$ in Proposition 5.6. The fact that $\nabla_n\kappa/\nabla_1\kappa$ is parallel is equivalent to

$$\frac{\lambda_k^{n-1}(x_2)}{\lambda_k^{n-1}(x_1)}\lambda_k^{n-1}(x_1)\frac{D_n\kappa(x_1)}{D_1\kappa(x_1)}=\lambda_k^{n-1}(x_2)\frac{D_n\kappa(x_2)}{D_1\kappa(x_2)}$$

by Proposition 5.2 part 1; in other words, that $D_n\kappa(x)/D_1\kappa(x)$ is constant on $(-1,1)$.

This is related to the following invariance of the Koebe function:

$$\kappa(z)=\frac{\kappa\circ T_x(z)-\kappa\circ T_x(0)}{(\kappa\circ T_x)'(0)} \tag{5.2}$$

for any $T_x(z)=(z+x)/(1+xz)$. Differentiating (5.2) and setting $z=0$, using Proposition 4.6 we have that

$$\kappa^{(n)}(0)=\frac{D_n\kappa(x)}{D_1\kappa(x)}$$

and so $D_n\kappa(x)/D_1\kappa(x)$ is constant. Conversely, it can be shown that the fact that $D_2\kappa(x)/D_1\kappa(x)$ is constant along $(-1,1)$ implies the invariance property (5.2).

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