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PRIMARY SOLUTIONS OF GENERAL BELTRAMI EQUATIONS

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Dedicated to Oleńka.

Abstract. In the paper the positive answer is given to a conjecture formulated in [10], [17] and partially answered there.

The concept of a pair of primary solutions of the general Beltrami equation

$$\frac{\partial w}{\partial \bar z} = \mu(z)\,\frac{\partial w}{\partial z} + \nu(z)\overline{\frac{\partial w}{\partial z}}, \quad w\colon \Omega \to \mathbf{C}, \tag{1}$$

in a planar domain $\Omega \subseteq \mathbf{C}$ with complex-valued measurable μ and ν satisfying the ellipticity condition

$$|\mu(z)| + |\nu(z)| \le k < 1 \quad \text{a.e. in } \Omega \tag{2}$$

for some k, usually written as

$$k = \frac{K-1}{K+1} < 1, \quad 1 \le K < \infty,$$

was introduced by Iwaniec et al. in [10], [17].

In this paper, for simplicity, we restrict our considerations mainly to the case

$$\Omega \equiv \mathbf{C}, \quad \mu, \nu \ \text{compactly supported.} \tag{3}$$

Thus in the terminology of the theory of quasiconformal mappings we consider k-quasiconformal mappings $f\colon \Omega \to \mathbf{C}$ with conformal univalent extensions outside a compact subset of the complex plane $\mathbf{C}$ (also understood as the Riemann sphere $S^2 = \mathbf{P}^1$). In this geometric context we consider quasiconformal mappings in the class $W^{1,2}_{\mathrm{loc}}(\mathbf{C})$ with the condition

$$w(\infty) = \infty \tag{4}$$

and conformal near ∞.

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Definition. A pair of homeomorphic solutions $\phi, \psi\colon \Omega \to \mathbf{C}$ is called *a primary pair* if

$$\mathrm{Im}\Big(\frac{\partial\phi}{\partial z}\,\frac{\partial\overline{\psi}}{\partial z}\Big) \neq 0 \quad \text{almost everywhere in } \Omega. \tag{5}$$

In a series of recent papers by Iwaniec, written in cooperation with his collaborators, [2], [17], [10], the primary solutions of (1) have been investigated from various points of view. They turn out to play a fundamental role in describing the G-convergence phenomena in the theory of planar Beltrami operators, G-compactness results for general two-dimensional second order elliptic operators of divergence type with not necessarily symmetric coefficient matrices, homogenisation problems for second order elliptic differential operators. They are also crucial for recent applications of the two-dimensional quasiconformal mappings to material science, theory of composites, particularly laminates, and phase transitions [3], [15], [16]. In this context Iwaniec et al. also initiated, [10], [17], the very interesting theory of linear (over the real number field $\mathbf{R}$) families of k-quasiconformal mappings.

The fundamental problem here is the existence theorem of primary pairs for the general Beltrami equation (1). It is proved in [17] that under the condition

$$K < 3 \quad (k < \tfrac{1}{2}) \tag{6}$$

the existence holds for general Beltrami equations (1). For some special classes of Beltrami systems (e.g., Hölder regularity of the coefficients of (1), convexity of Ω with some boundary conditions on $\partial\Omega$) the dilatation condition (6) was also shown to be redundant for the existence theorem.

Our main result in this paper is that for the general Beltrami equation the primary pair exists without the restriction (6).

In what follows we shall freely use the basic analytical and geometrical properties of solutions of Beltrami equations (1) and planar k-quasiconformal mappings as established in detail in our papers [4], [5], [6] and Vekua's paper [20] and the monograph [21]. In particular the usual rules of differential and integral calculus hold, sets of zero Lebesgue measure are preserved under taking images and counter-images etc. All these properties are essentially connected with the basic fact, first established in [4], that the $W^{1,2}_{\mathrm{loc}}$ solutions of (1) are in the class $W^{1,p}_{\mathrm{loc}}$ for some $p > 2$, depending only on the ellipticity constant k.

Strictly following the analytical ideas and formulas developed in [4], [21], we represent the considered homeomorphic solutions of (1) or the k-quasiconformal mappings in the form

$$f(z) \equiv az - \frac{1}{\pi}\iint\limits_{\mathbf{C}} \frac{\omega(t)}{t-z}\,d\sigma_t \equiv az + T\omega(z) \tag{7}$$

for some complex a and $\omega \in L^p(\mathbf{C})$, where $d\sigma_t$ is the Lebesgue measure on $\mathbf{C}$ or

$$d\sigma_t = -\frac{1}{2i}\,dt \wedge d\bar{t}. \tag{8}$$

Then the system (1) for $f(z)$ reduces to the system of linear singular equations

$$\omega - \mu S\omega - \nu\overline{S\omega} = h, \quad h = a\mu + \overline{a}\nu \tag{9}$$
$$S\omega \equiv -\frac{1}{\pi}\iint\limits_{\mathbf{C}} \frac{\omega(t)}{(t-z)^2}\, d\sigma_t$$

in the space $L^p(\mathbf{C})$ for p satisfying

$$|p-2| < \varepsilon \quad \text{for some } \varepsilon > 0 \tag{10}$$

depending on the ellipticity constant k in (2).

For the most important simplest case of the classical Beltrami equation (1), i.e., when the coefficient $\nu \equiv 0$,

$$\frac{\partial w}{\partial \overline{z}} - \mu(z)\,\frac{\partial w}{\partial z} = 0, \tag{11}$$

the equation (9) reduces to

$$\omega - \mu S\omega = h \tag{12}$$

which is linear over the complex field $\mathbf{C}$. The equation (9) has to be considered over the real field $\mathbf{R}$ only. However in both cases the ellipticity conditions (2) ensure the unique solvability in the form of convergent Neumann series of the non-homogeneous equations (9) and (12). All these facts have been discussed and used in [4], [6], [5] and are fundamental in the literature on quasiconformal mappings and planar elliptic partial differential equations in the following years. As shown in [4] and [6], the basic existence theorems of planar quasiconformal mappings related with linear, quasilinear and non-linear two-dimensional elliptic systems are consequences of the discussed methods.

In particular, for the Beltrami equation (11) and the general Beltrami equation (1) the global homeomorphism exists. By this we mean solutions of (11) realising the homeomorphic mapping $w\colon \mathbf{C} \to \mathbf{C}$ of the whole complex plane $\mathbf{C}$ onto itself, with the condition $w(\infty) = \infty$. Properly normalised the global solutions are unique. If the coefficients μ and ν are compactly supported, the normalisation can be taken as in formula (7). Homeomorphic solutions are measurable in the sense that (Lebesgue) measurable sets have measurable images and null sets are mapped on null sets and the set function $\operatorname{mes} w(e)$ defined for Borel sets $e \subset \mathbf{C}$ is absolutely continuous:

$$\operatorname{mes} w(e) = \iint\limits_{e} \Big(\Big|\frac{\partial w}{\partial z}\Big|^2 - \Big|\frac{\partial w}{\partial \overline{z}}\Big|^2\Big)\, dx\, dy$$

where $J_z^w \equiv \big|\frac{\partial w}{\partial z}\big|^2 - \big|\frac{\partial w}{\partial \overline{z}}\big|^2$ is the Jacobian of the mapping $w = w(z)$. The same holds for the inverse function $z = w^{-1}(\zeta)$, $\zeta = w(z)$. It follows that for a non-constant homeomorphic solution of (1) the Jacobian $J_z^w > 0$ a.e. and consequently $\frac{\partial w}{\partial z} \neq 0$ a.e. Also the normalised global homeomorphism depends smoothly (real analytically) on any parameters $t = (t_1, \ldots, t_n)$ smoothly (real analytically) included in the coefficients μ and ν of the equation (1). For the Beltrami equations (11) the dependence on the parameters t may be complex analytic. All these facts have been proved in detail in [6], or are direct consequences of the formulas and methods presented there; see also [18], [8], [9]. Except the dependence on parameters, all the facts mentioned above are crucial in the proof of our main theorem below.

The main theorem

We present here the proof of our main theorem for the special case of Beltrami equations (1) in the form

$$\frac{\partial w}{\partial \overline{z}} - q(z)\Big(\frac{\partial w}{\partial z} - \frac{\partial \overline{w}}{\partial z}\Big) = 0 \tag{13}$$

or $\mu + \nu \equiv 0$ in (1). The ellipticity condition (2) has then the form

$$|q(z)| < k_0 < \tfrac{1}{2}, \quad 2k_0 = k. \tag{14}$$

The general case is reduced to (13) by a change of variable in the image domain of w, see [6], §6, pp. 490–492. A characteristic feature of the equations (13) is that the identity map $w(z) \equiv z$ is a solution of (13) for *any* dilatation $q(z)$.

Probably the class (13) of Beltrami equations was discussed for the first time in our paper [6] and appeared there in connection with the uniqueness problem in the Riemann mapping theorem for the general Beltrami equations (1).

If we take in (5) $\phi(z) \equiv z$, then the existence problem of primary pair for (13) takes the form: Does the equation (13) admit a global homeomorphic solution $\psi\colon \mathbf{C} \to \mathbf{C}$ of the form (7) such that

$$\operatorname{Im} \frac{\partial \psi}{\partial z} \neq 0 \quad \text{a.e.?} \tag{15}$$

This implies in particular that the coefficient a in (7) satisfies the condition

$$\operatorname{Im} a \neq 0. \tag{16}$$

Theorem. *The Beltrami equation*

$$\frac{\partial w}{\partial \overline{z}} - q(z)\Big(\frac{\partial w}{\partial z} - \overline{\frac{\partial w}{\partial z}}\Big) = 0$$

with the compactly supported measurable dilatation ($q \equiv 0$ *in the neighbourhood of* ∞) *satisfying the ellipticity condition* (14) *admits a global solution* $\psi\colon \mathbf{C} \to \mathbf{C}$ *satisfying condition* (15). *Then the pair* $(z, \psi(z))$ *is a primary pair for the Beltrami equation* (13).

Proof. The main idea of the proof is the "factorisation" of solutions of (13) through the solutions of a Beltrami equation

$$\frac{\partial w_0}{\partial \overline{z}} - q_1(z)\,\frac{\partial w_0}{\partial z} = 0 \tag{17}$$

and the "conjugate" or inverse Beltrami equation

$$\frac{\partial w}{\partial \overline{\zeta}} + \widetilde{q}(\zeta)\,\overline{\frac{\partial w}{\partial \zeta}} = 0 \tag{18}$$

with a special choice of the auxiliary dilatations $q_1(z)$ and $\widetilde{q}(\zeta)$ in (17) and (18). The function $q_1(z)$ in (17) is determined by $q(z)$ in (13) by the formula

$$q(z) = \frac{q_1(z)}{1 + |q_1(z)|^2}\,.$$

The uniform condition (14) ensures that (17) is uniformly elliptic also. The choice of $q_1(z)$ is "canonical": it depends on $q(z)$ in (13) only.

The choice of $\widetilde{q}(\zeta)$ for the "conjugate" Beltrami equation depends on the selected global homeomorphic solution of (17). If w_0 is a normalised—"principal"—homeomorphism of (17) (i.e. it is determined by putting $a = 1$ in (7)) then any other global homeomorphism $\zeta(z)$ is determined by the formula

$$\zeta(z) = L(w_0(z)) \tag{19}$$

for some linear $L(w_0) = bw_0 + c$.

Assuming a $\zeta(z)$ fixed we determine $\widetilde{q}(\zeta)$ in the equation (18) by

$$\widetilde{q}(\zeta) \equiv q_1\big(z^{-1}(\zeta)\big) \quad \text{or} \quad \widetilde{q}(\zeta(z)) \equiv q_1(z). \tag{20}$$

If $\widetilde{w}_0(\zeta)$ is the principal homeomorphism of (18) then the following two crucial facts hold.

Proposition 1. *The composition mapping $w(z) = F(z) = \widetilde{w}_0(\zeta(z))$ is a solution of* (13).

Proposition 2. *For any pair $w(z), \widetilde{w}(\zeta)$ of global solutions of equations* (17) *and* (18) *respectively, such that the relation* (20) ($with\ \zeta(z) \equiv w(z)$) *holds the mapping $G(\zeta) = w\big(\widetilde{w}(\zeta)\big)$ is a conformal homeomorphism of the ζ-plane preserving ∞.*

Obviously the linear asymptotic behaviour of $G(\zeta)$ at $\zeta = \infty$ is determined by the asymptotic behaviour of the components $\widetilde{w}(\zeta)$ and $w(z)$. We recall that both $q_1(z)$ and $\widetilde{q}(\zeta)$ are $\equiv 0$ near infinity and consequently $w(z)$ and $\widetilde{w}(\zeta)$ are conformal and asymptotically linear at ∞.

Proof of Proposition 1. Indeed,

$$\frac{\partial w}{\partial z} = \frac{\partial \widetilde{w}_0}{\partial \zeta}\frac{\partial \zeta}{\partial z} + \frac{\partial \widetilde{w}_0}{\partial \overline{\zeta}}\frac{\partial \overline{\zeta}}{\partial z} = \frac{\partial \widetilde{w}_0}{\partial \zeta}\frac{\partial \zeta}{\partial z} - q_1(\zeta)\overline{q_1(z)}\,\overline{\frac{\partial \zeta}{\partial z}}\,\overline{\frac{\partial \widetilde{w}_0}{\partial \zeta}}$$

and taking into account (18), we get

$$\frac{\partial w}{\partial z} - \frac{\partial \overline{w}}{\partial z} = \left(\frac{\partial \widetilde{w}_0}{\partial \zeta}\frac{\partial \zeta}{\partial z} - \frac{\partial \widetilde{w}_0}{\partial \overline{\zeta}}\frac{\partial \overline{\zeta}}{\partial z}\right)\big(1 + |q_1(z)|^2\big).$$

Also

$$\frac{\partial w}{\partial \overline{z}} = q_1(z)\left(\frac{\partial \widetilde{w}_0}{\partial \zeta}\frac{\partial \zeta}{\partial z} - \frac{\partial \widetilde{w}_0}{\partial \overline{\zeta}}\frac{\partial \overline{\zeta}}{\partial z}\right).$$

Hence

$$\frac{\partial w}{\partial \overline{z}} = \frac{q_1(z)}{1 + |q_1(z)|^2}\left(\frac{\partial w}{\partial z} - \frac{\partial \overline{w}}{\partial z}\right). \qquad \square$$

Proof of Proposition 2. We have

$$\frac{\partial G}{\partial \overline{\zeta}} = \frac{\partial w_0}{\partial \overline{z}}\,\frac{\partial \overline{\widetilde{w}_0}}{\partial \overline{\zeta}} + \frac{\partial w_0}{\partial z}\,\frac{\partial \widetilde{w}_0}{\partial \overline{\zeta}} = q_1(z)\,\frac{\partial w_0}{\partial z}\,\frac{\partial \overline{\widetilde{w}_0}}{\partial \overline{\zeta}} + \frac{\partial w_0}{\partial z}\cdot(-\widetilde{q}(\zeta))\,\overline{\frac{\partial \widetilde{w}_0}{\partial \zeta}}$$
$$= \big[q_1(z) - \widetilde{q}(\zeta)\big]\,\frac{\partial w_0}{\partial z}\cdot\overline{\frac{\partial \widetilde{w}_0}{\partial \zeta}} = 0$$

since by (20) $q_1(z) - \widetilde{q}(\zeta(z)) \equiv 0$. □

Now we apply Proposition 1 and Proposition 2 to two distinct cases. In both cases $w_0(z)$ is the same principal solution of (17) normalised at ∞ by (7) with $a = 1$.

Case I. By $\widetilde{w}(\zeta)$ we denote the global solution of equation (18) normalised at ∞ by the condition

$$\widetilde{w}(\zeta) \sim \zeta + O\big(\tfrac{1}{|\zeta|}\big). \tag{21}$$

By the basic existence theorem for systems (9) it exists, is unique and may be represented by formula (7) with $a = 1$. By Proposition 2 $G(\zeta) = w_0\big(\widetilde{w}(\zeta)\big)$ is conformal on the complex ζ-plane. Also near $\zeta = \infty$ it behaves like (21).

Hence $G(\zeta) = w_0\big(\widetilde{w}(\zeta)\big) \equiv \zeta$. Then also $F(z) \equiv \widetilde{w}\big(w_0(z)\big) \equiv z$ and we recover the equations (17) and (18) as a well known [6], [21] pair of classical Beltrami and inverse Beltrami equations for a quasiconformal mapping and its inverse. In this case the identity $F(z) = \widetilde{w}\big(w_0(z)\big) \equiv z$ may also be obtained from the Liouville theorem for the equation (13) since by Proposition 1 $F(z)$ and $w(z) \equiv z$ are both global solutions of (13) and

$$F(z) - z = o(1) \quad \text{for} \quad |z| \to \infty.$$

Case II. Denote by $\widehat{w}(\zeta)$ the global solution of the equation (18) normalised at ∞ by the condition

$$\widehat{w}(\zeta) = i\zeta + O\big(\tfrac{1}{|\zeta|}\big)$$

or represented by (7) with $a = i$. It exists and is unique. By Proposition 2 we get then analogously

$$G(\zeta) = w_0\big(\widehat{w}(\zeta)\big) \equiv i\zeta, \quad \zeta \in \mathbf{C} \tag{22}$$

on the whole ζ-plane.

Again we consider the composition $F(z) = \widehat{w}\big(w_0(z)\big)$. By Proposition 1 it is a global solution of equation (13). For $\frac{\partial F}{\partial z} - \overline{\frac{\partial F}{\partial z}}$ we have the formula

$$2i\,\mathrm{Im}\,\frac{\partial F}{\partial z} = \left(\frac{\partial \widehat{w}}{\partial \zeta}\,\overline{\frac{\partial \widehat{w}}{\partial \zeta}}\,\overline{\frac{\partial \zeta}{\partial z}}\right)\big(1 + |q_1(z)|^2\big), \quad \zeta(z) \equiv w_0(z).$$

Differentiating (22) with respect to ζ and $\overline{\zeta}$ we have

$$\frac{\partial w_0}{\partial z}\frac{\partial \widehat{w}}{\partial \zeta}+\frac{\partial w_0}{\partial \overline{z}}\frac{\partial \overline{\widehat{w}}}{\partial \zeta}=i$$

$$\frac{\partial w_0}{\partial z}\frac{\partial \widehat{w}}{\partial \overline{\zeta}}+\frac{\partial w_0}{\partial \overline{z}}\frac{\partial \overline{\widehat{w}}}{\partial \overline{\zeta}}=0.$$

Eliminating $\frac{\partial w_0}{\partial \overline{z}}$ we get

$$\frac{\partial w_0}{\partial z}\left(\frac{\partial \widehat{w}}{\partial \zeta}\,\overline{\frac{\partial \widehat{w}}{\partial \zeta}}-\frac{\partial \widehat{w}}{\partial \overline{\zeta}}\,\overline{\frac{\partial \widehat{w}}{\partial \overline{\zeta}}}\right)=i\,\overline{\frac{\partial \widehat{w}}{\partial \zeta}}$$

and

$$\frac{\partial w_0}{\partial z}\frac{\partial \widehat{w}}{\partial \zeta}J_{\zeta}^{\hat{w}}=i\left|\frac{\partial \widehat{w}}{\partial \zeta}\right|^2$$

where $J_{\zeta}^{\hat{w}}$ is the Jacobian of the quasiconformal mapping $\widehat{w}=\widehat{w}(\zeta)$.

Finally

$$\operatorname{Im}\frac{\partial F}{\partial z}\equiv\left(1+|q_1(z)|^2\right)\left|\frac{\partial \widehat{w}}{\partial \zeta}\right|^2 J_{\hat{w}}^{\zeta}\neq 0 \quad \text{a.e. on the } z\text{-plane}$$

and the proof of the theorem is complete.

Corollary. *The general Beltrami equation* (1)–(2) *with the compactly supported coefficients* μ *and* ν *admits a primary pair.*

This solves in the positive Conjecture 1 in [10].

Final remarks

Since the primary pair (ϕ,ψ) for (1) allows to recover the coefficients μ and ν of the general Beltrami equation (1),

$$\mu(z)=\frac{\phi_{\overline{z}}\overline{\psi_z}-\psi_{\overline{z}}\overline{\phi_z}}{2i\operatorname{Im}(\phi_z\overline{\psi_z})},\quad \nu(z)=-\frac{\phi_{\overline{z}}\psi_z-\psi_{\overline{z}}\phi_z}{2i\operatorname{Im}(\phi_z\overline{\psi_z})},$$

it plays such a fundamental role in describing the totality (infinite dimensional) of all solutions of (1), in particular in discussing the G-convergence phenomenon and the G-closed classes of elliptic operators in the complex plane.

As is well known (see e.g. Vekua's monograph [21]), the auxiliary equations (17) and (18) admit global solutions without necessarily assuming that $|\mu|+|\nu|$ is compactly supported. The factorisation procedure used in the proof of our theorem then can be extended to the general Beltrami equation (1) in the whole complex plane $\mathbf{C}$.

Geometric meaning of the class of Beltrami equations (13) was explained in our paper [6] and can be understood, as well as the geometric meaning of the general Beltrami equations (1), on the basis of the Lavrentiev theory of characterisics of quasiconformal mappings discussed in detail in the unduly somehow neglected interesting book of Volkovysky [22], which should be useful in discussing the important problem of invariant infinitesimal meaning of the condition (5).

There are also many other interesting analytical and geometrical problems related with the important class of Beltrami equations (13). Some of these will be discussed in a subsequent publication.

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