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SETS OF FINITE $\mathscr{H}^1$ MEASURE THAT INTERSECT POSITIVELY MANY LINES IN INFINITELY MANY POINTS

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Abstract. We construct a planar Borel set A of finite $\mathscr{H}^1$-measure such that through positively many points of A, positively many lines meet A at infinitely many points. This answers a question of Mattila.

Recall that a set $A \subset \mathbf{R}^2$ is called rectifiable if $\mathscr{H}^1$-almost all of A can be covered by countably many C^1 curves; a set A is purely unrectifiable if it intersects every rectifiable set (or, equivalently, every C^1-curve) in a set of $\mathscr{H}^1$-measure zero. In this note, unless it is otherwise specified, by 'almost every point' we always understand $\mathscr{H}^1$-a.e. point, by a 'null set of lines through a given point' we mean a set of lines passing through the point whose directions form a nullset (according to the natural measure on the set of directions) and finally by a 'null set of lines of a given direction' we mean a collection of parallel lines of that direction, whose intersection with another (non-parallel) line has zero measure.

It is not difficult to see that for every rectifiable set C of finite measure, almost every line through almost every point of C intersects C only in finitely many points (we will recall the proof in the next section). Mattila asked (see 10.12 in [4], or Problem 12 in [5]) whether the same statement is true for every Borel set A with $\mathscr{H}^1(A) < \infty$. We answer his question negatively:

Theorem 1. *For every purely unrectifiable Borel set $E \subset \mathbf{R}^2$ with $\mathscr{H}^1(E) < \infty$ there is a rectifiable set F with $\mathscr{H}^1(F) < \infty$ such that through a.e. point of E a.e. line intersects F in an infinite set.*

By choosing $A = E \cup F$ we obtain a counterexample to Mattila's question mentioned above.

By Besicovitch's projection theorem (see [1], [3] or [4]), if E is purely unrectifiable and $\mathscr{H}^1(E) < \infty$, then almost every projection of E has measure zero. Therefore Theorem 1 is an immediate corollary of the following, slightly more general theorem:

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Theorem 2. *Let μ be a Borel measure on $\mathbf{R}^2$, and let $E \subset \mathbf{R}^2$ be a Borel set such that a.e. projection of E has measure zero, and $\mu(E) < \infty$. Then there exists a collection of vertical segments whose total length is finite, such that through μ-a.e. point of E a.e. line intersects infinitely many segments.*

It seems to be a much harder question, and it is still open, whether there exists a purely unrectifiable counterexample to Mattila's question. The conjecture is that if A is purely unrectifiable and $\mathscr{H}^1(A) < \infty$, then through a.e. point $x \in A$, a.e. line meets A only at x. If this conjecture were true then the proof of Besicovitch's projection theorem could be slightly simplified (see Remark 18.10 in [4]). The conjecture was proved by Simon and Solomyak in the special case when A is self-similar (see [6]).

1. Preliminaries

In the proof of Theorem 2 it is convenient to study the projective plane instead of $\mathbf{R}^2$; in essence, a projective plane may be thought of as an extension of the Euclidean geometry of $\mathbf{R}^2$ in which the 'direction' of each line is subsumed within the line as an extra point, and in which the set of all extra points is regarded as an extra line. We call these extra points and line 'infinite points' and 'infinite line', respectively. In this note we denote the infinite line by D, and the natural measure on D by m, and say that a subset of D is 'null' if its m-measure is zero. Notice that when one considers a 'null set of lines of a given direction' to be a nullset of lines through the infinite point belonging to that direction, then every projective transformation maps a null set of lines through a point of the projective plane to another null set of lines through the image of that point.

We will use the standard duality mapping between lines and points of the projective plane (that maps the points $(a, b) \in \mathbf{R}^2$ to the lines $y = ax + b$ and the directions $d \in D$ to the vertical lines $x = d$), allowing any theorem to be transformed by swapping 'point' and 'line', 'is contained by' and 'contains'. We say that a set of lines L is Borel if its dual set of points $\hat{L}$ in the projective plane is a Borel set, and for every Borel measure μ we define a dual measure $\hat{\mu}(L) = \mu(\hat{L})$. We denote the Lebesgue measure on $\mathbf{R}^2$ by λ (and its dual by $\hat{\lambda}$). By Fubini's theorem, one can see easily (see e.g. in [2]) that a set of lines L has zero measure with respect to $\hat{\lambda}$ if and only if L contains only a null set of lines in a.e. direction.

Another measure on the space of lines is μ_C, where C is a rectifiable set, and $\mu_C(L)$ is the $(\mathscr{H}^1 \times m)$-measure of the set

$$\{(x, d) \in C \times D : \text{the line through } x \text{ in direction } d \text{ belongs to } L\}.$$

It was shown in Theorem 16 in [2] that for every rectifiable set C, μ_C is absolutely continuous with respect to $\hat{\lambda}$. From this it follows easily (as it was claimed in the introduction) that:

Claim 1. *Through a.e. point of a rectifiable set C with $\mathscr{H}^1(C) < \infty$, a.e. line intersects C only at finitely many points.*

Indeed, the set of those lines that intersect C in infinitely many points has $\hat{\lambda}$-measure zero, since for any fixed direction there is only a nullset of lines of that direction that meet C in infinitely many points (moreover, if P_d denotes the orthogonal projection onto a line ℓ of direction d, then $\int_\ell \# P_d^{-1}(x)\, dx \le \mathscr{H}^1(C) < \infty$). It follows from the absolute continuity that the set of these lines is μ_C-null, and then applying Fubini's theorem the claim is proved.

2. Proof of Theorem 2

Let $E \subset \mathbf{R}^2$ be a Borel set such that a.e. projection of E has measure zero, and let μ be a finite Borel measure on E. Then $\hat{\mu}$ is a finite Borel measure on the dual set of lines $\hat{E}$. Let $\tilde{E}$ be the set of points covered by the lines of $\hat{E}$, and let $\tilde{\mu}$ be the measure on $\tilde{E}$ defined by

$$\tilde{\mu}(B) = \int_{\hat{E}} \mathscr{H}^1(\ell \cap B)\, d\hat{\mu}(\ell) \tag{1}$$

where $B \subset \mathbf{R}^2$ is a Borel set.

Note that $\tilde{\mu}$ is a singular measure on the plane. Indeed, since a.e. projection of E has measure zero therefore (by definition) the set of all lines intersecting E has zero measure with respect to the dual of the Lebesgue measure, and then, by duality, the set of all points covered by the lines of $\hat{E}$ has measure zero.

It requires a bit more work to understand the dual of the statement that through a.e. point of E a.e. line meets infinitely many vertical segments and the total length of the vertical segments is finite.

The dual of a vertical line segment is a 'strip', i.e. a collection of parallel lines that meet every other (non-parallel) line in an interval. The dual of the statement 'a.e. line through a given point meets infinitely many vertical segments' is 'a.e. point of a given line is covered by infinitely many strips'. The dual of 'a vertical segment through a given point' is 'a strip containing a given line'. In what follows, we will only use strips whose middle line is one of the lines of $\hat{E}$.

We can assume without loss of generality that E is bounded. Then the lines of $\hat{E}$ and the strips will have bounded slopes, and the length of a vertical segment is comparable to the width of its dual strip. Therefore our aim is to show that $\tilde{\mu}$-a.e. point of $\tilde{E}$ can be covered by infinitely many strips, such that the total width of the strips is finite. Of course it is enough to show that $\tilde{\mu}$-a.e. point can be covered by (at least one) strip such that the total width is arbitrarily small. More generally, we will show the following:

Proposition 1. *Let ν be an arbitrary non-zero singular Borel measure with the property*

$$\nu(B) > 0 \quad \Longrightarrow \quad \mathscr{H}^1(B \cap \ell) > 0 \text{ for some line } \ell.$$

Then for any $\varepsilon > 0$ there exists a $w > 0$ and a strip S of width w, so that $w < \varepsilon \nu(S)$.

Proof of Theorem 2. Let B be an arbitrary Borel set of positive and finite $\tilde{\mu}$-measure, and fix an $\varepsilon_0 > 0$. Let $\varepsilon = \varepsilon_0/\tilde{\mu}(B)$. By applying Proposition 1 inductively we define sets B_α and strips S_α of width w_α, where $B_0 = B$, $B_\alpha = B \setminus \bigcup_{\beta<\alpha} S_\beta$, and we apply the proposition to the measure ν_α which is the restriction of $\tilde{\mu}$ to the set B_α (provided that $\tilde{\mu}(B_\alpha) > 0$; otherwise we stop the construction). We infer from (1) that $\tilde{\mu}$ and hence also ν_α satisfy the requirements of Proposition 1.

Then $w_\alpha < \varepsilon\tilde{\mu}(B \cap S_\alpha \setminus \bigcup_{\beta<\alpha} S_\beta)$ for each α, and hence

$$\sum_\alpha w_\alpha < \varepsilon \sum_\alpha \tilde{\mu}(B \cap S_\alpha \setminus \bigcup_{\beta<\alpha} S_\beta).$$

Since the sets $S_\alpha \setminus \bigcup_{\beta<\alpha} S_\beta$ are pairwise disjoint and $\tilde{\mu}(B) < \infty$, therefore after countably many steps the induction will stop, and we will have $\tilde{\mu}(B \setminus \bigcup S_\alpha) = 0$ and $\sum w_\alpha < \varepsilon \sum \tilde{\mu}(B \cap S_\alpha \setminus \bigcup_{\beta<\alpha} S_\beta) = \varepsilon\tilde{\mu}(B) = \varepsilon_0$. That is, $\tilde{\mu}$-a.e. point of B is covered by strips whose total width is at most ε_0. Since B was an arbitrary set of finite measure and ε_0 was arbitrary small, Theorem 2 is proved. □

Proof of Proposition 1. Let $B(x,r)$ denote the open disc of centre x and radius r. Since ν is a singular measure, $\nu(B(x,r))/r^2 \to \infty$ as $r \to 0$ for ν-a.e. x. Let B be a Borel set of positive ν-measure on which $\nu(B(x,r))/r^2 \to \infty$ uniformly.

Since $\nu(B) > 0$ there is a line ℓ that intersects B in a set of positive length. We choose a positive and finite $m \le \mathscr{H}^1(B \cap \ell)$, and choose an r so small that $\nu(B(x,r)) > 8r^2/m\varepsilon$ for any $x \in B$. At least half of $B \cap \ell$ can be covered by disjoint discs of centres in $B \cap \ell$ and of radius r. The number of discs needed to cover a set of measure $m/2$ is at least $m/4r$, and each of these discs has measure larger than $8r^2/m\varepsilon$. Therefore the total measure of the discs is larger than $2r/\varepsilon$, and they are contained in a strip S of width $w = 2r$ with centre line ℓ. □

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