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QUASIHYPERBOLIC GEOMETRY OF DOMAINS IN HILBERT SPACES

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Abstract. The paper deals with basic smoothness and bilipschitz properties of geodesics, balls and spheres in the quasihyperbolic metric of a domain in a Hilbert space.

1. Introduction

1.1. Let E be a real Hilbert space with $\dim E \geq 2$ and let $G \subsetneq E$ be a domain. We recall that the *quasihyperbolic length* of a rectifiable arc $\gamma \subset G$ or a path γ in G is the number

$$l_k(\gamma) = \int_\gamma \frac{|dx|}{\delta(x)},$$

where $\delta(x) = d(x, E \setminus G) = d(x, \partial G)$. For $a, b \in G$, the *quasihyperbolic distance* $k(a,b) = k_G(a,b)$ is defined by

$$k(a,b) = \inf l_k(\gamma)$$

over all rectifiable arcs γ joining a and b in G. An arc γ from a to b is a *quasihyperbolic geodesic* or briefly a *geodesic* if $l_k(\gamma) = k(a,b)$.

This paper deals with basic smoothness and bilipschitz properties of geodesics, balls and spheres in the quasihyperbolic metric of a domain in a Hilbert space.

The quasihyperbolic metric of a domain in $\mathbf{R}^n$ was introduced by Gehring and Palka [GP] in 1976, and it has turned out to be a useful tool, for example, in the theory of quasiconformal maps. However, several questions on the basic quasihyperbolic geometry remain open. Important results on quasihyperbolic geodesics in domains $G \subset \mathbf{R}^n$ were obtained by Martin [Ma] in 1985. For example, he proved that the geodesics, which always exist in domains of finite-dimensional spaces by [GO], are C^1 smooth.

We start by giving in Section 2 a new proof for Martin's smoothness result, valid in all Hilbert spaces. We next show that there is a universal positive constant r_0 such that each quasihyperbolic ball of radius $r < r_0$ is strictly starlike and can be mapped onto a round ball by an $M(r)$-bilipschitz map of E onto itself. Moreover, $M(r) \to 1$ as $r \to 0$. The easier case $\dim E < \infty$ is considered in Section 3 and the general case in Section 4. Tangential properties of a quasihyperbolic sphere S are considered in Section 5. For example, if $\dim E < \infty$, then S has an inner normal

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vector (defined in 5.2) at each point, and for $r < r_0$, a normal vector and therefore a tangent hyperplane in a dense set of points.

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1.2. Notation and terminology. Throughout the paper, E is a real Hilbert space with $\dim E \geq 2$ and $G \subsetneq E$ is a domain. The inner product is written as $x \cdot y$ and the norm as $|x| = \sqrt{x \cdot x}$. For nonzero vectors $x, y \in E$, we let $\operatorname{ang}(x, y)$ denote the angle between x and y. Thus

$$x \cdot y = |x||y| \cos \operatorname{ang}(x, y), \quad 0 \leq \operatorname{ang}(x, y) \leq \pi.$$

For open and closed balls and for spheres in E we use the customary notation $B(x, r)$, $\bar{B}(x, r)$, $S(x, r)$, and the center x may be omitted if $x = 0$. Thus $B(1)$ is the open unit ball of E. The distance between nonempty sets $A, B \subset E$ is $d(A, B)$. For real numbers s, t we write $s \wedge t = \min\{s, t\}$, $s \vee t = \max\{s, t\}$.

An *arc* in E is a homeomorphic image of a real interval, which is assumed to be closed unless otherwise indicated. Arcs are assumed to be *oriented*, that is, equipped with one of the two possible orderings, written as $x \leq y$. We write $\gamma\colon a \curvearrowright b$ if γ is an arc with first point a and last point b. The length of an arc γ is $l(\gamma)$. We let $\gamma[x, y]$ denote the closed subarc of γ between points $x, y \in \gamma$. For half open and open subarcs we use the natural notation $\gamma[x, y)$, $\gamma(x, y]$, $\gamma(x, y)$.

2. Smoothness of geodesics

Martin [Ma] proved that a quasihyperbolic geodesic in a domain $G \subset \mathbf{R}^n$ is C^1 smooth. It is not always C^2 but the derivative satisfies a Lipschitz condition. In this section we extend this result to all Hilbert spaces. Some parts of Martin's proof [Ma, 4.3] make use of finite dimensionality, but it would be possible to replace these by new arguments to make the proof valid in the general case. However, the proof of the present paper differs considerably from the proof of [Ma]. It is shorter and more straightforward, does not involve normal hyperplanes and gives a slightly better Lipschitz constant. Moreover, it gives without extra work smoothness at the endpoints, which were ignored in [Ma].

On the other hand, we shall make substantial use of the important idea of [Ma] on ball convexity of quasihyperbolic geodesics. In fact, this is all that is needed for the smoothness proof.

2.1. Ball convexity. An arc γ in a domain G is said to be *ball convex* in G if $\gamma \cap \bar{B}$ is connected for every open ball $B \subset G$. In other words, $a, b \in \gamma \cap \bar{B}$ implies that $\gamma[a, b] \subset \bar{B}$. Trivially, every subarc of a ball convex arc is ball convex.

If $B \subset G$ is a ball and if $\gamma\colon a \curvearrowright b$ is a ball convex arc in G with $a, b \in \bar{B}$, then $\gamma \subset \bar{B}$. If also $\bar{B} \subset G$, we can say slightly more:

2.2. Lemma. *Let B be an open ball with $\bar{B} \subset G$ and let $\gamma\colon a \curvearrowright b$ be a ball convex arc with $a, b \in \bar{B}$. Then $\gamma(a, b) \subset B$.*

Proof. If there is a point $z \in \gamma(a,b) \cap \partial B$, it is easy to find a ball $B' \subset G$ such that $a, b \in \bar{B}'$ and $z \notin \bar{B}'$; see the proof of [Ma, 2.2].This contradicts the ball convexity of γ. □

We next prove that quasihyperbolic geodesics are ball convex. The proofs of 2.3 and 2.4 are essentially from [Ma, pp. 170-171]. The following auxiliary result is needed also in 4.3.

2.3. Lemma. *Let G be a domain containing the unit ball $B(1)$ and let $\gamma\colon a \curvearrowright b$ be an arc in G with $\gamma \cap \bar{B}(1) = \{a, b\}$. Then*

$$l_k(u\gamma) \le \int_\gamma \frac{|dx|}{|x|\delta(x)} < l_k(\gamma),$$

where u is the inversion $ux = x/|x|^2$.

Proof. Let $x, z \in E \setminus B(1)$. As $(|x|^2 - 1)(|z|^2 - 1) \ge 0$, the identity

$$|x|^2 |ux - z|^2 = 1 - 2x \cdot z + |x|^2 |z|^2$$

implies that $|x - z| \le |x||ux - z|$. For each pair $x \in G \setminus B(1), z \in \partial G$ we thus have $\delta(x) \le |x||ux - z|$, whence $\delta(x) \le |x|\delta(ux)$ for all $x \in G \setminus B(1)$. As u is conformal and the norm of the derivative is $|u'(x)| = 1/|x|^2$, we obtain

$$|u'(x)| \le \delta(ux)/|x|\delta(x)$$

for all $x \in \gamma$. Hence

$$l_k(u\gamma) = \int_{u\gamma} \frac{|dx|}{\delta(x)} = \int_\gamma \frac{|u'(x)|}{\delta(ux)} |dx| \le \int_\gamma \frac{|dx|}{|x|\delta(x)} < \int_\gamma \frac{|dx|}{\delta(x)} = l_k(\gamma). \qquad \square$$

2.4. Theorem. *Every quasihyperbolic geodesic in a domain G is ball convex in G.*

Proof. Assume that the theorem is false. Then there are a ball $B \subset G$ and a geodesic $\gamma\colon a \curvearrowright b$ such that $\gamma \cap \bar{B} = \{a, b\}$. We may assume that $B = B(1)$. Since $u\gamma\colon a \curvearrowright b$ and since γ is a geodesic, Lemma 2.3 gives a contradiction. □

2.5. Shuttles. Let $a, b \in E$ and let $R \ge |a - b|/2$. The set

$$Y(a, b; R) = \bigcap \{B(z, R) \colon |z - a| = |z - b| = R\}$$

is the open *shuttle* with chord $[a, b]$ and radius R and its closure

$$\bar{Y}(a, b; R) = \bigcap \{\bar{B}(z, R) \colon |z - a| = |z - b| = R\}$$

is the closed shuttle. The shuttle Y is obtained by rotating a circular arc of radius R keeping its endpoints fixed. The *angle* of Y is

$$\alpha = \sup\{\operatorname{ang}(b - a, x - a) \colon x \in Y(a, b; R)\}.$$

Thus

$$\sin \alpha = |a - b|/2R, \quad 0 < \alpha \le \pi/2.$$

We also write

$$Y^*(a,b;R) = \bigcup\{B(z,R)\colon |z-a| = |z-b| = R\},$$
$$\bar{Y}^*(a,b;R) = \bigcup\{\bar{B}(z,R)\colon |z-a| = |z-b| = R\}.$$

From the definition of ball convexity and from 2.2 we readily obtain:

2.6. Theorem. *Let γ be a ball convex arc in a domain G, for example, a quasihyperbolic geodesic. Let $x, y \in \gamma$ and let $|x-y| \le 2R$. If $Y^*(x,y;R) \subset G$ then $\gamma[x,y] \subset \bar{Y}(x,y;R)$. In particular, this holds if $2R \le \delta(x) \vee \delta(y)$.*

If $\bar{Y}^*(x,y;R) \subset G$, then $\gamma(x,y) \subset Y(x,y;R)$. In particular, this holds if $2R < \delta(x) \vee \delta(y)$. □

As a preparation for the smoothness theorem 2.8 we consider arcs in a Hilbert space. Recall from 1.2 that an arc $\gamma\colon a \curvearrowright b$ is considered as an ordered set.

2.7. Smooth arcs. Let $\gamma\colon a \curvearrowright b$ be an arc in E and let $z \in \gamma[a,b)$. A unit vector $v \in E$ is the *right tangent vector* of γ at z if

$$v = \lim_{\substack{x\to z\\ x\in\gamma, x>z}} \frac{x-z}{|x-z|}.$$

Similarly, if the left limit of $(z-x)/|z-x|$ exists, it is the *left tangent vector* of γ at $z \in \gamma(a,b]$ (observe the sign). If both of these exist and are equal, their common value is the *tangent vector* of γ at x.

Since $|u-v|^2 = 2(1-u\cdot v)$ for unit vectors $u, v \in E$, the following conditions are equivalent for a unit vector v and for a point $z \in \gamma[a,b)$:

(1) v is the right tangent vector of γ at z,

(2) the right limit of $\operatorname{ang}(x-z, v)$ is 0 as $x \to z$ on γ.

(3) the right limit of $((x-z)\cdot v)/|x-z|$ is 1 as $x \to z$ on γ.

The corresponding result holds for $z \in \gamma(a,b]$ if "right" is replaced by "left" and $x-z$ by $z-x$.

We say that an arc $\gamma\colon a \curvearrowright b$ is C^1 *smooth* or briefly *smooth* if

(1) the tangent vector $v(z)$ exists at each interior point $z \in \gamma$,

(2) the right tangent vector $v(a)$ exists at a and the left tangent vector $v(b)$ exists at b,

(3) the function $v\colon \gamma \to E$ is continuous.

It is probably more common to call an arc γ smooth if it has a C^1 parametrization $\varphi\colon [t_1, t_2] \to \gamma$ such that $\varphi'(t) \ne 0$ for all t. Clearly this condition implies smoothness with $v(\varphi(t)) = \varphi'(t)/|\varphi'(t)|$. The converse must be well known but it is not quite trivial and not easy to find in the literature. We give a proof in the appendix. In fact, a smooth arc is rectifiable with a C^1 length parametrization $g\colon [0, l(\gamma)] \to \gamma$ such that $g'(t) = v(g(t))$ for $0 \le t \le l(\gamma)$.

We next give the main result of this section.

2.8. Theorem. *Suppose that $\gamma \subset G$ is a ball convex arc, for example, a quasihyperbolic geodesic. Then γ is smooth. The tangent vectors $v(x)$ (one-sided at* *endpoints) satisfy the local Lipschitz condition*

$$|v(x) - v(y)| \le 2|x - y|/\delta(x) \tag{2.9}$$

for all $x, y \in \gamma$.

Proof. We first show that the right tangent vector $v(x)$ exists at each $x \in \gamma[a, b)$. Set $R = \delta(x)/2$, let $0 < t < R$, and choose a point $z \in \gamma(x, b)$ with $|x - z| \le t$. By the shuttle theorem 2.6 we have $\gamma[x, z] \subset \bar{Y}(x, z; R)$.

If $y_1, y_2 \in \gamma(x, z)$, then $\operatorname{ang}(y_1 - x, y_2 - x) \le 2\alpha$ where α is the angle of the shuttle. Setting $u_i = (y_i - x)/|y_i - x|$ we have $\operatorname{ang}(u_1, u_2) \le 2\alpha$, whence $|u_1 - u_2| \le 2 \sin\alpha \le t/R \to 0$ as $t \to 0$. As E is complete, the right tangent vector exists. Similarly, the left tangent vector $w(x)$ exists at each $x \in \gamma(a, b]$.

Let $x \in \gamma(a, b)$. We must show that $w(x) = v(x)$. Set $R = \delta(x)/4$ and let $t < |x - a| \wedge |x - b| \wedge R$. Let $y, z \in \gamma$ be points such that $y < x < z$ and $|y - x| = |z - x| = t$. Then $\delta(y) \ge \delta(x) - R = 3R$, whence $|y - z| \le 2t < 2R < \delta(y)$. By 2.6 we have $x \in \bar{Y}(y, z; R)$, and thus $\operatorname{ang}(z - x, x - y) \le 2\alpha$ where $\sin\alpha = |y - z|/2R \le t/R$. Hence

$$\operatorname{ang}(v(x), w(x)) \le \operatorname{ang}(v(x), z - x) + 2\alpha + \operatorname{ang}(x - y, w(x)).$$

Letting $t \to 0$ we get $w(x) = v(x)$.

It remains to verify the local Lipschitz condition (2.9). Let $x, y \in \gamma$. We may assume that $|x - y| < \delta(x)$, since otherwise (2.9) is trivially true. Setting $R = \delta(x)/2$ we have $\gamma[x, y] \subset \bar{Y}(x, y; R)$ by 2.6, and the angle α of the shuttle satisfies $\sin\alpha = |x - y|/2R$. It follows that $\operatorname{ang}(v(x), v(y)) \le 2\alpha$, which implies that

$$|v(x) - v(y)| \le 2 \sin\alpha = 2|x - y|/\delta(x). \qquad \square$$

2.10. Convex domains. Let us say that an arc γ in a domain G is *strongly ball convex* in G if $\gamma \cap \bar{B}$ is connected whenever B is a ball with center in G. Trivially, such an arc is ball convex in G. In [Vä2, 4.4] we proved that a quasihyperbolic geodesic in a convex domain is always strongly ball convex.

The following result gives "strong versions" of 2.6 and 2.8.

2.11. Theorem. *Let γ be a strongly ball convex arc in a domain G, for example, a quasihyperbolic geodesic in a convex domain. Let $x, y \in \gamma$ and let $|x - y|/2 \le R < \delta(x) \vee \delta(y)$. Then $\gamma[x, y] \subset \bar{Y}(x, y; R)$.*

Furthermore, γ is smooth and the tangent vectors satisfy the local Lipschitz condition

$$|v(x) - v(y)| \le |x - y|/\delta(x) \tag{2.12}$$

for all $x, y \in \gamma$.

Proof. In the first part it suffices to observe that $z \in G$ whenever $|x - z| = |y - z| = R$.

To prove (2.12) we may assume that $|x - y| < 2\delta(x)$. For $|x - y|/2 < R < \delta(x)$, the first part implies that $\gamma[x, y] \subset \bar{Y}(x, y; R)$. As in 2.8 we get

$$|v(x) - v(y)| \le 2 \sin\alpha = |x - y|/R,$$

which gives (2.12) as $R \to \delta(x)$. $\square$

2.13. Remark. The case where G is a half plane shows that the Lipschitz constant in (2.12) is sharp.

3. Quasihyperbolic balls

3.1. Notation. Let $G \subset E$ be a domain and let $a \in G$, $r > 0$. For quasihyperbolic balls and spheres in G we use the notation

$$B_k(a,r) = \{x \in G\colon k(x,a) < r\},\ \bar{B}_k(a,r) = \{x \in G\colon k(x,a) \le r\},$$
$$S_k(a,r) = \{x \in G\colon k(x,a) = r\}.$$

We shall show that for r less than a universal constant r_0, the quasihyperbolic ball $B_k(a,r)$ is strictly starlike and bilipschitz equivalent to a round ball. In this section we prove this in the case $\dim E < \infty$, which is easier than the general case, because one can make use of quasihyperbolic geodesics. In the general case these must be replaced by quasigeodesics, which makes the theory somewhat more complicated. The general case will be treated in Section 4.

We shall make use of the following standard estimates for the quasihyperbolic metric.

3.2. Lemma. *Let $G \subset E$ be a domain.*
(1) $k(a,b) \ge \log\left(1 + \dfrac{|a-b|}{\delta(a) \wedge \delta(b)}\right)$ *for all $a,b \in G$.*
(2) *If $a,x \in G$ with $k(a,x) = r$, then $1 - e^{-r} \le |x-a|/\delta(a) \le e^r - 1$.*
(3) *If $a \in G$, $0 < q < 1$, and $x,y \in \bar{B}(a, q\delta(a))$, then*

$$k(x,y) \le \frac{1}{1-q}\frac{|x-y|}{\delta(a)}.$$

(4) *If, in addition, $q \le 1/2$, then*

$$k(x,y) \ge \frac{1}{1+2q}\frac{|x-y|}{\delta(a)}.$$

Proof. Proofs of (1),(3),(4) are given, for example, in [Vä1, 3.7], and (2) is given in [Vu, (3.9)]. $\square$

3.3. Notation. For $a,b \in E$, $a \ne b$, we let $A(a,b)$ denote the open ball with center $(a+b)/2$ and radius $|a-b|/2$. Thus a and b are diametrically opposite points of the sphere $\partial A(a,b)$.

Let $G \subset E$ be a domain and let $a \in G$. For each unit vector $v \in E$ we set

$$\lambda(v) = \sup\{t > 0\colon A(a, a+tv) \subset G\}.$$

Then $\delta(a) \le \lambda(v) \le \infty$. The case $\lambda(v) = \infty$ occurs iff G contains the half plane $\{x\colon (x-a)\cdot v > 0\}$. We write

$$\begin{aligned} J(a,v) &= \{a + tv\colon 0 \le t < \lambda(v)\},\\ U(a) &= \bigcup\{J(a,v)\colon |v| = 1\},\\ F(a) &= \{a + \lambda(v)v\colon |v| = 1,\ \lambda(v) < \infty\}. \end{aligned}$$

Observe that if $\lambda(v) < \infty$, then $A(a, a+\lambda(v)v) \subset G$. Hence each $J(a,v)$ is contained in G and $U(a)$ is a connected (even starlike) subset of G. Moreover,

$$B(a,\delta(a)) \subset U(a). \tag{3.4}$$

3.5. Lemma. *The set $U(a)$ is open and $F(a) = \partial U(a)$. Moreover, $U(a)$ is a component of $G \setminus F(a)$.*

Proof. We may assume that $a = 0$. It suffices to show that the sets $U(0)$ and $V = E \setminus (U(0) \cup F(0))$ are open. Let $x \in U(0)$, $x \ne 0$. Then $x = tv$ for some $v \in S(1)$, $0 < t < \lambda(v)$. Choose a number $t_1 \in (t, \lambda(v))$. Since $A(0, \lambda(v)v) \subset G$ and since $\delta(0) > 0$, we have $\delta(t_1 v/2) > t_1/2$. Hence there is a number $r > 0$ such that $A(0, t_1 v') \subset G$ whenever $v' \in S(1)$ and $|v' - v| < r$. For these v' we have $[0, t_1 v') \subset J(0, v') \subset U(0)$, whence x is an interior point of $U(0)$. In view of (3.4) this implies that $U(0)$ is open.

Let $x \in V$. Now $x = tv$ for some $v \in S(1)$ and $t > \lambda(v)$. Choose a point $y \in A(0,x) \setminus G$. There is $r > 0$ such that $y \in A(0, tv')$ for all $v' \in S(1)$ with $|v' - v| < r$. As $\lambda(v') < t$ for these v', the set V is open. □

3.6. Theorem. *Let $G \subset \mathbf{R}^n$ and let $a \in G$, $|v| = 1$. Then the function $t \mapsto k(a, a+tv)$ is strictly increasing on the interval $[0, \lambda(v))$.*

Proof. For $0 \le t < \lambda(v)$ we set $x_t = a + tv$ and $f(t) = k(a, x_t)$. Assume that f is differentiable at a point $t \in (0, \lambda(v))$. As f is locally Lipschitz, it suffices to show that $f'(t) > 0$. Since $t < \lambda(v)$, there is $R > t/2$ such that $Y^*(a, x_t; R) \subset G$ where Y^* is defined in 2.5. Let $\gamma\colon a \curvearrowright x_t$ be a quasihyperbolic geodesic. Then $\gamma \subset \bar{Y}(a, x_t; R)$ by 2.6. The angle of Y is $\alpha = \arcsin(t/2R) < \pi/2$.

Let $0 < s < t$. There is a point $y = y(s) \in \gamma$ such that $(y - x_s)\cdot v = 0$. We have

$$f(s) \le k(a,y) + k(y,x_s) = f(t) - k(y,x_t) + k(y,x_s).$$

From 3.2 it follows that the numbers

$$K_1(s) = \frac{k(y,x_t)\delta(x_t)}{|y - x_t|}, \quad K_2(s) = \frac{k(y,x_s)\delta(x_t)}{|y - x_s|}$$

converge to 1 as $s \to t$. Setting $\beta = \operatorname{ang}(y - x_t, -v)$ we have

$$\beta \le \alpha,\ |y - x_t| = (t-s)/\cos\beta,\ |y - x_s| = (t-s)\tan\beta,$$

whence

$$\frac{f(t) - f(s)}{t - s} \ge \frac{K_1(s) - K_2(s)\sin\beta}{\delta(x_t)\cos\beta} \ge \frac{K_1(s) - K_2(s)\sin\alpha}{\delta(x_t)},$$

assuming that $t-s$ is so small that $K_1(s)-K_2(s)\sin\alpha>0$. As $s\to t$, this gives $f'(t)\ge(1-\sin\alpha)/\delta(x_t)>0$ as desired. □

3.7. Strictly starlike domains. A domain $D\subset E$ is *strictly starlike* with respect to a point $a\in D$ if D is bounded and if each ray from a meets ∂D at precisely one point. Equivalently, there is a bounded continuous function $h=h_D\colon S(1)\to(0,\infty)$ such that $D=\{a+tv\colon 0\le t<h(v),\ v\in S(1)\}$. To simplify notation we normalize $a=0$ and assume that

$$D=\{tv\colon 0\le t<h(v),\ v\in S(1)\}.$$

Let $s>0$ and define the radial map $F_s\colon E\to E$ by

$$F_s(x)=h(x/|x|)x/s,\quad F_s(0)=0. \tag{3.8}$$

The map F_s is a bijection, which maps each ray from the origin linearly onto itself. Moreover,

$$F_sB(s)=D,\quad F_sS(s)=\partial D.$$

The inverse mapping is given by

$$F_s^{-1}(y)=\frac{sy}{h(y/|y|)},\quad F_s^{-1}(0)=0.$$

The maps F_s and F_s^{-1} are clearly continuous in $E\setminus\{0\}$. Since h is bounded away from 0 and ∞, they are continuous also at the origin. Hence $F_s\colon E\to E$ is a *homeomorphism.*

We next study the bilipschitz property of F_s. For $v\in S(1)$, the upper derivative of h at v is written as

$$L(v,h)=\limsup_{u\to v,u\in S(1)}\frac{|h(u)-h(v)|}{|u-v|}.$$

3.9. Lemma. *Let $0<c_1\le 1\le c_2$ and suppose that*

$$c_1s\le h(v)\le c_2s,\quad L(v,h)\le Hs \tag{3.10}$$

for all $v\in S(1)$. Then F_s is M-bilipschitz with $M=(c_2+H)/c_1^2$.

Proof. Let $p\colon E\setminus\{0\}\to S(1)$ be the central projection $px=x/|x|$. Then p is differentiable with $|p'(x)|=1/|x|$. Since $F_sx=h(px)x/s$, standard differentiation rules (see [Vä1, 5.3]) give

$$L(x,F_s)\le(h(px)+|x|L(px,h)/|x|)/s\le c_2+H\le M$$

for all $x\ne 0$, and similarly

$$L(y,F_s^{-1})\le\frac{h(py)+|y|L(py,h)/|y|}{h(py)^2}s\le 1/c_1+H/c_1^2\le M$$

for $y\ne 0$. Hence F_s is locally M-bilipschitz in $E\setminus\{0\}$, and the lemma follows. □

We next show that quasihyperbolic balls with sufficiently small radius are strictly starlike.

3.11. Theorem. *If $0 < r < \pi/2$, then every quasihyperbolic ball $B_k(a,r)$ in a domain $G \subset \mathbf{R}^n$ is strictly starlike with respect to a, and $\bar{B}_k(a,r) \subset U(a)$ where $U(a)$ is the neighborhood of a defined in 3.3.*

Proof. We may assume that $a = 0$. If $z \in \partial U(0)$, then $z = \lambda(v)v$ for some $v \in S(1)$ by 3.5. By the definition of $\lambda(v)$ in 3.3 we have $\delta(z/2) = |z|/2$. Let $b \in \partial G$ be a point with $|b - z/2| = |z|/2$. By [MO, p. 38], we obtain

$$k_G(0,z)^2 \geq k_{\mathbf{R}^n \setminus \{b\}}(0,z)^2 = \left(\log \frac{|b|}{|b-z|}\right)^2 + \operatorname{ang}(b, b-z)^2 \geq \operatorname{ang}(b, b-z)^2.$$

Since $|z|^2/4 = |b - z/2|^2 = b \cdot (b-z) + |z|^2/4$, we have $b \cdot (b - z) = 0$, whence

$$k(0,z) \geq \pi/2 \tag{3.12}$$

for each $z \in \partial U(0)$. As $B_k(0,r)$ is connected, this implies that $\bar{B}_k(0,r) \subset U(0)$. If $\lambda(v) < \infty$, the segment $J(0,v)$ meets $S_k(0,r)$ at exactly one point by 3.6. This is also true if $v \in S(1)$ and $\lambda(v) = \infty$, in which case $k(0,x)$ increases strictly from 0 to ∞ on the ray $J(0,v)$. Hence $B_k(0,r)$ is strictly starlike. □

3.13. Remark. The condition $r < \pi/2$ is presumably not sharp. By Klén [Kl], the best bound for the punctured plane is $2.832\ldots$. It is possible that this holds for all domains.

We shall apply Lemma 3.9 to show that for $0 < r < \pi/2$, each quasihyperbolic ball $B_k(a,r)$ in a domain $G \subset \mathbf{R}^n$ can be mapped onto the round ball $B(a, r\delta(a))$ by a radial bilipschitz homeomorphism $F\colon \mathbf{R}^n \to \mathbf{R}^n$. An auxiliary result is needed.

3.14. Lemma. *For each $r \in (0, \pi/2)$, there is a number $\sigma = \sigma(r) < 1$ such that if $G \subset \mathbf{R}^n$ is a domain and if $a, x \in G$ with $0 < k(a,x) = r$, then $Y^*(a,x;R) \subset G$ where $R = |x-a|/2\sigma$ and Y^* is defined in 2.5. Moreover, $\sigma(r) \to 0$ as $r \to 0$.*

Proof. We may assume that $a = 0$. We first consider the special case where $r < \log 2$. Now 3.2(2) gives $|x| \leq (e^r - 1)\delta(0) < \delta(0)$. Since $Y^*(0,x;R) \subset G$ for $R = \delta(0)/2$, we can take $\sigma(r) = e^r - 1$, which has the property $\sigma(r) \to 0$ as $r \to 0$.

The rest of the proof is valid for every $r < \pi/2$, but the estimate for $\sigma(r)$ has not the desired behavior as $r \to 0$. Set $v = x/|x|$. From 3.6, 3.5 and (3.12) it follows that there is a unique point $z \in J(0,v)$ such that $k(0,z) = \pi/2$ and $|x| < |z|$. By 3.2(1) we have

$$\pi/2 = k(0,z) \geq \log\left(1 + \frac{|z|}{\delta(0) \wedge \delta(z)}\right),$$

whence

$$\delta(0) \wedge \delta(z) \geq |z|/(e^{\pi/2} - 1) > |z|/4 > |x|/4. \tag{3.15}$$

Furthermore, $A(0,z) \subset G$ yields $\delta(z/2) \geq |z|/2$, whence $\delta(w) \geq |z|/5$ for all $w \in [0,z]$. Consequently,

$$\pi/2 - r \leq k(x,z) \leq 5(1 - |x|/|z|),$$

which gives

$$|x| \le (1-q)|z| \tag{3.16}$$

where $q = q(r) = (\pi/2 - r)/5 > 0$. We consider two cases.

Case 1. $|z| \le \delta(0)$. Now $Y^*(0, x; R) \subset G$ for $R = \delta(0)/2$ and hence for $R = |x|/2(1-q)$. Thus we can take $\sigma(r) = 1 - q(r)$.

Case 2. $|z| > \delta(0)$. Since G contains $B(\delta(0)) \cup A(0, z)$, it also contains the ball $B(x/2, R_1)$ where $R_1 = |y - x/2|$ for an arbitrary point $y \in S(\delta(0)) \cap \partial A(0, z)$. Write $y = y_1 v + y_2 v_2$ where v_2 is a unit vector with $v_2 \cdot v = 0$. Now

$$\begin{aligned} \delta(0)^2 &= y_1^2 + y_2^2, \\ |z|^2/4 &= (|z|/2 - y_1)^2 + y_2^2, \\ R_1^2 &= (|x|/2 - y_1)^2 + y_2^2. \end{aligned}$$

It follows that

$$R_1^2 = |x|^2/4 + \delta(0)^2(1 - |x|/|z|). \tag{3.17}$$

Since $B(x/2, R_1) \subset G$, we have $Y^*(0, x; R) \subset G$ where $R^2 = |x|^2/4 + (R_1 - R)^2$, and therefore

$$R = R_1/2 + |x|^2/8R_1. \tag{3.18}$$

Set

$$\varrho_1 = 2R_1/|x|, \quad \varrho = 2R/|x|.$$

By (3.17), (3.16) and (3.15) we get

$$\varrho_1^2 = 1 + 4\delta(0)^2(1 - |x|/|z|)/|x|^2 \ge 1 + q/4.$$

Since the function $t \mapsto t + 1/t$ is increasing for $t \ge 1$, this and (3.18) give the lower bound

$$\varrho \ge (\varrho_1 + 1/\varrho_1)/2 \ge \varrho_0(r) > 1$$

where

$$2\varrho_0 = (1 + q/4)^{1/2} + (1 + q/4)^{-1/2}.$$

Hence the lemma holds with $\sigma(r) = \min\{1/\varrho_0(r), e^r - 1\}$. □

3.19. Theorem. *Let $G \subset \mathbf{R}^n$ be a domain and let $a \in G$, $0 < r < \pi/2$. Then there is a radial M-bilipschitz homeomorphism $F\colon \mathbf{R}^n \to \mathbf{R}^n$ which maps $\bar{B}_k(a, r)$ onto $\bar{B}(a, r\delta(a))$, where $M = M(r) \to 1$ as $r \to 0$.*

Proof. We may assume that $a = 0$. By 3.11, the domain $D = B_k(0, r)$ is strictly starlike with respect to the origin. Set $s = r\delta(0)$ and let $F_s\colon \mathbf{R}^n \to \mathbf{R}^n$ be the radial homeomorphism defined by (3.8) where $h\colon S(1) \to (0, \infty)$ is defined by $h(v)v \in \partial D$ for $v \in S(1)$. By 3.2(2), the first inequalities of (3.10) hold with

$$c_1 = c_1(r) = (1 - e^{-r})/r, \quad c_2 = c_2(r) = (e^r - 1)/r = e^r c_1.$$

It suffices to find an estimate $L(v,h) \le Hs$ where $H = H(r) \to 0$ as $r \to 0$. Indeed, from 3.9 it then follows that F_s is M-bilipschitz with

$$M = M(r) = \frac{c_2(r) + H(r)}{c_1(r)^2} \to 1$$

as $r \to 0$.

Fix $v \in S(1)$. Let $u \in S(1)$, $u \ne v$, and write $t = |u - v|$. We must find an estimate

$$\limsup_{t\to 0} |h(u) - h(v)|/t \le Hs. \tag{3.20}$$

Write

$$x = h(v)v,\ y = h(u)u,\ \beta = |\pi/2 - \operatorname{ang}(x - y, x)|.$$

Let $\sigma = \sigma(r) < 1$ be the number given by 3.14 and set

$$\alpha = \alpha(r) = \arcsin \sigma < \pi/2.$$

Then $\alpha(r) \to 0$ as $r \to 0$.

It suffices to show that

$$\limsup_{t\to 0} \beta \le \alpha. \tag{3.21}$$

Indeed, an easy geometric argument shows that (3.21) implies (3.20) with

$$H = c_2 \tan \alpha.$$

Assume that (3.21) is false. Then there is a sequence of unit vectors $u_j \ne v$ such that $u_j \to v$ and such that setting $y_j = h(u_j)u_j$ and $\beta_j = |\pi/2 - \operatorname{ang}(x - y_j, x)|$ we have $\beta_j \ge \beta_0 > \alpha$ for some β_0 and for all j. Passing to a subsequence we may assume that $(x - y_j) \cdot x$ does not change sign. Since h is continuous, $y_j \to x$.

Case 1. $(x - y_j) \cdot x \ge 0$ for all j. Now $\beta_j = \pi/2 -$ $\operatorname{ang}(x - y_j, x)$. Let T_j be the normal hyperplane of $y_j - x$ through y_j. Since $y_j \to x$ and since $\beta_j \ge \beta_0$, we may assume that T_j separates the points x and 0 for each j.

Set $R = |x|/2\sigma$. The angle of the shuttle $Y = Y(0, x; R)$ is α.

Let $\gamma\colon 0 \curvearrowright x$ be a quasihyperbolic geodesic. Then $\gamma \subset \bar{Y}$ by 3.14 and 2.6. For each j we choose a point $z_j \in \gamma_j \cap T_j$. Setting $\psi_j = \operatorname{ang}(y_j - x, z_j - x)$ we have

$$\psi_j \le \operatorname{ang}(y_j - x, -x) + \operatorname{ang}(z_j - x, -x) \le \pi/2 - \beta_j + \alpha \le \pi/2 - \beta_0 + \alpha, \tag{3.22}$$

whence

$$|z_j - x| = |y_j - x|/\cos \psi_j \le |y_j - x|/\sin(\alpha - \beta_0).$$

Hence $|z_j - x| \to 0$ as $j \to \infty$. Furthermore, as $r = k(0, x) = k(0, y_j)$, we have

$$k(z_j, x) = r - k(0, z_j) \le k(z_j, y_j).$$

Since

$$|z_j - y_j| = |z_j - x| \sin \psi_j \le |z_j - x| \cos(\alpha - \beta_0)$$

by (3.22), we obtain

$$\frac{k(z_j, x)|z_j - y_j|}{k(z_j, y_j)|z_j - x|} \le \cos(\alpha - \beta_0).$$

As $j \to \infty$, the left hand side tends to 1 by 3.2, and we obtain the contradiction $1 \le \cos(\beta_0 - \alpha)$.

Case 2. $(x - y) \cdot x < 0$ for all j. The proof is rather similar to that of Case 1. Now $\beta_j = \pi/2 - \operatorname{ang}(y_j - x, x)$. Let now T_j be the normal hyperplane of $y_j - x$ through x. The angle of the shuttles $Y_j = Y(0, y_j; R_j)$, $R_j = |y_j|/2\sigma$, is again α.

Let $\gamma_j : 0 \curvearrowright y_j$ be a quasihyperbolic geodesic and choose a point $z_j \in \gamma_j \cap T_j$. Then $\gamma_j \subset \bar{Y}_j$ by 2.6. Setting $\omega_j = \operatorname{ang}(z_j - y_j, x - y_j)$ we have

$$\omega_j \le \operatorname{ang}(z_j - y_j, -y_j) + \operatorname{ang}(x - y_j, -y_j) \le \alpha + \pi/2 - \beta_j \le \alpha + \pi/2 - \beta_0.$$

Consequently, $|z_j - x| = |y_j - x| \tan \omega_j \to 0$ as $j \to \infty$.

Proceeding almost as in Case 1 we get

$$k(z_j, y_j) \le k(z_j, x), \quad |z_j - y_j| = |z_j - x|/\sin \omega_j \ge |z_j - x|/\cos(\beta_0 - \alpha),$$

and we again obtain the contradiction $1 \le \cos(\beta_0 - \alpha)$. $\square$

3.23. Remark. The proofs of the present section are valid in a domain G in an arbitrary Hilbert space if points of G can be joined by a quasihyperbolic geodesic. Hence all results are true for convex domains in all Hilbert spaces by [Vä2, 2.1]. However, in this case they can be obtained directly by the results of [MV], even in an improved form.

Let G be a convex domain in a Hilbert space E. Then every quasihyperbolic ball $B_k(a, r)$ in G is convex by [MV, 2.13] and hence strictly starlike. Moreover, if $a \in G$ and $|e| = 1$, then the function $t \mapsto k(a, a + te)$ increases strictly from 0 to ∞ on the whole interval $\{t \ge 0 : a + te \in G\}$. For the angle β in 3.19 we get the estimate $\cos \beta \ge c_1/c_2$, which replaces (3.21).

3.24. Question. Does there exist a constant $r_0 > 0$ such that for $r < r_0$, every quasihyperbolic ball $B_k(a, r)$ is convex?

4. Quasihyperbolic balls, $\dim E = \infty$

The proofs of Theorems 3.6 and 3.19 made use of quasihyperbolic geodesics, which are not always available in an infinite-dimensional space. In the proof of 3.11 we made use of finite dimensionality by choosing a point $b \in \partial G \cap S(z/2, \delta(z/2))$. Other proofs of Section 3 are valid in all Hilbert spaces. In this section we indicate how the proofs of these results can be modified so as to remain valid in the general case. In the proofs of 3.6 and 3.19 we replace the geodesics by quasigeodesics. We give the proof of Lemma 3.6 about increasingness in detail and sketches for the other two results.

4.1. Quasigeodesics. For $c \ge 1$, an arc $\gamma \subset G$ is a *c-quasigeodesic* (called near-geodesic in some of my earlier papers) if γ is c-quasiconvex in the quasihyperbolic metric, that is,

$$l_k(\gamma[x, y]) \le ck(x, y)$$

for all $x, y \in \gamma$.

Quasigeodesics act as a substitute for geodesics in infinite-dimensional spaces where geodesics are not always available. Unfortunately, they make the proofs somewhat more complicated.

We recall the basic result on the existence of quasigeodesics; see [Vä1, 9.4].

4.2. Lemma. *If a, b are points in a domain G and if $c > 1$, then there is a c-quasigeodesic $\gamma\colon a \curvearrowright b$ in G.* □

The following result states, roughly speaking, that c-quasigeodesics with c close to one are "ball λ-quasiconvex" with λ close to one. We use the notation $\lambda B(x,r) = B(x, \lambda r)$ for balls in E.

4.3. Lemma. *For each $\lambda > 1$ there is $c > 1$ with the following property: Let $G \subset E$ be a domain and let B be a ball with $\lambda B \subset G$. Let $\gamma\colon a \curvearrowright b$ be a c-quasigeodesic in G with $a, b \in \bar{B}$. Then $\gamma \subset \lambda B$.*

Proof. Assume that $\gamma\colon a \curvearrowright b$ is a c-quasigeodesic with $a, b \in \bar{B}$, $\gamma \not\subset \lambda B$. We must find a lower bound $c \geq c_1(\lambda) > 1$. We may assume that $B = B(1)$. Replacing γ by a subarc we may assume that $\gamma \cap \bar{B} = \{a, b\}$. Set $\mu = (\lambda+1)/2$ and $A_1 = \gamma \cap \bar{B}(\mu)$, $A_2 = \gamma \setminus \bar{B}(\mu)$. For each Borel set $A \subset \gamma$ we write

$$l_k(A) = \int_A \frac{d\mathscr{H}^1 x}{\delta(x)},$$

where $\mathscr{H}^1$ is the Hausdorff 1-measure. By 2.3 we obtain

$$k(a,b) \leq \int_{A_1} \frac{d\mathscr{H}^1 x}{|x|\delta(x)} + \int_{A_2} \frac{d\mathscr{H}^1 x}{|x|\delta(x)} \leq l_k(A_1) + l_k(A_2)/\mu.$$

As $l_k(A_1) \leq ck(a,b) - l_k(A_2)$, this gives

$$l_k(A_2) \leq (c-1)\mu k(a,b)/(\mu - 1).$$

Setting $A_3 = \{x \in \gamma\colon \mu < |x| \leq \lambda\}$ we have $A_3 \subset A_2$ and $\mathscr{H}^1(A_3) \geq 2(\lambda - \mu) = 2(\mu - 1)$. Each $x \in A_3$ satisfies $\delta(x) \leq \delta(0) + |x| \leq \delta(0) + \lambda$, whence

$$l_k(A_2) \geq l_k(A_3) \geq \frac{2(\mu-1)}{\delta(0)+\lambda}.$$

As $\delta(x) \geq \delta(0) - |x| \geq \delta(0) - 1$ for $x \in [a,b]$, we have

$$k(a,b) \leq l_k([a,b]) \leq 2/(\delta(0) - 1).$$

Combining the estimates and substituting $\mu = (\lambda+1)/2$ we get

$$c - 1 \geq \frac{(\lambda-1)^2}{2(\lambda+1)} \frac{\delta(0)-1}{\delta(0)+\lambda}.$$

Since $\delta(0) \geq \lambda$ and since $(t-1)/(t+\lambda)$ is increasing in t, this yields $c - 1 \geq (\lambda-1)^3/4(\lambda+1)$. Hence the lemma holds with $c = 1 + (\lambda-1)^3/5(\lambda+1)$. □

From 4.3 we obtain as a corollary the following quasi version of the shuttle theorem 2.6:

4.4. Theorem. *Let $G \subset E$ be a domain, let $a, b \in G$ and let $R \geq |a-b|/2$ be such that $d(Y^*(a,b;R), \partial G) > 0$. Then for each $\varepsilon > 0$ there is $c_0 > 1$ such that if $\gamma\colon a \curvearrowright b$ is a c_0-quasigeodesic, then $\gamma \subset Y(a,b;R) + B(\varepsilon)$, where the shuttle $Y(a,b;R)$ is defined in 2.5.* □

We next prove Theorem 3.6 for general Hilbert spaces. Recall the notation $\lambda(v)$ from 3.3.

4.5. Theorem. *Let $a \in G$ and let $v \in E$ be a unit vector. Then the function $t \mapsto k(a, a+tv)$ is strictly increasing on the interval $[0, \lambda(v))$.*

Proof. For $0 \leq t < \lambda(v)$ we set $x_t = a + tv$ and $f(t) = k(a, x_t)$. Assume that f is differentiable at a point $t \in (0, \lambda(v))$. It suffices to show that $f'(t) > 0$. Since $t < \lambda(v)$ and $\delta(a) > 0$, there is $R > t/2$ such that $d(Y^*(a, x_t; R), G) > 0$. Let $0 < s < t$, $0 < \varepsilon < t - s$ and let $c_0 > 1$ be the number given by 4.4 for $b = x_t$. Let $1 < c < c_0$ and let $\gamma\colon a \curvearrowright x_t$ be a c-quasigeodesic. Then $\gamma \subset Y(a, x_t; R) + B(\varepsilon)$ by 4.4. The angle of Y is $\alpha = \arcsin(t/2R) < \pi/2$.

There is a point $y = y(s) \in \gamma$ with $(y - x_s)\cdot v = 0$ and a point $y' \in Y$ with $|y - y'| < \varepsilon$. Then $\beta := \operatorname{ang}(y' - x_t, -v) \leq \alpha$ and

$$|y - x_t| \leq |y' - x_t| + \varepsilon \leq (t - s + \varepsilon)/\cos\beta + \varepsilon < 2(t-s)/\cos\alpha + t - s \to 0$$

as $s \to t$.

By 3.2, the numbers

$$K_1(s) = \frac{k(y, x_t)\delta(x_t)}{|y - x_t|}, \quad K_2(s) = \frac{k(y, x_s)\delta(x_t)}{|y - x_s|}$$

converge to 1 as $s \to t$. As γ is a c-quasigeodesic, we have

$$k(a, y) + k(y, x_t) \leq l_k(\gamma) \leq cf(t),$$

whence

$$f(s) \leq k(a, y) + k(y, x_s) = cf(t) - k(y, x_t) + k(y, x_s).$$

Furthermore,

$$|y - x_t| \leq (t - s - \varepsilon)/\cos\beta - \varepsilon, \ |y - x_s| \geq (t - s + \varepsilon)\tan\beta + \varepsilon.$$

Combining the estimates and assuming that $t-s$ is so small that $K_1(s) - K_2(s)\sin\alpha > 0$ we obtain

$$\frac{f(t) - f(s)}{t - s} \geq \frac{K_1(s) - K_2(s)\sin\alpha}{\delta(x_t)} - \frac{\eta}{\delta(x_t)(t-s)}$$

where

$$\eta = (K_1 + K_2)(1/\cos\alpha + 1)\varepsilon + (c-1)f(t)\delta(x_t).$$

Letting first $c \to 1$, then $\varepsilon \to 0$ and then $s \to t$ we obtain $f'(t) \geq (1 - \sin\alpha)/\delta(x_t) > 0$ as desired. □

4.6. Theorem. *Theorem 3.11 (about starlikeness) holds in every Hilbert space.*

Proof. We can follow the proof of 3.11, but instead of choosing a point $b \in \partial G$ we choose a sequence of points $b_j \in \partial G$ with $|b_j - z/2| \to |z|/2$. □

4.7. Theorem. *Theorem 3.19 (about bilipschitz property) holds in every Hilbert space.*

Proof. The proof is again carried out by following the proof of 3.19, replacing quasihyperbolic geodesics by c-quasigeodesics with c close to one and applying Theorem 4.3 with a very small ε. The details are omitted. $\square$

5. Smoothness of quasihyperbolic spheres

In this section we study the smoothness properties of a quasihyperbolic sphere $S_k(a, r)$ in a domain $G \subset E$. We start with an example which shows that this sphere need not have a tangent hyperplane at each point.

5.1. Example. Let G be the domain $\mathbf{R}^2 \setminus \{0\}$. Using complex notation we let $f\colon \mathbf{R}^2 \to G$ be the covering map $fz = e^z$. This domain was studied by Martin and Osgood [MO, p. 38], who made the important observation that f maps euclidean lengths to quasihyperbolic lengths. In particular, f maps each euclidean disk $B(z, r)$ onto the quasihyperbolic disk $B_k(fz, r)$. It follows that for $r > \pi$, the quasihyperbolic sphere $S_k(1, r) = \partial B_k(1, r)$ consists of two Jordan curves, each of which has a corner point on the negative real axis. Both corners are directed "into" $B_k(1, r)$.

5.2. Normal vectors. We shall show that, intuitively speaking, there cannot exist any "outward directed corners" in $S_k(a, r)$, at least in the finite-dimensional case. To formulate the result rigorously, we give the following definitions.

Let $G \subset E$ be a domain, let $a \in G$, $r > 0$, and set $S = S_k(a, r)$. We say that a unit vector e is an *inner normal vector* of S at a point $b \in S$ if the following mutually equivalent conditions are true:

$$\liminf_{x\to b,\ k(x,a)\ge r} \operatorname{ang}(x-b, e) \ge \pi/2, \qquad \limsup_{x\to b,\ k(x,a)\ge r} \frac{(x-b)\cdot e}{|x-b|} \le 0.$$

Similarly, a unit vector u is an *outer normal vector* of S at b if

$$\liminf_{x\to b,\ k(x,a)\le r} \operatorname{ang}(x-b, u) \ge \pi/2, \qquad \limsup_{x\to b,\ k(x,a)\le r} \frac{(x-b)\cdot u}{|x-b|} \le 0.$$

If b is an isolated point of S, the definition of an inner normal vector does not make sense, and we agree that in this case each unit vector $e \in E$ is an inner normal vector of S at b.

Observe that these definitions do not rule out the possibility that S has several inner or outer normal vectors at some point. In fact, at the corner points of Example 5.1, there are an infinite number of inner normal vectors and no outer normal vectors. The next lemma implies that if S has a normal vector in both directions at some point $b \in S$, then they are unique, and we say that the outer normal vector u is the *normal vector* of S at b. In this case, the hyperplane $T = b + u^\perp$ is the *tangent* *hyperplane* of S at b, which means that

$$\lim_{x\to b,\ x\in S} \frac{d(x,T)}{|x-b|} = 0.$$

5.3. Lemma. *If a quasihyperbolic sphere S has an inner normal vector e and an outer normal vector u at a point $b \in S$, then $u = -e$.*

Proof. Assume that $u \neq -e$. Let v be the unit vector with $\operatorname{ang}(v,e) = \operatorname{ang}(v,u) = \operatorname{ang}(e,u)/2 < \pi/2$. There is a sequence (t_j) of positive numbers such that either $k(a, b + t_j v) \geq r$ for all j or $k(a, b + t_j v) \leq r$ for all j. But both cases give a contradiction by the definitions of inner and outer normal vectors. $\square$

We next give the main result on normal vectors of quasihyperbolic spheres.

5.4. Theorem. *Let $\gamma\colon a \curvearrowright b$ be a quasihyperbolic geodesic in G and let v be the left tangent vector of γ at b, given by Theorem 2.8. Then the vector $e = -v$ is an inner normal vector of $S = S_k(a,r)$ at b where $r = k(a,b)$. If u is an outer normal vector of S at b, then $u = v$ and u is the normal vector of S at b.*

Proof. It suffices to show that e is an inner normal vector of S at b, because this implies the last statement of the theorem by 5.3. If this is not true, then there is a sequence of points $x_j \neq b$ such that

$$x_j \to b,\ k(x_j, a) \geq r,\ \beta_j := \operatorname{ang}(x_j - b, e) \leq \beta < \pi/2$$

for some β and for all j. Setting $t_j = |x_j - b|/\cos\beta_j$ and $y_j = b + t_j e$ we have $t_j \to 0$ and $(y_j - x_j)\cdot(b - x_j) = 0$. Since $-e$ is the left tangent vector of γ at b, we may assume that for each j there is a point $z_j \in \gamma$ such that $(z_j - y_j)\cdot e = 0$ and such that $|z_j - y_j| = \varepsilon_j t_j$ where $\varepsilon_j \to 0$.

We can now use an argument similar to that in 3.19. We have

$$r \leq k(a, x_j) \leq k(a, z_j) + k(z_j, x_j) = r - k(b, z_j) + k(z_j, x_j),$$

whence $k(b, z_j) \leq k(z_j, x_j)$. Furthermore, $|z_j - b| \geq t_j$ and

$$|z_j - x_j| \leq |y_j - x_j| + |z_j - y_j| = t_j \sin\beta_j + t_j\varepsilon_j \leq t_j(\sin\beta + \varepsilon_j).$$

It follows that

$$\frac{k(z_j,b)}{k(z_j,x_j)}\frac{|z_j - x_j|}{|z_j - b|} \leq \sin\beta + \varepsilon_j.$$

By 3.2, the left-hand side converges to 1 as $j \to \infty$, and we obtain the contradiction $1 \leq \sin\beta$. $\square$

5.5. Remark. Similar results have been independently obtained by Klén [Kl].

5.6. Geodesic points. Let $G \subset E$ be a domain and let $a \in G$, $r > 0$, $S = S_k(a,r)$. We say that a point $b \in S$ is a *geodesic point* of S if there is a quasihyperbolic geodesic $\gamma\colon a \curvearrowright b$. If there is a quasihyperbolic geodesic $\gamma\colon a \curvearrowright b_1$ such that b is an interior point of γ, then b is a *strongly geodesic point* of S.

The corner points of the quasihyperbolic sphere $S_k(1,r)$ for $r > \pi$ in Example 5.1 are geodesic but not strongly geodesic. For convex domains the situation is simple:

5.7. Lemma. *In a convex domain every point of a quasihyperbolic sphere is strongly geodesic.*

Proof. This follows from [Vä2, 2.1] and [MV, 3.12]. $\square$

In nonconvex domains of arbitrary Hilbert spaces, no general existence theorems for quasihyperbolic geodesics are known, and we restrict ourselves to the case $\dim E < \infty$.

5.8. Lemma. *Let $S = S_k(a,r)$ be a quasihyperbolic sphere in a domain $G \subset \mathbf{R}^n$. Then every point of S is geodesic. If b is a boundary point of $\bar{B}_k(a,r)$, then there is a sequence of strongly geodesic points b_j of S converging to b. If $r < \pi/2$, then the strongly geodesic points are dense in S.*

Proof. As each pair of points in G can be joined by a geodesic by [GO, Lemma 1], the first statement is clear. Assume that $b \in \partial\bar{B}_k(a,r)$. Let $\varepsilon > 0$ and let $x \in G \setminus \bar{B}_k(a,r)$ be a point with $k(x,b) < \varepsilon$. Choose a quasihyperbolic geodesic $\gamma\colon a \curvearrowright x$. Then γ meets S at a point b' with $k(b',x) = k(a,x) - r \le k(x,b) < \varepsilon$, whence $k(b,b') < 2\varepsilon$. As b' is a strongly geodesic point of S, the second part of the lemma is proved.

Finally, if $r < \pi/2$, then $B_k(a,r)$ is strictly starlike by Theorem 3.11, and the last statement follows. $\square$

5.9. Questions. (1) Is the condition $b \in \partial\bar{B}_k(a,r)$ always true for $b \in S_k(a,r)$? In other words, can the function $x \mapsto k(a,x)$ have a local maximum at some point of G?

(2) Does there exist a universal constant $r_1 > 0$ such that for $r < r_1$, every point of a quasihyperbolic sphere $S_k(a,r)$ in a domain $G \subset \mathbf{R}^n$ is strongly geodesic?

(3) Is some part of Lemma 5.8 true in all Hilbert spaces?

(4) Is the answer to (2) affirmative in all Hilbert spaces?

5.10. Theorem. *Let b be a geodesic point of $S = S_k(a,r)$ in a domain G. Then S has at least one inner normal vector e and at most one outer normal vector u at b. If u exists, then u is the normal vector of S at b.*

If b is a strongly geodesic point of S, then S has a normal vector at b.

Proof. The first part of the theorem follows from 5.4. If b is a strongly geodesic point of S, then there is a quasihyperbolic geodesic $\gamma_1\colon a \curvearrowright b_1$ such that $b \in \gamma_1$ and $k(a,b_1) = r + r_1$ with $r_1 > 0$. Applying Theorem 5.4 to the subarc of γ_1 from b_1 to b we see that the tangent vector of γ_1 at b is an inner normal vector of $S_k(b_1,r_1)$ and hence an outer normal vector of S at b. $\square$

5.11. Convex domains. We finally consider quasihyperbolic balls in a convex domain $G \subset E$. We recall that then each quasihyperbolic ball is strictly convex by [MV, 2.13]. We shall show that, moreover, each quasihyperbolic sphere S in G

is a C^1 smooth surface, that is, S has a normal vector at each point $b \in S$ and this normal vector is a continuous function of b. As a preparation, we consider the following situation.

Fix a unit vector $e_0 \in E$, and let e be another unit vector with $e \cdot e_0 > 0$. For each $h \in e_0^\perp$ there is a unique real number t such that $h + te_0 \in e^\perp$; explicitly $t = -(e \cdot h)/(e \cdot e_0)$. Thus t is linear in h, and we write $t = L(e)h$ where $L(e)\colon e_0^\perp \to \mathbf{R}$ is linear. We show that

$$\operatorname{ang}(e, e_0) \le |L(e)| \tag{5.12}$$

where $|L(e)|$ is the operator norm.

We may assume that $e \neq e_0$. Let v be a unit vector in $e_0^\perp \cap \operatorname{span}(e, e_0)$. Setting $\alpha = \operatorname{ang}(e, e_0)$ and $y = v + (L(e)v)e_0$ we have $\operatorname{ang}(y, v) = \alpha$. Hence

$$|L(e)| \ge |L(e)v| = |y - v| = \tan\alpha \ge \alpha,$$

which is (5.12).

5.13. Theorem. *Let $S = S_k(a, r)$ be a quasihyperbolic sphere in a convex domain $G \subset E$. Then S has a normal vector $u(b)$ at each point of S, and the map $u\colon S \to S(1)$ is continuous.*

Proof. As each point $b \in S$ is strongly geodesic by [MV, 3.12], the normal vector $u(b)$ exists by 5.10. We show that the inner normal vector $e = -u$ is continuous at an arbitrary point $b_0 \in S$. We may assume that $b_0 = 0$, and we set $e_0 = e(b_0)$. Since $B_k(a, r)$ is convex by [MV, 2.13], there is a ball $U = e_0^\perp \cap B(t)$ in $e_0^\perp$ and a convex function $f\colon U \to \mathsf{R}$ such that the set $V = \{x + f(x)e_0 \colon x \in U\}$ is a neighborhood of 0 in S.

Let $x \in U$ and set $b = x + f(x)e_0 \in V$. Since $u(b)$ is the normal vector of S at b, it is easy to see that f is Fréchet differentiable at x with derivative $Df(x) = L(e(b))$ where L is defined in 5.11. In particular, $Df(0) = 0$. By a general property of convex functions [BL, 4.7], the function Df from U to the dual space of $e_0^\perp$ is continuous. Let $P\colon E \to e_0^\perp$ be the orthogonal projection. By (5.12) we obtain

$$\operatorname{ang}(e(b), e_0) \le |L(e(b))| = |Df(Pb)| \to |Df(0)| = 0$$

as $b \to 0$, whence e is continuous at 0. $\square$

5.14. Corollary. *If $G \subset \mathbf{R}^2$ is a convex domain, then each quasihyperbolic circle in G is a C^1 smooth Jordan curve.* $\square$

Appendix. Smoothness of arcs

We show that an arc $\gamma \subset E$ is smooth in the sense of 2.7 if and only if it has a C^1 parametrization with nonvanishing derivative. The result is probably well known but difficult to find in the literature. A related result is in [Gl, 10.1]. As before, E is a Hilbert space.

In one direction the result is very elementary:

A.1. Theorem. *Let $\varphi\colon [t_1, t_2] \to E$ be an injective C^1 map with $\varphi'(t) \neq 0$ for all $t \in [t_1, t_2]$ (one-sided derivatives at the endpoints). Then $\gamma = \operatorname{im} \varphi$ is a smooth arc with tangent vector $v(\varphi(t)) = \varphi'(t)/|\varphi'(t)|$.*

Proof. Let $t_1 \le t < t_2$. For $0 < h < t_2 - t$ we can write

$$\varphi(t+h) - \varphi(t) = h\varphi'(t) + ha(h), \ |\varphi(t+h) - \varphi(t)| = h|\varphi'(t)| + h\alpha(h),$$

where $a(h)$ and $\alpha(h)$ tend to zero as $h \to 0$. Hence

$$\frac{\varphi(t+h) - \varphi(t)}{|\varphi(t+h) - \varphi(t)|} = \frac{\varphi'(t) + a(h)}{|\varphi'(t)| + \alpha(h)} \to \frac{\varphi(t)}{|\varphi'(t)|}$$

as $h \to 0$. Thus $\varphi'(t)/|\varphi'(t)|$ is the right tangent vector of γ at $\varphi(t)$. Similarly we see that it is the left tangent vector at $\varphi(t)$, $t_1 < t \le t_2$. □

In the converse part we make use of the following elementary property of real functions:

A.2. Lemma. *Suppose that $f\colon (a, b) \to \mathbf{R}$ is continuous and that for each $x \in (a, b)$ there is $h > 0$ such that $f(y) > f(x)$ whenever $x < y < x + h$. Then f is strictly increasing.*

Proof. The condition implies that f cannot have a local maximum. □

A.3. Theorem. *If $\gamma \subset E$ is a smooth arc, then it is rectifiable and its length parametrization $\varphi\colon [0, l(\gamma)] \to \gamma$ is of class C^1.*

Proof. There is a continuous map $v\colon \gamma \to E$ such that $v(z)$ is the tangent vector at z (one-sided at the endpoints). Let z be an interior point of γ. Setting $fx = x \cdot v(z)$ we obtain a 1-Lipschitz map $f\colon \gamma \to \mathbf{R}$. Let $0 < \varepsilon < 1/2$. Choose an open arc neighborhood β_z of z in γ such that $|v(x) - v(z)| < \varepsilon$ for all $x \in \beta_z$. We show that

$$|fx - fy| \ge (1 - 2\varepsilon)|x - y| \tag{A.4}$$

for all $x, y \in \beta_z$.

Let $x \in \beta_z$. There is an open arc neighborhood α_x of x in β_z such that $|(y - x)/|y - x| - v(x)| < \varepsilon$ for all $y \in \alpha_x$ with $y > x$. For these y we have $(y - x) \cdot v(x) > (1 - \varepsilon)|y - x|$, whence

$$fy - fx = (y - x) \cdot v(x) + (y - x) \cdot (v(z) - v(x)) > (1 - 2\varepsilon)|x - y| > 0.$$

By A.2 this implies that $f|\beta_z$ is an embedding onto an open interval $J \subset \mathbf{R}$. Let $g\colon J \to \beta_z$ be its inverse. Considering similarly the case $y < x$ we see that

$$\limsup_{s \to t} \frac{|g(s) - g(t)|}{|s - t|} \le M_\varepsilon = \frac{1}{1 - 2\varepsilon}$$

for all $t \in J$. Hence g is M_ε-Lipschitz (see [Fe, 2.2.7]), and A.4 is proved.

From A.4 it follows that β_z is rectifiable. Treating similarly the endpoints of γ we see that γ is rectifiable. Set $\lambda = l(\gamma)$ and let $\varphi\colon [0, \lambda] \to \gamma$ be the length parametrization of γ. We assume that $0 \le s < \lambda$, set $z = \varphi(s)$ and show that

$v(z)$ is the right derivative of φ at s. The left case can then be treated by obvious modifications.

Let β_z, f, g be as above, and assume that $s' > s$ with $z' = \varphi(s') \in \beta_z$. Then $s' - s = l(\gamma[z, z'])$. Write $t = fz,\ t' = fz'$. As g is M_ε-Lipschitz, we get $l(\gamma[z, z']) \le M_\varepsilon(t' - t) \le M_\varepsilon|z' - z|$. Since $(z' - z)/|z' - z| \to v(z)$ and since ε is arbitrary, these estimates imply that

$$\lim_{t' \searrow t} \frac{\varphi(s') - \varphi(s)}{s' - s} = v(z). \qquad \square$$

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