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ON BI-LIPSCHITZ TYPE INEQUALITIES FOR QUASICONFORMAL HARMONIC MAPPINGS

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Dedicated to Professor Yoshihiro Mizuta on the occasion of his sixtieth birthday.

Abstract. In the paper [13] Pavlović proved that any quasiconformal and harmonic self-mapping F of the unit disk is bi-Lipschitz with respect to the Euclidean metric. We find explicit estimations of bi-Lipschitz constants for F that are expressed by means of the maximal dilatation K of F and $|F^{-1}(0)|$. Under the additional assumption $F(0) = 0$ the estimations are asymptotically sharp as $K \to 1$, so F behaves almost like a rotation for sufficiently small K.

Introduction

Set $\mathbf{D} := \{z \in \mathbf{C} : |z| < 1\}$, $\mathbf{T}_r := \{z \in \mathbf{C} : |z| = r\}$ for $r > 0$ and $\mathbf{T} := \mathbf{T}_1$. Given $K \geq 1$ and domains Ω_1 and Ω_2 in $\mathbf{C}$ write $\mathrm{QC}(\Omega_1, \Omega_2; K)$ for the class of all K-quasiconformal mappings of Ω_1 onto Ω_2 and let $\mathrm{QCH}(\Omega_1, \Omega_2; K)$ be the class of all mappings in $\mathrm{QC}(\Omega_1, \Omega_2; K)$ that are harmonic on Ω_1. In case $\Omega_1 = \Omega_2$ we write shortly $\mathrm{QC}(\Omega_1; K) := \mathrm{QC}(\Omega_1, \Omega_2; K)$ and $\mathrm{QCH}(\Omega_1; K) := \mathrm{QCH}(\Omega_1, \Omega_2; K)$.

There are a lot of results providing intrinsic characterizations of the boundary valued mapping f for a mapping $F \in \mathrm{QC}(\mathbf{D}) := \bigcup_{K\geq 1} \mathrm{QC}(\mathbf{D}; K)$; cf. e.g. [5], [9] and [17]. A similar problem may be posed in the case where $F \in \mathrm{QCH}(\mathbf{D}) := \bigcup_{K\geq 1} \mathrm{QCH}(\mathbf{D}; K)$. In the papers [8] and [10] several results were established that provide intrinsic characterizations of f in terms of the Cauchy and Cauchy–Stieltjes singular integrals involving f. The results also provide motivation for the further study of such integrals, which is a task for this paper. It is naturally divided into three sections. In the first one we express the Cauchy singular integral of the derivative f' by means of two functions $\mathrm{V}[f]$ and $\mathrm{V}^*[f]$ defined in (1.13) and (1.14), respectively; cf. Theorem 1.2. It is done in the case where f is a sense-preserving homeomorphic self-mapping of $\mathbf{T}$ and f is absolutely continuous on $\mathbf{T}$. The rest part of the section deals with estimating $\mathrm{V}[f]$ under the additional assumption that f is the boundary valued mapping of $F \in \mathrm{QC}(\mathbf{D})$ and $F(0) = 0$; see Theorem 1.4

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and Corollary 1.6. In the second section we gather a few results related to formal derivatives ∂F and $\bar{\partial} F$ of $F \in \mathrm{QCH}(\mathbf{D})$ in the context of Hardy spaces $\mathrm{H}^1(\mathbf{D})$ and $\mathrm{H}^\infty(\mathbf{D})$. They seem to be known, but we prove them for the sake of completeness of our considerations in the next section, where we present applications of Corollary 1.6. As the first one we prove Theorem 3.1 which gives asymptotically sharp estimations for $\mathrm{V}[f]$ and $\mathrm{V}^*[f]$ as $K \geq 1$ tends to 1, provided $F \in \mathrm{QCH}(\mathbf{D}; K)$. We use them for the bi-Lipschitz type estimations for f (Theorem 3.2) and F (Theorems 3.3 and 3.4) under the additional assumption $F(0) = 0$. All the estimations are asymptotically sharp as $K \to 1$. These theorems combined with [10, Corollary 4.3] essentially improve the eminent results by Pavlović [13].

The main results of the paper were presented by the second named author on the conference "The Ninth Conference on Real and Complex Analysis in Hiroshima", Higashi-Hiroshima (Japan), December 15–17, 2005.

1. On boundary properties for quasiconformal self-mappings of the unit disk

We recall that the Cauchy singular integral $\mathrm{C}_\mathbf{T}[f]$ of a function $f\colon \mathbf{T} \to \mathbf{C}$ Lebesgue integrable on $\mathbf{T}$ is defined for every $z \in \mathbf{T}$ as follows:

$$\mathrm{C}_\mathbf{T}[f](z) := \mathrm{PV}\,\frac{1}{2\pi i}\int_\mathbf{T} \frac{f(u)}{u-z}\,du := \lim_{\varepsilon\to 0^+}\frac{1}{2\pi i}\int_{\mathbf{T}\setminus\mathbf{T}(z,\varepsilon)} \frac{f(u)}{u-z}\,du \tag{1.1}$$

whenever the limit exists and $\mathrm{C}_\mathbf{T}[f](z) := 0$ otherwise, where $\mathbf{T}(e^{ix},\varepsilon) := \{e^{it} \in \mathbf{T} : |t-x| < \varepsilon\}$. Here and subsequently, integration along any arc $I \subset \mathbf{T}$ is understood under counterclockwise orientation and the limit operator is understood in $\mathbf{C}$ with the euclidian distance. Given a function $f\colon \mathbf{T} \to \mathbf{C}$ and $z \in \mathbf{T}$ we define

$$f'(z) := \lim_{u\to z}\frac{f(u)-f(z)}{u-z} \tag{1.2}$$

provided the limit exists and $f'(z) := 0$ otherwise. Write $\mathrm{Hom}^+(\mathbf{T})$ for the class of all sense-preserving homeomorphic self-mappings of $\mathbf{T}$. Each $f \in \mathrm{Hom}^+(\mathbf{T})$ defines a unique continuous function $\hat{f}$ satisfying $0 \leq \hat{f}(0) < 2\pi$ and

$$f(e^{it}) = e^{i\hat{f}(t)}, \quad t \in \mathbf{R}. \tag{1.3}$$

Actually, $\hat{f}$ is an increasing homeomorphism of $\mathbf{R}$ onto itself satisfying

$$\hat{f}(t+2\pi) - \hat{f}(t) = 2\pi, \quad t \in \mathbf{R}. \tag{1.4}$$

Moreover, from (1.3) it follows that for every $t \in \mathbf{R}$, f is differentiable at e^{it} iff $\hat{f}$ is differentiable at t, and for every such point t,

$$f'(e^{it})e^{it} = \hat{f}'(t)e^{i\hat{f}(t)} = |f'(e^{it})|f(e^{it}). \tag{1.5}$$

Thus by Lebesgue's classical theorem on the differentiation of a monotonic function, for each $f \in \mathrm{Hom}^+(\mathbf{T})$ the limit in (1.2) exists for a.e. $z \in \mathbf{T}$.

Lemma 1.1. *Suppose that $f \in \mathrm{Hom}^+(\mathbf{T})$ is absolutely continuous on $\mathbf{T}$ and that f is differentiable at a point $z \in \mathbf{T}$. Then both the following limits exist and*

$$\lim_{\varepsilon\to 0^+} \mathrm{Re}\left[\frac{z\overline{f(z)}}{\pi i}\int_{\mathbf{T}\setminus\mathbf{T}(z,\varepsilon)} \frac{f'(u)}{u-z}\,du\right] = \lim_{\varepsilon\to 0^+}\frac{1}{2\pi}\int_{\mathbf{T}\setminus\mathbf{T}(z,\varepsilon)} \frac{|f(u)-f(z)|^2}{|u-z|^2}\,|du|. \tag{1.6}$$

Moreover, both the following limits simultaneously exist or not and in the first case

$$\lim_{\varepsilon\to 0^+} \mathrm{Im}\left[\frac{z\overline{f(z)}}{\pi i}\int_{\mathbf{T}\setminus\mathbf{T}(z,\varepsilon)} \frac{f'(u)}{u-z}\,du\right] = -\lim_{\varepsilon\to 0^+}\frac{1}{\pi}\int_{\mathbf{T}\setminus\mathbf{T}(z,\varepsilon)} \frac{\mathrm{Im}[f(u)\overline{f(z)}]}{|u-z|^2}\,|du|. \tag{1.7}$$

Proof. Fix $z = e^{ix} \in \mathbf{T}$ and $\varepsilon \in (0;\pi)$. Since f is absolutely continuous on $\mathbf{T}$, we see, integrating by parts, that

$$\begin{aligned}\int_{\mathbf{T}\setminus\mathbf{T}(z,\varepsilon)} \frac{f'(u)}{u-z}\,du &= \int_{\mathbf{T}\setminus\mathbf{T}(z,\varepsilon)} \frac{d}{du}[f(u)-f(z)]\frac{1}{u-z}\,du\\ &= \frac{f(z''_\varepsilon)-f(z)}{z''_\varepsilon - z} - \frac{f(z'_\varepsilon)-f(z)}{z'_\varepsilon - z} + \int_{\mathbf{T}\setminus\mathbf{T}(z,\varepsilon)} \frac{f(u)-f(z)}{(u-z)^2}\,du,\end{aligned} \tag{1.8}$$

where $z'_\varepsilon := e^{i(x+\varepsilon)}$ and $z''_\varepsilon := e^{i(x-\varepsilon)}$. Furthermore,

$$\begin{aligned}\frac{z\overline{f(z)}}{\pi i}\int_{\mathbf{T}\setminus\mathbf{T}(z,\varepsilon)} \frac{f(u)-f(z)}{(u-z)^2}\,du &= -\frac{1}{2\pi}\int_{\mathbf{T}\setminus\mathbf{T}(z,\varepsilon)} \frac{2-2\overline{f(z)}f(u)}{\overline{z}(u-z)^2\overline{u}}\,|du|\\ &= \frac{1}{2\pi}\int_{\mathbf{T}\setminus\mathbf{T}(z,\varepsilon)} \frac{|f(u)|^2 - 2\overline{f(z)}f(u) + |f(z)|^2}{|u-z|^2}\,|du|.\end{aligned} \tag{1.9}$$

Thus combining (1.8) and (1.9) we obtain

$$\begin{aligned}\frac{z\overline{f(z)}}{\pi i}\int_{\mathbf{T}\setminus\mathbf{T}(z,\varepsilon)} \frac{f'(u)}{u-z}\,du &= \frac{z\overline{f(z)}}{\pi i}\left[\frac{f(z''_\varepsilon)-f(z)}{z''_\varepsilon - z} - \frac{f(z'_\varepsilon)-f(z)}{z'_\varepsilon - z}\right]\\ &+ \frac{1}{2\pi}\int_{\mathbf{T}\setminus\mathbf{T}(z,\varepsilon)} \frac{|f(u)-f(z)|^2}{|u-z|^2}\,|du| - \frac{i}{\pi}\int_{\mathbf{T}\setminus\mathbf{T}(z,\varepsilon)} \frac{\mathrm{Im}[f(u)\overline{f(z)}]}{|u-z|^2}\,|du|.\end{aligned} \tag{1.10}$$

Assume now that f is differentiable at z. Then

$$\lim_{\varepsilon\to 0^+}\left[\frac{f(z''_\varepsilon)-f(z)}{z''_\varepsilon - z} - \frac{f(z'_\varepsilon)-f(z)}{z'_\varepsilon - z}\right] = f'(z) - f'(z) = 0 \tag{1.11}$$

as well as

$$\lim_{\varepsilon\to 0^+}\frac{1}{2\pi}\int_{\mathbf{T}\setminus\mathbf{T}(z,\varepsilon)} \frac{|f(u)-f(z)|^2}{|u-z|^2}\,|du| = \frac{1}{2\pi}\int_{\mathbf{T}} \frac{|f(u)-f(z)|^2}{|u-z|^2}\,|du| < +\infty. \tag{1.12}$$

Thus combining (1.10) with (1.11) and (1.12) we obtain the assertion of the lemma, which ends the proof. □

Given a continuous function $f\colon \mathbf{T} \to \mathbf{C}$ and $z \in \mathbf{T}$ set

$$\mathrm{V}[f](z) := \lim_{\varepsilon\to 0^+} \frac{1}{2\pi} \int_{\mathbf{T}\setminus\mathbf{T}(z,\varepsilon)} \frac{|f(u)-f(z)|^2}{|u-z|^2}\,|du|, \tag{1.13}$$

$$\mathrm{V}^*[f](z) := -\lim_{\varepsilon\to 0^+} \frac{1}{\pi} \int_{\mathbf{T}\setminus\mathbf{T}(z,\varepsilon)} \frac{\operatorname{Im}[f(u)\overline{f(z)}]}{|u-z|^2}\,|du|, \tag{1.14}$$

provided the limits exist as well as $\mathrm{V}[f](z) := +\infty$ and $\mathrm{V}^*[f](z) := 0$ otherwise.

Theorem 1.2. *If $f \in \mathrm{Hom}^+(\mathbf{T})$ is absolutely continuous on $\mathbf{T}$, then for a.e. $z \in \mathbf{T}$ the limit in* (1.1) *with f replaced by f' and the limits in* (1.13) *and* (1.14) *exist, and*

$$2\,\mathrm{C}_{\mathbf{T}}[f'](z) = \overline{z} f(z)\big(\mathrm{V}[f](z) + i\,\mathrm{V}^*[f](z)\big). \tag{1.15}$$

Proof. Since $f(\mathbf{T}) = \mathbf{T}$ is a rectifiable curve, it follows that f' is a Lebesgue integrable function on $\mathbf{T}$. Then by [3, Lemma 1.2, p. 103] we see that the limit

$$\frac{1}{2\pi} \lim_{\varepsilon\to 0^+} \int_{\varepsilon<|t-x|\le\pi} f'(e^{it}) \cot\frac{x-t}{2}\,dt \tag{1.16}$$

exists for a.e. $z = e^{ix} \in \mathbf{T}$. Moreover, as shown in the proof of [8, Theorem 1.2], the following equality

$$\begin{aligned} &\frac{1}{2\pi i} \int_{\mathbf{T}\setminus\mathbf{T}(z,\varepsilon)} \frac{f'(u)}{u-z}\,du \\ &= \frac{1}{4\pi} \int_{\varepsilon<|t-x|\le\pi} f'(e^{it})\,dt + \frac{i}{4\pi} \int_{\varepsilon<|t-x|\le\pi} f'(e^{it}) \cot\frac{x-t}{2}\,dt \end{aligned} \tag{1.17}$$

holds for all $z = e^{ix} \in \mathbf{T}$ and $\varepsilon \in (0;\pi]$. Thus the limit in (1.1) with f replaced by f' exists for a.e. $z \in \mathbf{T}$ and the theorem follows directly from Lemma 1.1. $\square$

We recall that for each $K > 0$ the Hersch–Pfluger distortion function Φ_K is defined by the equalities

$$\Phi_K(r) := \mu^{-1}(\mu(r)/K), \quad 0 < r < 1; \quad \Phi_K(0) := 0,\ \Phi_K(1) := 1, \tag{1.18}$$

where μ stands for the module of the Grötzsch extremal domain $\mathbf{D} \setminus [0;r]$; cf. [4] and [6, pp. 53 and 63].

Lemma 1.3. *For every $K \ge 1$ the following inequalities hold:*

$$1 \le M_K := \frac{4}{\pi} \int_0^{1/\sqrt{2}} \left(\frac{\Phi_K(r)}{r}\right)^{1+1/K} \frac{dr}{\sqrt{1-r^2}} \le K^2 2^{5(1-1/K^2)/2} \tag{1.19}$$

and

$$1 \ge L_K := \frac{4}{\pi} \int_0^{1/\sqrt{2}} \left(\frac{\Phi_{1/K}(r)}{r}\right)^{1+1/K} \frac{dr}{\sqrt{1-r^2}} \ge \frac{K 2^{5(1-K^2)/(2K)}}{K^2+K-1}. \tag{1.20}$$

In particular, $L_K \to 1$ and $M_K \to 1$ as $K \to 1^+$.

Proof. Fix $K \geq 1$. Substituting $r := \sin t$ we have

$$M_K = \frac{4}{\pi} \int_0^{\pi/4} \left(\frac{\Phi_K(\sin t)}{\sin t} \right)^{1+1/K} dt \tag{1.21}$$

and

$$L_K = \frac{4}{\pi} \int_0^{\pi/4} \left(\frac{\Phi_{1/K}(\sin t)}{\sin t} \right)^{1+1/K} dt. \tag{1.22}$$

Since $\sin t \geq (4t)/(\pi\sqrt{2})$ for $t \in [0; \pi/4]$, we conclude from the Hübner inequality (cf. [1, (3.2)] or [6, p. 65, (3.6)])

$$r^{1/K} \leq \Phi_K(r) \leq 4^{1-1/K} r^{1/K}, \quad 0 \leq r \leq 1,\ K \geq 1, \tag{1.23}$$

that

$$1 \leq \frac{\Phi_K(\sin t)}{\sin t} \leq 4^{1-1/K} \left(\frac{4t}{\pi\sqrt{2}} \right)^{1/K-1}, \quad 0 < t \leq \pi/4.$$

This together with (1.21) yields (1.19). From (1.18) it follows that the composition $\Phi_K \circ \Phi_{1/K}$ is the identity function on $[0; 1]$. Hence and by (1.23),

$$r^K \geq \Phi_{1/K}(r) \geq 4^{1-K} r^K, \quad 0 \leq r \leq 1,\ K \geq 1. \tag{1.24}$$

Using once more the estimation $\sin t \geq (4t)/(\pi\sqrt{2})$ for $t \in [0; \pi/4]$ we conclude from (1.24) that

$$1 \geq \frac{\Phi_{1/K}(\sin t)}{\sin t} \geq 4^{1-K} \left(\frac{4t}{\pi\sqrt{2}} \right)^{K-1}, \quad 0 < t \leq \pi/4.$$

This together with (1.22) yields (1.20). From the estimations (1.19) and (1.20) it easily follows that $L_K \to 1$ and $M_K \to 1$ as $K \to 1^+$, which ends the proof. $\square$

Given a continuous function $f\colon \mathbf{T} \to \mathbf{C}$ and $z \in \mathbf{T}$ set

$$f^+(z) := \sup_{u \in \mathbf{T}\setminus\{z\}} \left| \frac{f(u) - f(z)}{u - z} \right| \in [0; +\infty], \tag{1.25}$$

$$f^-(z) := \inf_{u \in \mathbf{T}\setminus\{z\}} \left| \frac{f(u) - f(z)}{u - z} \right| \in [0; +\infty). \tag{1.26}$$

Theorem 1.4. *Given $K \geq 1$ and $F \in \mathrm{QC}(\mathbf{D}; K)$ let f be the boundary valued function of F. If $F(0) = 0$, then*

$$L_K (f^-(z))^{1-1/K} \leq \mathrm{V}[f](z) \leq M_K (f^+(z))^{1-1/K}, \quad z \in \mathbf{T}. \tag{1.27}$$

Proof. Since $F \in \mathrm{QC}(\mathbf{D}; K)$ and $F(0) = 0$, we see by the quasi-invariance of the harmonic measure that for every $t \in [\theta - \pi; \theta + \pi]$,

$$\Phi_{1/K}\left(\cos \frac{|\hat{f}(t) - \hat{f}(\theta)|}{4} \right) \leq \cos \frac{t - \theta}{4} \leq \Phi_K \left(\cos \frac{|\hat{f}(t) - \hat{f}(\theta)|}{4} \right); \tag{1.28}$$

see e.g. [7, (2.3.9), p. 51]. Applying now the identity ([1, Thm. 3.3])

$$\Phi_K(r)^2 + \Phi_{1/K}(\sqrt{1-r^2})^2 = 1, \quad 0 \le r \le 1, \tag{1.29}$$

we obtain for every $t \in [\theta - \pi; \theta + \pi]$,

$$\Phi_{1/K}\left(\sin \frac{|\hat{f}(t) - \hat{f}(\theta)|}{4}\right) \le \sin \frac{|t-\theta|}{4} \le \Phi_K\left(\sin \frac{|\hat{f}(t) - \hat{f}(\theta)|}{4}\right). \tag{1.30}$$

Given $\theta \in \mathbf{R}$ and $t \in [\theta - \pi; \theta + \pi]$ set $\alpha := (t-\theta)/2$ and $\beta := (\hat{f}(t) - \hat{f}(\theta))/2$. Then $|\alpha| \le \pi/2$ and $|\beta| \le \pi$. From (1.28) and (1.30) it follows that

$$|\alpha| \le |\beta| \Longrightarrow 1 \le \frac{\sin\frac{|\beta|}{2}}{\sin\frac{|\alpha|}{2}} \le \frac{\Phi_K\left(\sin\frac{|\alpha|}{2}\right)}{\sin\frac{|\alpha|}{2}} \quad \text{and} \quad \frac{\Phi_{1/K}\left(\cos\frac{|\alpha|}{2}\right)}{\cos\frac{|\alpha|}{2}} \le \frac{\cos\frac{|\beta|}{2}}{\cos\frac{|\alpha|}{2}} \le 1$$

and

$$|\alpha| \ge |\beta| \Longrightarrow \frac{\Phi_{1/K}\left(\sin\frac{|\alpha|}{2}\right)}{\sin\frac{|\alpha|}{2}} \le \frac{\sin\frac{|\beta|}{2}}{\sin\frac{|\alpha|}{2}} \le 1 \quad \text{and} \quad 1 \le \frac{\cos\frac{|\beta|}{2}}{\cos\frac{|\alpha|}{2}} \le \frac{\Phi_K\left(\cos\frac{|\alpha|}{2}\right)}{\cos\frac{|\alpha|}{2}}.$$

Hence

$$\begin{aligned} &\min\left\{\frac{\Phi_{1/K}\left(\sin\frac{|\alpha|}{2}\right)}{\sin\frac{|\alpha|}{2}}, \frac{\Phi_{1/K}\left(\cos\frac{|\alpha|}{2}\right)}{\cos\frac{|\alpha|}{2}}\right\} \\ &\le \left|\frac{\sin\beta}{\sin\alpha}\right| \le \max\left\{\frac{\Phi_K\left(\sin\frac{|\alpha|}{2}\right)}{\sin\frac{|\alpha|}{2}}, \frac{\Phi_K\left(\cos\frac{|\alpha|}{2}\right)}{\cos\frac{|\alpha|}{2}}\right\}. \end{aligned} \tag{1.31}$$

From [1, Thm. 3.18] it follows that for any fixed $K \ge 1$, $(0;1] \ni t \mapsto \Phi_K(t)t^{-1/K}$ is a decreasing function and $(0;1] \ni t \mapsto \Phi_{1/K}(t)t^{-K}$ is an increasing function. Then (1.31) yields

$$\frac{\Phi_{1/K}\left(\sin\frac{|\alpha|}{2}\right)}{\sin\frac{|\alpha|}{2}} \le \left|\frac{\sin\beta}{\sin\alpha}\right| \le \frac{\Phi_K\left(\sin\frac{|\alpha|}{2}\right)}{\sin\frac{|\alpha|}{2}}. \tag{1.32}$$

Fix $z = e^{i\theta} \in \mathbf{T}$. If $f^+(z) = +\infty$, then the second inequality in (1.27) is obvious. So we may assume that $f^+(z) < +\infty$. Applying (1.32), (1.13) and (1.25) we obtain

$$\begin{aligned}
\mathrm{V}[f](z) &= \frac{1}{2\pi}\int_{\mathbf{T}} \left|\frac{f(u)-f(z)}{u-z}\right|^2 |du| \\
&\le \frac{1}{2\pi}\int_{\mathbf{T}} (f^+(z))^{1-1/K} \left|\frac{f(u)-f(z)}{u-z}\right|^{1+1/K} |du| \\
&= (f^+(z))^{1-1/K}\frac{1}{2\pi}\int_{\theta-\pi}^{\theta+\pi} \left|\frac{\sin\frac{\hat{f}(t)-\hat{f}(\theta)}{2}}{\sin\frac{t-\theta}{2}}\right|^{1+1/K} dt \\
&\le (f^+(z))^{1-1/K}\frac{1}{2\pi}\int_{\theta-\pi}^{\theta+\pi} \left(\frac{\Phi_K\left(\sin\frac{|t-\theta|}{4}\right)}{\sin\frac{|t-\theta|}{4}}\right)^{1+1/K} dt.
\end{aligned} \tag{1.33}$$

Thus substituting $s := \frac{t-\theta}{4}$ and using (1.21) we derive the second inequality in (1.27). Applying now (1.32), (1.13), (1.26) and following calculations from (1.33) we obtain

$$\mathrm{V}[f](z) \ge (f^-(z))^{1-1/K}\frac{1}{2\pi}\int_{\theta-\pi}^{\theta+\pi} \left(\frac{\Phi_{1/K}\left(\sin\frac{|t-\theta|}{4}\right)}{\sin\frac{|t-\theta|}{4}}\right)^{1+1/K} dt.$$

Thus substituting $s := \frac{t-\theta}{4}$ and using (1.22) we derive the first inequality in (1.27), which completes the proof. $\square$

Lemma 1.5. *Suppose that* $f \in \mathrm{Hom}^+(\mathbf{T})$ *is absolutely continuous on* $\mathbf{T}$*. Then*

$$\sup_{z\in\mathbf{T}} f^+(z) = e_f := \operatorname*{ess\,sup}_{z\in\mathbf{T}} |f'(z)|. \tag{1.34}$$

as well as

$$\inf_{z\in\mathbf{T}} f^-(z) = d_f := \operatorname*{ess\,inf}_{z\in\mathbf{T}} |f'(z)|. \tag{1.35}$$

Proof. From (1.25) and (1.26) it follows that

$$f^-(z) \le |f'(z)| \le f^+(z) \tag{1.36}$$

for each $z \in \mathbf{T}$ such that the limit (1.2) exists. Hence

$$\inf_{z\in\mathbf{T}} f^-(z) \le d_f \le e_f \le \sup_{z\in\mathbf{T}} f^+(z). \tag{1.37}$$

Assume now that f is absolutely continuous on $\mathbf{T}$. If $e_f = +\infty$, then (1.37) yields (1.34). Thus we may confine considerations to the case $e_f < +\infty$. Then

$$|\hat{f}(t) - \hat{f}(x)| = \left|\int_x^t \hat{f}'(s)\,ds\right| \le e_f|t-x|, \quad t, x \in \mathbf{R}. \tag{1.38}$$

Fix $u = e^{it}$, $z = e^{ix} \in \mathbf{T}$. Since $e_f \geq 1$ and the function sin is increasing and concave on $[0; \pi/2]$, we conclude from (1.38) that

$$|f(u) - f(z)| = 2\sin\frac{|\hat{f}(t) - \hat{f}(x)|}{2} \leq 2\sin\frac{e_f|t-x|}{2} \leq 2e_f\sin\frac{|t-x|}{2} = e_f|u-z|$$

provided $|t - x| \leq \pi/e_f$. If $\pi/e_f \leq |t-x| \leq \pi$, then

$$e_f|u-z| = 2e_f\sin\frac{|t-x|}{2} \geq 2e_f\frac{2}{\pi}\frac{|t-x|}{2} \geq 2 \geq |f(u) - f(z)|.$$

Thus

$$|f(u) - f(z)| \leq e_f|u-z|, \quad u, z \in \mathbf{T}. \tag{1.39}$$

Combining (1.39) with (1.37) we obtain (1.34).

If $d_f = 0$, then (1.37) yields (1.35). So we may assume that $d_f > 0$. Then

$$|\hat{f}(t) - \hat{f}(x)| = \left|\int_x^t \hat{f}'(s)\,ds\right| \geq d_f|t-x|, \quad t, x \in \mathbf{R},$$

and so the inverse mapping f^{-1} is also absolutely continuous on $\mathbf{T}$. Then for a.e. $z \in \mathbf{T}$, $(f^{-1})'(z) = 1/f'(f^{-1}(z))$ and, in consequence, $e_{f^{-1}} = 1/d_f$. Applying now (1.39) with f replaced by f^{-1} we get for any $u, z \in \mathbf{T}$,

$$d_f|u-z| = d_f|f^{-1}(f(u)) - f^{-1}(f(z))| \leq d_f e_{f^{-1}}|f(u) - f(z)| = |f(u) - f(z)|. \tag{1.40}$$

Combining (1.40) with (1.37) we obtain (1.35), which completes the proof. □

Corollary 1.6. *Given $K \geq 1$ and $F \in \mathrm{QC}(\mathbf{D}; K)$ let f be the boundary valued function of F. If $F(0) = 0$ and f is absolutely continuous on $\mathbf{T}$, then*

$$L_K d_f^{1-1/K} \leq \mathrm{V}[f](z) = 2\,\mathrm{Re}\left[z\overline{f(z)}\,\mathrm{C}_{\mathbf{T}}[f'](z)\right] \leq M_K e_f^{1-1/K} \tag{1.41}$$

for a.e. $z \in \mathbf{T}$, where M_K, L_K, e_f and d_f are defined by (1.19), (1.20), (1.34) *and* (1.35), *respectively.*

Proof. The corollary follows directly from Theorems 1.4 and 1.2 and Lemma 1.5. □

2. Derivatives of quasiconformal harmonic mappings and Hardy spaces

In this section we collect results that seem to be known. However, we prove them for the sake of completeness of our considerations in the next section.

Lemma 2.1. *Given $K \geq 1$ and a domain Ω in $\mathbf{C}$ let $F \in \mathrm{QCH}(\mathbf{D}, \Omega; K)$. If Ω is bounded by a rectifiable Jordan curve Γ, then*

$$\sup_{0<r<1}\int_{\mathbf{T}_r}|\partial F(z)|\,|dz| \leq \frac{K+1}{2}|\Gamma|_1, \tag{2.1}$$

and

$$\sup_{0<r<1}\int_{\mathbf{T}_r}|\bar{\partial} F(z)|\,|dz| \leq \frac{K-1}{2}|\Gamma|_1, \tag{2.2}$$

where $|\Gamma|_1$ *is the length of* Γ. *In particular,* $\partial F, \overline{\bar\partial F} \in \mathrm{H}^1(\mathbf{D})$.

Proof. Write f for the boundary valued function of F. Then

$$F(z) = \mathrm{P}[f](z) := \int_0^{2\pi} f(e^{is})\,\mathrm{P}_r(t-s)\,ds, \quad z = re^{it} \in \mathbf{D}, \tag{2.3}$$

where

$$\mathrm{P}_r(\theta) := \frac{1}{2\pi}\,\mathrm{Re}\,\frac{1+re^{i\theta}}{1-re^{i\theta}}, \quad 0 \le r < 1,\ \theta \in \mathbf{R}, \tag{2.4}$$

is the Poisson kernel function. Since the function P_r is symmetric, we get

$$\frac{\partial}{\partial t}\,\mathrm{P}_r(t-s) = -\frac{\partial}{\partial s}\,\mathrm{P}_r(t-s), \quad t, s \in \mathbf{R}.$$

Then integrating by parts we conclude from (2.3) that

$$\begin{aligned}\frac{\partial}{\partial t}F(re^{it}) &= \int_0^{2\pi} f(e^{is})\frac{\partial}{\partial t}\,\mathrm{P}_r(t-s)\,ds = -\int_0^{2\pi} f(e^{is})\frac{\partial}{\partial s}\,\mathrm{P}_r(t-s)\,ds\\ &= \int_0^{2\pi} \mathrm{P}_r(t-s)\,df(e^{is}), \quad 0 \le r < 1,\ t \in \mathbf{R},\end{aligned} \tag{2.5}$$

because the function $s \mapsto f(e^{is})$ is of bounded variation on $[0; 2\pi]$; the last integral is regarded as the Stieltjes one. Fix $r \in (0;1)$. Then by (2.5),

$$\frac{\partial}{\partial t}F(re^{it}) = \lim_{n\to\infty}\sum_{k=1}^{n} \mathrm{P}_r(t-2\pi k/n)[f(e^{2\pi ki/n}) - f(e^{2\pi(k-1)i/n})], \quad t \in \mathbf{R}. \tag{2.6}$$

Hence, applying Fatou's limiting integral lemma, we obtain

$$\begin{aligned}&\int_0^{2\pi}\left|\frac{\partial}{\partial t}F(re^{it})\right| dt\\ &= \int_0^{2\pi}\left|\lim_{n\to\infty}\sum_{k=1}^{n} \mathrm{P}_r(t-2\pi k/n)[f(e^{2\pi ki/n}) - f(e^{2\pi(k-1)i/n})]\right| dt\\ &\le \liminf_{n\to\infty}\int_0^{2\pi}\left|\sum_{k=1}^{n} \mathrm{P}_r(t-2\pi k/n)[f(e^{2\pi ki/n}) - f(e^{2\pi(k-1)i/n})]\right| dt.\end{aligned} \tag{2.7}$$

Since

$$\begin{aligned}&\int_0^{2\pi}\left|\sum_{k=1}^{n} \mathrm{P}_r(t-2\pi k/n)[f(e^{2\pi ki/n}) - f(e^{2\pi(k-1)i/n})]\right| dt\\ &\le \sum_{k=1}^{n}\left[|f(e^{2\pi ki/n}) - f(e^{2\pi(k-1)i/n})|\int_0^{2\pi} \mathrm{P}_r(t-2\pi k/n)\,dt\right]\\ &\le \sum_{k=1}^{n}|f(e^{2\pi ki/n}) - f(e^{2\pi(k-1)i/n})| \le |\Gamma|_1\end{aligned}$$

and since for $z = re^{it}$,

$$\frac{\partial}{\partial t}F(re^{it}) = i[z\partial F(z) - \bar{z}\bar{\partial}F(z)], \tag{2.8}$$

we conclude from (2.7) that

$$\int_{\mathbf{T}_r} (|\partial F(z)| - |\bar{\partial}F(z)|)|\,dz| \leq \int_0^{2\pi} |z\partial F(z) - \bar{z}\bar{\partial}F(z)|\,dt \leq |\Gamma|_1. \tag{2.9}$$

By the assumption, the mapping F is K-quasiconformal, which means that

$$(K+1)|\bar{\partial}F(z)| \leq (K-1)|\partial F(z)|, \quad z \in \mathbf{D}. \tag{2.10}$$

Hence by (2.9),

$$\int_{\mathbf{T}_r} (|\bar{\partial}F(z)| + |\partial F(z)|)|\,dz| \leq K \int_{\mathbf{T}_r} (|\partial F(z)| - |\bar{\partial}F(z)|)|\,dz| \leq K|\Gamma|_1.$$

Combining this with (2.9) and (2.10) leads to (2.1) and (2.2). ☐

Corollary 2.2. *Given $K \geq 1$ and a domain Ω in $\mathbf{C}$ let $F \in \mathrm{QCH}(\mathbf{D}, \Omega; K)$. If Ω is bounded by a rectifiable Jordan curve Γ, then the boundary valued function f of F is absolutely continuous.*

Proof. From Lemma 2.1 it follows that $\partial F, \overline{\bar{\partial}F} \in \mathrm{H}^1(\mathbf{D})$. The classical result of Riesz [2, Theorem 3.11, p. 42] says that there exist functions $H, G\colon \overline{\mathbf{D}} \to \mathbf{C}$ continuous on $\overline{\mathbf{D}}$, holomorphic on $\mathbf{D}$ and absolutely continuous on $\mathbf{T}$ and such that $H'(z) = \partial F(z)$ and $G'(z) = \overline{\bar{\partial}F}(z)$, $z \in \mathbf{D}$, i.e. H and G are primitive functions to ∂F and $\overline{\bar{\partial}F}$ on $\mathbf{D}$, respectively. Moreover, F has a continuous extension to $\overline{\mathbf{D}}$. Hence for each $z \in \mathbf{T}$,

$$\begin{aligned} f(z) - F(0) &= \int_\gamma \partial F(u)\,du + \bar{\partial}F(u)\,\overline{du} \\ &= \int_\gamma H'(u)\,du + \overline{G'(u)}\,\overline{du} = H(z) - H(0) + \overline{G(z) - G(0)}, \end{aligned} \tag{2.11}$$

where $\gamma(t) := tz$, $t \in [0;1]$. From (2.11) we see that $f(z) = H(z) + \overline{G(z)} + F(0) - H(0) - \overline{G(0)}$ for $z \in \mathbf{T}$. Thus f is an absolutely continuous function on $\mathbf{T}$. ☐

Modifying the proof of Lemma 2.1 we may easily derive the following lemma.

Lemma 2.3. *Given $K \geq 1$ and a Jordan domain Ω in $\mathbf{C}$ let $F \in \mathrm{QCH}(\mathbf{D}, \Omega; K)$. If the boundary valued function f of F satisfies the inequality*

$$|f(u) - f(v)| \leq L|u - v|, \quad u, v \in \mathbf{T}, \tag{2.12}$$

for some positive constant L, then

$$\sup_{z \in \mathbf{D}} |\partial F(z)| \leq \frac{K+1}{2}L, \tag{2.13}$$

and

$$\sup_{z\in\mathbf{D}} |\bar\partial F(z)| \le \frac{K-1}{2} L. \tag{2.14}$$

In particular, $\partial F, \overline{\bar\partial F} \in \mathrm{H}^\infty(\mathbf{D})$.

Proof. From (2.12) it follows that Ω is bounded by a rectifiable Jordan curve Γ. Hence the function $s \mapsto f(e^{is})$ is of bounded variation on $[0; 2\pi]$ and, as in the proof of Lemma 2.1, the equality (2.6) holds. From (2.12) it also follows that for all $n \in \mathbf{N}$ and $k = 1, 2, \ldots, n$,

$$\begin{aligned} |f(e^{2\pi ki/n}) - f(e^{2\pi(k-1)i/n})| &\le L|e^{2\pi ki/n} - e^{2\pi(k-1)i/n}| \\ &= \frac{2\pi}{n} \cdot L(n/\pi)\sin(\pi/n). \end{aligned} \tag{2.15}$$

Fix $r \in (0; 1)$. Since $(n/\pi)\sin(\pi/n) \to 1$ as $n \to \infty$, we conclude from (2.6) and (2.15) that for every $t \in \mathbf{R}$,

$$\begin{aligned} \left|\frac{\partial}{\partial t} F(re^{it})\right| &\le L \lim_{n\to\infty} (n/\pi)\sin(\pi/n) \sum_{k=1}^{n} \mathrm{P}_r(t - 2\pi k/n)\frac{2\pi}{n} \\ &= L\int_0^{2\pi} \mathrm{P}_r(t-s)\, ds = L. \end{aligned} \tag{2.16}$$

Since for $z = re^{it}$ the equality (2.8) holds, we conclude from (2.16) that

$$r(|\partial F(z)| - |\bar\partial F(z)|) \le |z\partial F(z) - \bar z\bar\partial F(z)| \le L. \tag{2.17}$$

By the assumption, the mapping F is K-qc., which means that (2.10) holds. Hence by (2.17),

$$r(|\bar\partial F(z)| + |\partial F(z)|) \le rK(|\partial F(z)| - |\bar\partial F(z)|) \le KL. \tag{2.18}$$

Combining the inequalities (2.17) and (2.18) with (2.10) we obtain the inequalities (2.13) and (2.14), because both the functions $\overline{\bar\partial F}$ and ∂F are holomorphic on $\mathbf{D}$. □

3. The bi-Lipschitz property for quasiconformal harmonic self-mappings of the unit disk

Theorem 3.1. *Given* $K \ge 1$ *and* $F \in \mathrm{QCH}(\mathbf{D}; K)$ *let* f *be the boundary valued function of* F. *Then for a.e.* $z \in \mathbf{T}$,

$$\left|\mathrm{V}[f](z) + i\,\mathrm{V}^*[f](z) - \frac{1}{2}(K + \frac{1}{K})|f'(z)|\right| \le \frac{1}{2}(K - \frac{1}{K})|f'(z)|. \tag{3.1}$$

In particular, for a.e. $z \in \mathbf{T}$,

$$\frac{1}{K}|f'(z)| \le \mathrm{V}[f](z) \le K|f'(z)| \quad \textit{and} \quad |\mathrm{V}^*[f](z)| \le \frac{1}{2}(K - \frac{1}{K})|f'(z)|. \tag{3.2}$$

Proof. From [10, Theorem 3.1] it follows that $f'(z) \neq 0$ for a.e. $z \in \mathbf{T}$. By Corollary 2.2, f is absolutely continuous on $\mathbf{T}$. Hence and by [8, Corollary 2.2] we obtain

$$(K+1)\left|1 - 2\frac{\mathrm{C}_{\mathbf{T}}[f'](z)}{f'(z)}\right| \leq (K-1)\left|1 + 2\frac{\mathrm{C}_{\mathbf{T}}[f'](z)}{f'(z)}\right| \quad \text{for a.e. } z \in \mathbf{T},$$

which leads to

$$\left|2\frac{\mathrm{C}_{\mathbf{T}}[f'](z)}{f'(z)} - \frac{1}{2}(K + \frac{1}{K})\right| \leq \frac{1}{2}(K - \frac{1}{K}) \quad \text{for a.e. } z \in \mathbf{T}. \tag{3.3}$$

From Theorem 1.2 and (1.5) it follows that for a.e. $z \in \mathbf{T}$,

$$2\frac{\mathrm{C}_{\mathbf{T}}[f'](z)}{f'(z)} = \frac{\overline{z}f(z)}{f'(z)}\big(\mathrm{V}[f](z) + i\,\mathrm{V}^*[f](z)\big) = \frac{1}{|f'(z)|}\big(\mathrm{V}[f](z) + i\,\mathrm{V}^*[f](z)\big).$$

This combined with (3.3) yields (3.1). The inequalities (3.2) follow directly from (3.1), which ends the proof. □

Theorem 3.2. *Given* $K \geq 1$ *and* $F \in \mathrm{QCH}(\mathbf{D}; K)$ *let* f *be the boundary valued function of* F*. If* $F(0) = 0$*, then for a.e.* $z \in \mathbf{T}$,

$$\frac{2^{5(1-K^2)/2}}{(K^2+K-1)^K} \leq (L_K/K)^K \leq |f'(z)| \leq (M_K K)^K \leq K^{3K} 2^{5(K-1/K)/2}, \tag{3.4}$$

where M_K *and* L_K *are defined by* (1.19) *and* (1.20)*, respectively.*

Proof. By Corollary 2.2, f is absolutely continuous on $\mathbf{T}$. Then Corollary 1.6 and the first inequality in (3.2) show that for a.e. $z \in \mathbf{T}$, $L_K d_f^{1-1/K} \leq K|f'(z)|$ and $|f'(z)| \leq K M_K e_f^{1-1/K}$, where e_f and d_f are defined by (1.34) and (1.35), respectively. Hence $L_K d_f^{1-1/K} \leq K d_f$ and $e_f \leq K M_K e_f^{1-1/K}$, and consequently, we obtain the following implications

$$\big[0 < d_f \Rightarrow (L_K/K)^K \leq d_f\big] \quad \text{and} \quad \big[e_f < +\infty \Rightarrow e_f \leq (M_K K)^K\big]\,. \tag{3.5}$$

For any $n \in \mathbf{N}$ let $\mathbf{D}_n := \{z \in \mathbf{C} : |z| < n/(n+1)\}$ and φ_n be the conformal mapping from $\mathbf{D}$ onto $F^{-1}(\mathbf{D}_n)$ such that $\varphi_n(0) = 0$ and $\varphi_n'(0) > 0$. Then $F_n := (1 + 1/n)F \circ \varphi_n \in \mathrm{QCH}(\mathbf{D}; K)$ and $F_n(0) = 0$, $n \in \mathbf{N}$. Fix $n \in \mathbf{N}$. Since F is a C^2-diffeomorphic self-mapping of $\mathbf{D}$ we see that $F^{-1}(\mathbf{D}_n)$ is a domain bounded by a C^2-Jordan curve. Applying Kellogg–Warschawski theorem ([15, Theorem 3.5, p. 48], [16]) we see that φ_n' has a continuous extension ψ_n to the closed disk $\overline{\mathbf{D}}$ and $\psi_n(z) \neq 0$ for all $z \in \overline{\mathbf{D}}$. Thus the boundary valued function f_n of F_n is a C^1-diffeomorphic self-mapping of $\mathbf{T}$, and so $0 < d_{f_n} \leq e_{f_n} < +\infty$. By (3.5) and Lemma 1.5 we see that for all $u, z \in \mathbf{T}$, $u \neq z$,

$$(L_K/K)^K \leq d_{f_n} \leq \frac{|f_n(u) - f_n(z)|}{|u - z|} \leq e_{f_n} \leq (M_K K)^K, \quad n \in \mathbf{N}. \tag{3.6}$$

Setting $F_0 := F$ we conclude from [6, Theorem 3.2, p. 66] that

$$|F_n(z) - F_n(w)| \leq 16|z - w|^{1/K}, \quad z, w \in \overline{\mathbf{D}},\ n = 0, 1, 2, \ldots.$$

Hence for all $z \in \mathbf{T}$ and $w \in \mathbf{D}$,

$$(3.7)\qquad |f_n(z) - f(z)| \leq 32|z-w|^{1/K} + 16|\varphi_n(w) - w|^{1/K} + \frac{1}{n}, \quad n \in \mathbf{N}.$$

From [14, Theorem 1.8] it follows that $\varphi_n(w) \to w$ as $n \to \infty$ for each $w \in \mathbf{D}$. Thus given $\varepsilon > 0$ and $z \in \mathbf{T}$ we can choose $w \in \mathbf{D}$ and $n_\varepsilon \in \mathbf{N}$ such that the right hand side in (3.7) is less than ε as $n > n_\varepsilon$. This means that for every $z \in \mathbf{T}$, $f_n(z) \to f(z)$ as $n \to \infty$. Since f is absolutely continuous on $\mathbf{T}$, (3.6) and Lemma 1.5 then show that $(L_K/K)^K \leq d_f \leq e_f \leq (M_K K)^K$. This and Lemma 1.3 yield (3.4), which ends the proof. □

Theorem 3.3. *Given $K \geq 1$ and $F \in \mathrm{QCH}(\mathbf{D}; K)$ assume that $F(0) = 0$. Then for all $z, w \in \mathbf{D}$,*

$$(3.8)\qquad |F(z) - F(w)| \leq K(M_K K)^K |z-w| \leq K^{3K+1} 2^{5(K-1/K)/2} |z-w|$$

as well as

$$(3.9)\qquad |F(z) - F(w)| \geq \frac{L_K^{3K}}{K^{4K+1} M_K^K} |z-w| \geq \frac{2^{5(1-K^2)(3+1/K)/2}}{K^{3K+1}(K^2+K-1)^{3K}} |z-w|,$$

where M_K and L_K are defined by (1.19) and (1.20), respectively.

Proof. Fix $z, w \in \mathbf{D}$. Setting $\gamma(t) := z + t(w-z)$, $t \in [0;1]$, we get

$$(3.10)\qquad \begin{aligned} |F(z) - F(w)| &= \left| \int_0^1 \frac{d}{dt} F(\gamma(t))\, dt \right| \\ &= \left| \int_0^1 \Big(\partial F(\gamma(t)) \gamma'(t) + \bar\partial F(\gamma(t)) \overline{\gamma'(t)} \Big)\, dt \right| \\ &\leq \int_0^1 \big(|\partial F(\gamma(t))| + |\bar\partial F(\gamma(t))| \big)\, dt |z-w| \\ &\leq \sup_{u \in \mathbf{D}} \big(|\partial F(u)| + |\bar\partial F(u)| \big) |z-w|. \end{aligned}$$

From Corollary 2.2 and Lemmas 1.5 and 2.3 it follows that

$$(3.11)\qquad \sup_{u \in \mathbf{D}} \big(|\partial F(u)| + |\bar\partial F(u)| \big) \leq K e_f,$$

where f is the boundary valued function of F. Combining (3.10) and (3.11) we conclude from Theorem 3.2 that the estimation (3.8) holds. Setting now $\gamma(t) :=$

$F^{-1}(z+t(w-z))$, $t \in [0;1]$, we get

$$
\begin{aligned}
|z-w| &= \int_0^1 \left|\frac{d}{dt}F(\gamma(t))\right| dt = \int_0^1 |\partial F(\gamma(t))\gamma'(t) + \bar\partial F(\gamma(t))\overline{\gamma'(t)}|\, dt \\
&\geq \int_0^1 \Big(|\partial F(\gamma(t))||\gamma'(t)| - |\bar\partial F(\gamma(t))||\overline{\gamma'(t)}|\Big)\, dt \\
&\geq \inf_{u\in\mathbf{D}} \big(|\partial F(u)| - |\bar\partial F(u)|\big) \int_0^1 |\gamma'(t)|\, dt \\
&\geq \inf_{u\in\mathbf{D}} \frac{|\partial F(u)|^2 - |\bar\partial F(u)|^2}{|\partial F(u)| + |\bar\partial F(u)|} |F^{-1}(z) - F^{-1}(w)|.
\end{aligned} \tag{3.12}
$$

From [11, Theorem 0.2] it follows that $|\partial F(u)|^2 - |\bar\partial F(u)|^2 \geq d_f^3$ for all $u \in \mathbf{D}$. Hence and by (3.12) and (3.11) we get

$$
|F(z) - F(w)| \geq \frac{d_f^3}{Ke_f}|z-w|. \tag{3.13}
$$

Applying now Theorem 3.2 we obtain the estimation (3.9), which ends the proof. □

Applying a variant of Heinz's inequality from [12, Theorem 2.2] we derive an alternative estimation to (3.9) like below.

Theorem 3.4. *Given $K \geq 1$ and $F \in \mathrm{QCH}(\mathbf{D}; K)$ assume that $F(0) = 0$. Then for all $z, w \in \mathbf{D}$,*

$$
|F(z) - F(w)| \geq \frac{1}{K}\max\{\frac{2}{\pi},\ L_K^*\}|z-w|, \tag{3.14}
$$

where

$$
L_K^* := \frac{2}{\pi}\int_0^{\Phi_{1/K}(1/\sqrt{2})^2} \frac{dt}{\Phi_K\left(\sqrt{t}\,\right)\Phi_{1/K}\left(\sqrt{1-t}\,\right)}. \tag{3.15}
$$

Proof. From (3.12), (2.10) and [12, Theorem 2.2] we see that

$$
\begin{aligned}
|z-w| &\geq \inf_{u\in\mathbf{D}} \big(|\partial F(u)| - |\bar\partial F(u)|\big) \int_0^1 |\gamma'(t)|\, dt \\
&\geq \frac{2}{K+1} \inf_{u\in\mathbf{D}} |\partial F(u)||F^{-1}(z) - F^{-1}(w)| \\
&\geq \frac{1}{K}\max\{\frac{2}{\pi}\,,\ L_K^*\}|F^{-1}(z) - F^{-1}(w)|,
\end{aligned}
$$

which leads to (3.14). □

Remark 3.5. All the estimations in Theorems 3.2, 3.3 and 3.4 hold under the assumption $F(0) = 0$. In a general case where $a := F^{-1}(0) \in \mathbf{D}$ we may replace F by the composition $F_a := F \circ H_a$, where $H_a(z) := (z+a)(1+\bar a z)^{-1}$, $z \in \mathbf{D}$. Then $F_a(0) = 0$ and $F_a \in \mathrm{QCH}(\mathbf{D}; K)$. Applying the theorems to F_a we obtain variants of all the estimations (3.4), (3.8), (3.9) and (3.14) by multiplying some terms of them by constants $A := (1-|a|)/(1+|a|)$ or $B := (1+|a|)/(1-|a|)$ as follows:

(i) the first two terms in (3.4) by A and the last two ones by B;
(ii) the last two terms in (3.8) by B and the last two ones in (3.9) by A;
(iii) the right hand side in (3.14) by A.

Remark 3.6. Remark 3.5 yields an explicit variant of [13, Theorem 1.2] with L expressed by means of $|F^{-1}(0)|$ and K. Remark 3.5 combined with [10, Corollary 4.3] implies explicit variants of [13, Theorem 1.1] with 0 and ∞ replaced by constants expressed by means of $|F^{-1}(0)|$ and K. Moreover, under the normalized condition $F(0) = 0$ the explicit variants appear to be asymptotically sharp as K tends to 1, which is a consequence of the estimations (3.4), (3.8), (3.9) and (3.14) where all terms involving K tend to 1 as $K \to 1^+$. This means that $F \in \mathrm{QCH}(\mathbf{D}; K)$ keeping the origin fixed behaves almost like a rotation for sufficiently small K.

Remark 3.7. Numerical experiments show that $(L_K/K)^K < L_K^*/K$ for $K > 1$. Therefore the estimation [12, (2.1)] seems to be better than the lower bound in (3.4), and thereby the estimation (3.14) looks better as compared to the one (3.9). However, we decided to publish the estimations (3.4) and (3.9), too, because they were obtained in an alternative and more direct way without using the version of Heinz's inequality given by [12, Theorem 2.2]. Furthermore, the new approach sheds a new light on studying classes $\mathrm{QCH}(\Omega; K)$ for Jordan domains $\Omega \subset \mathbf{C}$.

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