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THE HESSIAN OF THE DISTANCE FROM A SURFACE IN THE HEISENBERG GROUP

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Abstract. Given a smooth surface S in the Heisenberg group, we compute the Hessian of the function measuring the Carnot–Charathéodory distance from S in terms of the mean curvature of S and of an "imaginary curvature" which was introduced in [2] in order to find the geodesics which are metrically normal to S. Explicit formulae are given when S is a plane or the metric sphere.

1. Introduction

In this article we continue to study the properties of the function "signed distance from a surface S", δ_S, in the Heisenberg group, started in [2]. In particular we are interested in the horizontal Hessian of δ_S and in the related notions of curvature for S. Let $\mathbf{H} = \mathbf{H}^1$ be $\mathbf{R}^3$ with the Heisenberg group structure, and let d be the associated Carnot–Charathéodory distance. See Section 2 for the definitions. If $S \subset \mathbf{H}$ is closed, the distance from a point P to S is

$$d_S(P) = \inf_{Q\in S} d(P,Q)$$

Here, we consider the case where $S = \partial\Omega$ is the C^3 boundary, in the Euclidean sense, of an open subset of $\mathbf{H}$. It would be natural to consider C^2 surfaces in the Euclidean sense, since such surfaces satisfy a internal/external Heisenberg sphere condition at any non-characteristic point, see [2]. However, our elementary approach requires a degree more of regularity. This is a minor nuisance, since here we are interested in geometric properties of surfaces.

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The *signed distance* from S is

$$\delta_S(P) = \begin{cases} -d_S(P) & \text{if } P \in \Omega, \\ d_S(P) & \text{if } P \notin \Omega. \end{cases} \tag{1}$$

Observe that, given S, the signed distance δ_S is defined modulo a sign, which corresponds to the choice of one of the two open sets having boundary S, or, equivalently, to the choice of an orientation for S.

It was shown in [24] that, if S is a closed subset of $\mathbf{H}$, then d_S is a.e. solution of the eikonal equation

$$|\nabla_{\mathbf{H}} d_S| = 1, \tag{2}$$

where $\nabla_{\mathbf{H}} = (X, Y)$ is the horizontal gradient. More on this line of investigation is in [3], [6], [12], [31]. Here X and Y are the left invariant vector fields

$$X = \partial_x + 2y\partial_t, \; Y = \partial_y - 2x\partial_t.$$

A sub-Riemannian metric is defined on the horizontal bundle $\mathscr{H} = \operatorname{span}\{X, Y\}$ in such a way that $\{X, Y\}$ is a orthonormal basis for $\mathscr{H}$.

Concerning the properties of the distance function from a point and the distance function from a surface, we recall the paper by Agrachev and Gautier, [1] and the contribution by Vershik and Gershkovich, which is surveyed in [33]. In [2], we proved several regularity results for the function δ_S, in particular that, if S is C^2 in the Euclidean sense, then $\nabla_{\mathbf{H}}\delta_S$ is C^1 in the Euclidean sense in an open neighborhood of $S \setminus \operatorname{Char}(S)$, where $\operatorname{Char}(S)$ is the characteristic set of S.

In this article, we move a step forward in the understanding of the higher regularity of δ_S. The *horizontal Hessian* of a smooth function f defined on an open subset of $\mathbf{H}$ is the matrix

$$\operatorname{Hess}_{\mathbf{H}} f = \begin{bmatrix} XXf & YXf \\ XYf & YYf \end{bmatrix}.$$

See, for instance, [10]. We denote by I and J the identity and the simplectic matrix, respectively,

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Theorem 1.1. *Let $S = \partial\Omega$ be boundary of a C^3 open subset of $\mathbf{H}$ and let P be a non-characteristic point of S. Then,*

$$\operatorname{Hess}_{\mathbf{H}} \delta_S(P) = v_P S \otimes v_P S \cdot (h_S(P)I + p_S(P)J), \tag{3}$$

where $v_P S$ is the unit horizontal vector tangent to S at P, $h_S(P)$ is the mean curvature of S at P and $p_S(P)$ is the curvature of the oriented metric normal to S at P.

We shall also refer to $h_S(P)$ as the *real curvature* and to $p_S(P)$ as the *imaginary curvature* of S at P, respectively. Some explanation about the terminology is in order. Complete definitions will be given in Section 2.

We can compare Theorem 1.1 with its Euclidean analog, see [17] Lemma 14.17, p. 355. Let $\delta_S^E(P)$ be the signed distance from P to $S=\partial\Omega$. The Euclidean Hessian of δ_S^E at $x_0 \in S$, $\mathrm{Hess}\,\delta_S^E(x_0)$, is a symmetric 3×3 metrics, having as eigenvalues 0 and the principal curvatures of S at P. The trace of $\mathrm{Hess}\,\delta_S^E(x_0)$ is the mean curvature of S at x_0. The quantity $p_S(P)$ in Theorem 1.1 seems to have no Euclidean analog.

The point $P \in S$ is characteristic if the tangent space of S at P coincides with the fiber $\mathscr{H}_P$. If P is a non-characteristic point of S, then the intersection of the space tangent to S at P, T_PS, with the horizontal fiber $\mathscr{H}_P$ is a one-dimensional linear space V_PS. We call V_PS the *horizontal direction tangent to S at P*. The *unit horizontal vector tangent to S at P* is the unit vector spanning V_PS, which is defined modulo a sign. There is a rich literature on the calculus on surfaces in $\mathbf{H}$. Just to mention a few references, we refer to [26], [13], [14], [15], [16], [28], [29]. In [14], in particular, surfaces are defined in terms of graphs of functions, with the aid of an implicit function theorem, an approach which is especially interesting in that it provides an intrinsic definition of C^1 surfaces in the Heisenberg sense.

If the open set $\Omega \subset \mathbf{H}$ and $S=\partial\Omega$ is, near P, the level set of a smooth function g and if locally they are given by $g(z,t)<0$ and $g(z,t)=0$, respectively, then the analytic expression for $p_S(P)$ is

$$p_S(P) = -\frac{[X,Y]g(P)}{|\nabla_{\mathbf{H}} g(P)|}. \tag{4}$$

We say that the function g is *compatible with the orientation* of S. Observe that $p_S(P)$ is a first order object in the Euclidean sense, since $[X,Y]g=-4\partial_t g$. On the other hand, from an intrinsic point of view, we can think of $p_S(P)$ as a second order quantity, since it involves the commutator of two intrinsic first order differential operators. It was shown in [2] how p_S is a quantity peculiarly related to the Heisenberg geometry. Let $\mathscr{N}_P^+S=\gamma$ be the *oriented metric normal to S at P*, i.e., the geodesic arc γ leaving Ω at P such that, for small $\sigma>0$, $\delta_S(\gamma(\sigma))=\sigma$. The lifetime τ of the geodesic γ, the maximum amount of time over which γ is length-minimizing, is $\tau=\frac{\pi}{|p_S(P)|}$. The sign of $p_S(P)$ is positive or negative according to the fact that $\mathscr{N}_P^+S$ points "upward" (the t-coordinate of $P^{-1}\cdot\mathscr{N}_P^+S(\sigma)$ increases with σ) or "downward". Notice that $p_S(P)=0$ if $\mathscr{N}_P^+S$ is a Euclidean straight line. The number $p_S(P)$ is a sort of curvature for $\mathscr{N}_P^+S$. Theorem 1.1 says, in a way, that $p_S(P)$ is also a sort of curvature of the surface S. Let $\mathrm{Hess}_{\mathbf{H}}^{sym}=\mathrm{Hess}_{\mathbf{H}}+\mathrm{Hess}_{\mathbf{H}}^{tr}$ be the *symmetrized horizontal Hessian* operator, where the superscript ^{tr} denotes transposition of matrices. Observe that the quantity p_S does not appear in the expression of $\mathrm{Hess}_{\mathbf{H}}^{sym}\,\delta_S$.

The *mean curvature* $h_S(P)$ of a surface S at P is an object much studied in the recent literature [7], [16], [28], [29], [5]. If S is given as the level set of the function g near the non-characteristic point P, which is compatible with the orientation of S, the mean curvature is defined by

$$h_S(P) = X\left(\frac{Xg(P)}{|\nabla_{\mathbf{H}} g(P)|}\right) + Y\left(\frac{Yg(P)}{|\nabla_{\mathbf{H}} g(P)|}\right). \tag{5}$$

Observe that, if $S = \{(z,t) : g(z,t) = 0\}$, and $\Omega = \{(z,t) :\ g(z,t) < 0\}$, then

$$h_S(P) = \operatorname{div}_{\mathbf{H}} \nu_P S,$$

where $\nu_P S = \frac{\nabla_{\mathbf{H}} g(P)}{|\nabla_{\mathbf{H}} g(P)|}$ is the intrinsic outer unit normal to S at P. By Theorem 1.1, it is tautological that

$$\operatorname{Trace}(\operatorname{Hess}_{\mathbf{H}} \delta_S(P)) = h_S(P).$$

In Section 4, with some applications in perspective, we show that there are other equivalent definitions for $h_S(P)$. On the one hand, $h_S(P)$ is the eigenvalue of a suitable *horizontal Weingarten map* on S at P, in analogy with the Euclidean case (Theorem 4.2). On the other hand, if Γ is the unique horizontal curve on S through P and $\pi(\Gamma)$ is its vertical projection onto the plane $t = 0$ in $\mathbf{H}$, then $h_S(P)$ is the *Euclidean* curvature of $\pi(\Gamma)$ at $\pi(P)$ (Theorem 4.1). Some of these facts have been independently proved in a similar form in [5], [16], [7], [8]. In [16] it is also proved that h_S is the trace of a second order matrix which is different from the horizontal Hessian of δ_S. For a more general treatment of the parallel transport see, e.g., [19] for the Riemannian case and [21] for the sub-Riemannian one.

As an application of Theorem 1.1 and Theorem 4.1, in Theorem 5.2 we give the explicit expression for the horizontal Hessian of the function measuring the Carnot–Charathéodory distance from a point.

There are two reasons why we restricted our analysis to the Heisenberg group. The first is out of necessity: the proofs in this paper are based on the results in [2], which are stated and proved in this setting. The second is that the Heisenberg group provides the simplest nontrivial example of Carnot–Charathéodory group and it is important in itself, for instance in modelling the visual cortex [9], [8], [30]. It is interesting that these new applications require a finer study of surfaces in the Heisenberg group and of their curvatures.

Here is a short overview of the article. In Section 2 we provide some background and preliminary results. In Section 3 we prove Theorem 1.1. Section 4 is devoted to a discussion of the mean curvature, while in Section 5 we discuss several examples.

Note on the proofs. We will always assume implicitely that our surface S is locally the graph of a function f from a subset of the plane $\{t = 0\}$ to $\mathbf{R}$, i.e. $0 = g(x, y, t) = t - f(x, y)$. In particular, without loss of generality, we assume that Ω lies below the graph of f. The cases when S is not locally the graph of function, i.e., in the "near-Euclidean case" when the plane tangent to S is vertical at some point, can be dealt with in a way similar to that employed in [2], see the proof of Theorem 6.1 (ii), pp. 680–681.

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2. Notation and preliminaries

In this section, we collect some basic definitions and known facts about the structure and the geometry of $\mathbf{H}$. There is a vast literature on sub-Riemannian geometry and Carnot groups. Justs to quote a few titles, we refer the reader to [4], [13], [18], [22], [25], [26], [27], [32].

The Heisenberg group $\mathbf{H} = \mathbf{H}^1$ is the Euclidean space $\mathbf{R}^3$ endowed with the noncommutative product

$$(x, y, t) \cdot (x', y', t') = (x + x', y + y', t + t' + 2(x'y - xy')).$$

Sometimes it is convenient to think of the elements of $\mathbf{H}$ as $(z, t) \in \mathbf{C} \times \mathbf{R}$. Consider the left invariant vector fields X and Y,

$$X = \partial_x + 2y\partial_t, \quad Y = \partial_y - 2x\partial_t.$$

The vector fields X, Y do not commute, $[X, Y] = -4\partial_t$. The span of the vector fields X and Y is called *horizontal distribution*, and it is denoted by $\mathscr{H}$. The *fiber* of $\mathscr{H}$ at a point P of $\mathbf{H}$ is $\mathscr{H}_P = \text{span}\{X_P, Y_P\}$. The inner product in $\mathscr{H}_P$, here denoted by $\langle \cdot, \cdot \rangle$, is the unique inner product making X_P and Y_P orthonormal.

An important element of $\mathbf{H}$'s structure is the *dilation group at the origin* $\{\delta_\lambda : \lambda \neq 0\}$,

$$\delta_\lambda(z, t) = (\lambda z, \lambda^2 t), \quad z = x + iy.$$

By left translation, a dilation group is defined at each point P of $\mathbf{H}$.

The Heisenberg group is also endowed with a *rotation group*, which is useful in simplifying some calculations. For $\theta \in \mathbf{R}$, let

$$R_\theta(z, t) = (e^{i\theta} z, t)$$

be the rotation by θ around the t-axis. Composing with left translation, one could define rotations around any vertical line $(x, y) = (a, b)$. R_θ is an isometry of $\mathbf{H}$ and its differential acts on the fiber $\mathscr{H}_O$ as a rotation by θ. Under the usual identification between the Riemannian tangent space of $\mathbf{H}$ at O, $T_O\mathbf{H}$, and the Lie algebra $\mathfrak{h}$ of $\mathbf{H}$, the differential of R_θ can be thought of as a rotation on $\text{span}\{X, Y\}$, the first stratum of $\mathfrak{h}$. With respect to the basis $\{X, Y\}$,

$$dR_\theta V = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} V$$

whenever $V \in \text{span}\{X, Y\}$. With some abuse of notation, we denote dR_θ by R_θ.

The *Carnot–Charathéodory distance* between two points P and Q in $\mathbf{H}$, here denoted by $d(P, Q)$, is defined as follows. Consider an absolutely continuous curve γ in $\mathbf{R}^3$, joining P and Q, which is *horizontal*. That is, $\dot{\gamma}(t) = a(t)X_{\gamma(t)} + b(t)Y_{\gamma(t)}$ lies in $\mathscr{H}_{\gamma(t)}$. Its Carnot–Charathéodory length is $l_{\mathbf{H}}(\gamma)$,

$$l_{\mathbf{H}}(\gamma) = \int (a(t)^2 + b(t)^2)^{1/2}\, dt.$$

The Carnot–Charathéodory distance between P and Q, $d(P, Q)$, is the infimum of the Carnot–Charathéodory lengths of such curves. The infimum is actually a minimum and the distance between P and Q is realized by the length of a geodesic.

By translation invariance, all geodesics are left translations of geodesics passing through the origin. The unit-speed geodesics at the origin [23], [22] are

$$(6)\qquad \gamma_{O,\phi,W}(\sigma)=\begin{cases} x(\sigma)=\sin(\alpha(W))\frac{1-\cos(\phi\sigma)}{\phi}+\cos(\alpha(W))\frac{\sin(\phi\sigma)}{\phi},\\ y(\sigma)=\sin(\alpha(W))\frac{\sin(\phi\sigma)}{\phi}-\cos(\alpha(W))\frac{1-\cos(\phi\sigma)}{\phi},\\ t(\sigma)=2\frac{\phi\sigma-\sin(\phi\sigma)}{\phi^2}.\end{cases}$$

Here, W is a unitary vector in $\mathscr{H}_O$ and $\alpha(W)\in[0,2\pi)$ is unique argument with the property $\dot\gamma_{O,\phi,W}(0)=W$. The geodesic is length minimizing over any interval of length $2\pi/|\phi|$. In the case $\phi=0$, the geodesic is a straight line in the plane $\{t=0\}$,

$$x(\sigma)=\cos(\alpha(W))\sigma,\quad y(\sigma)=\sin(\alpha(W))\sigma,$$

and we say that the geodesic is *straight*.

From these equations we deduce the parametric equations of the boundary of the ball $B(0,r)$. $(z,t)\in\partial B(0,r)$ if and only if there is $\phi\in[-2\pi/r,2\pi/r]$ so that

$$(7)\qquad \begin{cases} |z|=2\frac{\sin(\phi r/2)}{\phi},\\ t=2\frac{\phi r-\sin(\phi r)}{\phi^2}.\end{cases}$$

If $P=(z,t)$ and $z\neq 0$, then there exists a unique length minimizing geodesic connecting P and O. If $P=(0,t)$, $t\neq 0$, (i.e., if P belongs to the center of $\mathbf{H}$) then there is a one parameter family of length minimizing geodesics joining P and O, obtained by rotation of a single geodesic around the t-axis.

We call *curvature* of $\gamma_{O,\phi,W}$ the parameter $\kappa(\gamma_{O,\phi,W})=|\phi|$. We will see later how $\kappa(\gamma_{O,\phi,W})$ is related to the curvature of a surface in $\mathbf{H}$. The notion of curvature extends to all geodesics by left translations, $\kappa(L_P\gamma)=\kappa(\gamma)$, where L_P is left translation by $P\in\mathbf{H}$. Given points $P=(z,t)$ and $P'=(z',t')$, they are joined by a unique length minimizing geodesic, unless $z=z'$.

Let

$$\gamma_{P,\phi,\alpha}=L_P\gamma_{O,\phi,\alpha}.$$

The parameter ϕ is geometric in the following sense. The quantity $2\pi/|\phi|$ is the length of any sub-arc of $\gamma_{P,\phi,W}$ over which $\gamma_{P,\phi,W}$ is length minimizing. On the other hand, $sgn(\phi)$ is positive if and only if the t-coordinate increases with σ. In $\mathbf{H}$ the orientation of the t-axis is an intrinsic metric notion, unlike the Euclidean space. For instance, the Lie group of the isometries of $\mathbf{H}$ has two connected components, one containing the isometries which fix the direction of t-axis, the other containing the isometries which invert it. If $\gamma_{P,\phi,W}$ and $\gamma_{P',\phi',W'}$ have an arc in common, then $\phi=\pm\phi'$, while no such easy relation exists for the parameter W. To change the orientation of a geodesic, observe that

$$\gamma_{P,\phi,W}(\sigma)=\gamma_{P,-\phi,-W}(-\sigma).$$

Unlike the Euclidean case, a geodesic γ leaving O is not determined by its tangent vector at the origin, $\dot\gamma(0)=W=\cos(\alpha(W))X_O+\sin(\alpha(W))Y_O$. The extra parameter we need is $\kappa(\gamma)$.

The parameter $\kappa(\gamma)$, if $\phi = \kappa(\gamma) \neq 0$, is related to the dilation group as follows. $\delta_\lambda\left(\gamma_{O,\phi,W}\right)$ is a reparametrization of the geodesic $\gamma_{O,\phi/\lambda,W}$. That is, all geodesics γ leaving 0 and having fixed initial velocity $\dot\gamma(0) = v$ in $\mathscr{H}_O$ are dilated of each other. The case of the straight geodesics is the limiting one, corresponding to $\lambda \to 0$. In a precise sense, then, the set of geodesics at O is parametrized by the unit circle in $\mathscr{H}_O$ and by the dilation group, a feature of $\mathbf{H}$ with no Euclidean counterpart. This simple fact will be a recurrent theme in the following sections.

Let S be a smooth surface in $\mathbf{H}$. We need some geometric notions about S, some of which were studied in [2].

Definition 2.1. Let S be a closed subset of $\mathbf{H}$, $P \in S$. The *metric normal* to S at P is the set $\mathscr{N}_P S$ of the points $Q \in \mathbf{H}$ such that $d(Q,S) = d(Q,P)$.

Notice that, in general, the metric normal is not a geodesic arc. This can also be seen in the Euclidean case, considering the metric normal to a cone at its vertex. Nevertheless, if S is C^1 surface, then $\mathscr{N}_P S$ is a (possibly degenerate) geodesic arc. See [2] for the details, especially Sections 3 and 4.

We can now give the following definition of oriented metric normal.

Definition 2.2. Let S be a C^1 surface in the Euclidean sense in $\mathbf{H}$, which is the boundary of an open set Ω and of $\mathbf{H} - \overline{\Omega}$ and let $P \in S$. The *oriented metric normal to S at P*, $\mathscr{N}_P^+ S$, is the unique parametrization of $\mathscr{N}_P S$ such that $\delta_S(\mathscr{N}_P^+ S(\sigma), P) = \sigma$.

Definition 2.3. Let S be a smooth surface in $\mathbf{H}$ and let $P \in S$ be a non-characteristic point. The Euclidean tangent space to S at P is denoted by $T_P S$. The *direction tangent to S at P* is the 1-dimensional space $V_P S = T_P S \cap \mathscr{H}_P$. The *plane tangent to S at P*, $\Pi_P S$, is the Euclidean plane in $\mathbf{H}$, tangent to S at P in the Euclidean sense. The *direction normal to S* at P is $N_P S = \mathscr{H}_P \ominus V_P S$, where $\ominus$ is taken with respect to the sub-Riemannian metric.

The *exterior normal to* a C^1 surface S at P is

$$\nu_S(P) = \left(\mathscr{N}_P^+ S\right)'(0),$$

the vector tangent to $\mathscr{N}_P S$ at P.

Remark 2.1. The map $P \to \nu_S(P)$ from the non-characteristic surface S to the fiber $\mathscr{H}_P$ is known in literature as the *horizontal Gauss map*, see e.g. [16] and [28].

The *group tangent to S at P* [26] is the 2-dimensional vector space $G_P S = \mathscr{V}_P S \oplus \mathscr{T}$, where $\mathscr{T} = \{(0,0,t) : t \in \mathbf{R}\}$ is the center of $\mathbf{H}$ and $\mathscr{V}_P S$ is the one parameter subgroup of $\mathbf{H}$ generated by $V_P S$. One point we want to make in the present paper is that $G_P S$ does not seem to capture the complexity of the geodesics' set, while $T_P S$ does, in a precise sense.

Suppose that S is locally given by $g(x,y,t) = 0$. Let P be a point in S and let C be the characteristic point of $\Pi_P S$, the Euclidean plane in $\mathbf{H}$ which is tangent to

S at P. Then,

$$d(P,C)=\frac{|\nabla_{\mathbf{H}}g(P)|}{2|\partial_t g(P)|}. \tag{8}$$

The equation of $\mathscr{N}_P^+S$ can be written in terms of g's partial derivatives. $\mathscr{N}_P^+S = P\cdot\eta$ (left translation by P), where $\eta=(u,v,s)$ and

$$\eta(\sigma)=\begin{cases} x'(\sigma)=\frac{1}{4\partial_t g}\Big\{Yg(P)\left(1-\cos\left(\frac{4\partial_t g(P)\sigma}{|\nabla_{\mathbf{H}}g(P)|}\right)\right) \\ \qquad\qquad +Xg(P)\sin\left(\frac{4\partial_t g(P)\sigma}{|\nabla_{\mathbf{H}}g(P)|}\right)\Big\}, \\ y'(\sigma)=\frac{1}{4\partial_t g}\Big\{-Xg(P)\left(1-\cos\left(\frac{4\partial_t g(P)\sigma}{|\nabla_{\mathbf{H}}g(P)|}\right)\right) \\ \qquad\qquad +Yg(P)\sin\left(\frac{4\partial_t g(P)\sigma}{|\nabla_{\mathbf{H}}g(P)|}\right)\Big\}, \\ t'(\sigma)=\frac{|\nabla_{\mathbf{H}}g(P)|^2}{8(\partial_t g(P))^2}\left\{\frac{4\partial_t g(P)\sigma}{|\nabla_{\mathbf{H}}g(P)|}-\sin\left(\frac{4\partial_t g(P)\sigma}{|\nabla_{\mathbf{H}}g(P)|}\right)\right\}. \end{cases} \tag{9}$$

See p. 671 in [2]. When $g(z,t)=t-f(z)$, we have

$$\begin{cases} x'=\frac{1}{4}(Xf\sin\alpha+Yf(1-\cos\alpha)), \\ y'=\frac{1}{4}(Yf\sin\alpha-Xf(1-\cos\alpha)), \\ t'=\frac{|\nabla_{\mathbf{H}}f|^2}{8}(\alpha-\sin\alpha), \end{cases}$$

where $\alpha=\frac{4\tau}{|\nabla_{\mathbf{H}}f|}$.

The following facts are established by direct calculation.

Proposition 2.1. *Let S be a smooth surface in $\mathbf{H}$, implicitely defined by $g(x,y,t)=0$. Let $P\in S$ be non-characteristic. Then,*

$$\begin{aligned} V_PS&=\operatorname{span}\{Yg(P)\cdot X_P-Xg(P)\cdot Y_P\}, \\ N_PS&=\operatorname{span}\{Xg(P)\cdot X_P+Yg(P)\cdot Y_P\}=\operatorname{span}\{\nabla_{\mathbf{H}}g(P)\}. \end{aligned} \tag{10}$$

Proof. Let $U=Yg(P)\cdot X_P-Xg(P)\cdot Y_P$ and $W=Xg(P)\cdot X_P+Yg(P)\cdot Y_P$. Let $\cdot$ the Euclidean inner product and let ∇ the Euclidean gradient. Since $U\cdot\nabla g(P)=0$, then $U\in T_PS$. Since P is non-charachteristic, $U\neq 0\neq W$. The vectors U and W are orthogonal in $\mathscr{H}_P$, hence $\mathscr{H}_P=\operatorname{span}\{U,W\}$. □

We introduce an *exponential map* $F\colon (u,v,\sigma)\mapsto F(u,v,\sigma)$, defined from an open subset of $\mathbf{R}^2\times\mathbf{R}$ with values in $\mathbf{H}$,

$$F(u,v,\sigma)=\mathscr{N}^+_{(u,v,f(u,v))}S(\sigma). \tag{11}$$

We are going to use the coordinates

$$(x,y,t)=F(u,v,\sigma)=(u,v,f(u,v))\circ(x',y',t')$$

where $(x', y', t') = \gamma_{u,v}(\sigma)$ and $\gamma_{u,v}$ is the metric normal's left translate by P^{-1}. We have, then

$$\begin{cases} x_u = 1 + x'_u, \\ y_u = y'_u, \\ t_u = f_u(u, v) - 2y' + t'_u + 2(vx'_u - uy'_u). \end{cases}$$

See [2], p. 678, Lemma 6.5.

At p. 679 of [2] we find the following formulae, which we summarize in a vast lemma.

Lemma 2.1. *We have the following equalities.*

$$\begin{aligned} 4x'_u &= (Xf)_u \sin\alpha + (Yf)_u(1-\cos\alpha) + \alpha_u Xf\cos\alpha + \alpha_u Yf \sin\alpha \\ &= (Xf)_u \sin\alpha + (Yf)_u(1-\cos\alpha) \\ &\quad - \cos\alpha \frac{Xf}{2} \frac{4\tau}{\mid \nabla_{\mathbf{H}} f \mid^3} \big(2Xf(Xf)_u + 2Yf(Yf)_u\big) \\ &\quad - \sin\alpha \frac{Xf}{2} \frac{4\tau}{\mid \nabla_{\mathbf{H}} f \mid^3} \big(2Xf(Xf)_u + 2Yf(Yf)_u\big), \end{aligned}$$

and so $(x'_u)_{|\tau=0} = 0$*. Analogously*

$$\begin{aligned} 4x'_v &= (Xf)_v \sin\alpha + (Yf)_v(1-\cos\alpha) \\ &\quad - \Big(\cos\alpha Xf \frac{4\tau}{\mid \nabla_{\mathbf{H}} f \mid^3} + \sin\alpha Yf \frac{4\tau}{\mid \nabla_{\mathbf{H}} f \mid^3}\Big)\big(Xf(Xf)_u + Yf(Yf)_u\big), \end{aligned}$$

and $(x'_v)_{|\tau=0} = 0$*;*

$$4x'_\tau = Xf\cos\alpha \frac{4}{\mid \nabla_{\mathbf{H}} f \mid} + Yf \sin\alpha \frac{4}{\mid \nabla_{\mathbf{H}} f \mid},$$

and $(x'_\tau)_{|\tau=0} = \frac{Xf}{|\nabla_{\mathbf{H}} f|}$*;*

$$\begin{aligned} 4y'_u &= (Yf)_u \sin\alpha - (Xf)_u(1-\cos\alpha) \\ &\quad - \frac{4\tau}{\mid \nabla_{\mathbf{H}} f \mid^3}\big(\cos\alpha Yf - \sin\alpha Xf\big)\big(Xf(Xf)_u + Yf(Yf)_u\big), \end{aligned}$$

and $(y'_u)_{|\tau=0} = 0$*;*

$$\begin{aligned} 4y'_v &= (Yf)_v \sin\alpha - (Xf)_v(1-\cos\alpha) \\ &\quad - \frac{4\tau}{\mid \nabla_{\mathbf{H}} f \mid^3}\big(\cos\alpha Yf - \sin\alpha Xf\big)\big(Xf(Xf)_v + Yf(Yf)_v\big), \end{aligned}$$

and $(y'_v)_{|\tau=0} = 0$*;*

$$4y'_\tau = \frac{4\tau}{\mid \nabla_{\mathbf{H}} f \mid^3}\big(\cos\alpha Yf - \sin\alpha Xf\big),$$

and $(y'_\tau)_{|\tau=0} = \frac{Yf}{|\nabla_{\mathbf{H}} f|}$.

$$\begin{aligned} 8t'_u &= 2\big(Xf(Xf)_u + Yf(Yf)_u\big)(\alpha - \sin\alpha) \\ &\quad - \frac{4\tau}{|\nabla_{\mathbf{H}} f|}\big(Xf(Xf)_u + Yf(Yf)_u\big)(1-\cos\alpha), \end{aligned}$$

and $(t'_u)_{|\tau=0} = 0$;

$$\begin{aligned} 8t'_v &= 2\big(Xf(Xf)_v + Yf(Yf)_v\big)(\alpha - \sin\alpha) \\ &\quad - \frac{4\tau}{|\nabla_{\mathbf{H}} f|}\big(Xf(Xf)_v + Yf(Yf)_v\big)(1-\cos\alpha), \end{aligned}$$

and $(t'_v)_{|\tau=0} = 0$;

$$8t'_\tau = 4\,|\nabla_{\mathbf{H}} f|\,(1-\cos\alpha),$$

and

$$t'_\tau = 0.$$

3. The Hessian of the function measuring the distance from a smooth surface

Let g be a smooth function from $\mathbf{H}$ in $\mathbf{R}$. We define the *horizontal Hessian* $\mathrm{Hess}_{\mathbf{H}}\, g$ and the *symmetrized horizontal Hessian* $\mathrm{Hess}^*_{\mathbf{H}}\, g$ of g. See, e.g., [10], [20] for a discussion of second order differential operators in the Heisenberg group.

$$\mathrm{Hess}_{\mathbf{H}}\, g = \begin{pmatrix} XXg & YXg \\ XYg & YYg \end{pmatrix}. \tag{12}$$

Let $(XY)^*g = \frac{1}{2}(XYg + YXg)$.

$$\mathrm{Hess}^*_{\mathbf{H}}\, g = \begin{pmatrix} XXg & (XY)^*g \\ (XY)^*g & YYg \end{pmatrix}. \tag{13}$$

Next, we prove Theorem 1.1. For ease of calculation, we restate it in a less geometric form.

Theorem 3.1. *Let $S = \partial\Omega$ be a surface in $\mathbf{H}$, where Ω is locally defined by the inequality $t - f(x,y) < 0$. Let*

$$Q = XXf\ (Yf)^2 - 2(XY)^*f\ Xf\ Yf + YYf\ (Xf)^2.$$

Then, the horizontal Hessian of δ_S at $P \in S$, non-characteristic point, is

$$\begin{aligned} \mathrm{Hess}_{\mathbf{H}}\, \delta_S &= \begin{pmatrix} \frac{(Yf)^2}{|\nabla_{\mathbf{H}} f|^5} Q + 4\frac{Xf\ Yf}{|\nabla_{\mathbf{H}} f|^3} & -Yf\left[\frac{Xf}{|\nabla_{\mathbf{H}} f|^5} Q - 4\frac{Yf}{|\nabla_{\mathbf{H}} f|^3}\right] \\ -Xf\left[\frac{Yf}{|\nabla_{\mathbf{H}} f|^5} Q + 4\frac{Xf}{|\nabla_{\mathbf{H}} f|^3}\right] & \frac{(Xf)^2}{|\nabla_{\mathbf{H}} f|^5} Q - 4\frac{Xf\ Yf}{|\nabla_{\mathbf{H}} f|^3} \end{pmatrix} \\ &= v_P S \otimes v_P S \cdot (h_S(P) I + p_S(P) J). \end{aligned} \tag{14}$$

Hence, the symmetrized horizontal Hessian of the distance function δ_S, restricted to the surface S, is

$$-\operatorname{Hess}^*_{\mathbf{H}}\delta_S = \begin{pmatrix} -\frac{(Yf)^2}{|\nabla_{\mathbf{H}}f|^5}Q - 4\frac{Xf\ Yf}{|\nabla_{\mathbf{H}}f|^3} & \frac{Xf\ Yf}{|\nabla_{\mathbf{H}}f|^5}Q + 2\frac{(Xf)^2-(Yf)^2}{|\nabla_{\mathbf{H}}f|^3} \\ \frac{Xf\ Yf}{|\nabla_{\mathbf{H}}f|^5}Q + 2\frac{(Xf)^2-(Yf)^2}{|\nabla_{\mathbf{H}}f|^3} & -\frac{(Xf)^2}{|\nabla_{\mathbf{H}}f|^5}Q + 4\frac{Xf\ Yf}{|\nabla_{\mathbf{H}}f|^3} \end{pmatrix} \tag{15}$$

and, on S,

$$\Delta_{\mathbf{H}}\delta_S = \frac{Q}{|\nabla_{\mathbf{H}}f|^3} = \frac{XXf(Yf)^2 - 2(XY)^*fXfYf + YYf(Xf)^2}{|\nabla_{\mathbf{H}}f|^3}. \tag{16}$$

Remark 3.1. If S is smooth and C is a characteristic point of S, then $|\operatorname{Hess}_{\mathbf{H}} \delta_S(P)| = O(d(P,C)^{-1})$ as $P \to C$ in S. We will see in Section 5 that this estimate can fail if P goes to C in $\mathbf{H}$ without restrictions.

Elementary linear algebra applied to (14), or a direct calculation with (15), shows that the characteristic polynomial of $\operatorname{Hess}^*_{\mathbf{H}}\delta_S$ is

$$\mathscr{P}(\lambda) = \lambda^2 - h_S(P)\lambda - \frac{1}{4}p_S^2(P). \tag{17}$$

We could consider the roots $\lambda_{1,2}$ of $\mathscr{P}$, i.e., the eigenvalues of the symmetrized horizontal Hessian $\operatorname{Hess}^*_{\mathbf{H}}\delta_S$, to be the "horizontal principal curvatures" of S at P. It would be interesting to have a direct geometric interpretation for these two quantities.

The eigenvalues of $\operatorname{Hess}^*_{\mathbf{H}}\delta_S$ have opposite sign, unless $p_S(P) = 0$. On the other hand, if there is an open subset A of S s.t. $p_S(P) = 0$ whenever $P \in A$, then A must be a *vertical* subset of S, i.e., A is defined as $\{(x,y,t) : \varphi(x,y) = 0\}$. Following [10] (see also [20] and [11]), we say that a function k is *H-convex* if $\operatorname{Hess}^*_{\mathbf{H}} k$ is positive definite. As a consequence of the above, we have the following remark.

Remark 3.2. Let S be an orientable surface in $\mathbf{H}$, which is free of characteristic points, and let δ_S the signed distance from S associated to an orientation of S. Then, δ_S is H-convex or concave in a neighborhood of S if and only if S is a vertical set $\{(x,y,t) : \varphi(x,y) = 0\}$ and $\{(x,y) : \varphi(x,y) = 0\}$ is a curve in $\mathbf{R}^2$ which does not change the sign of its Euclidean curvature.

In particular the Carnot–Charathéodory distance in $\mathbf{H}$ *is not* H-convex, since the metric spheres are not vertical sets. This could be compared with the result in [10], where it is proved that the gauge distance in $\mathbf{H}$ is mildly convex.

In order to prove the theorem, we need some preliminary results. We denote by $\operatorname{Hess} k$ the Euclidean Hessian of a function k from $\mathbf{R}^3$ in $\mathbf{R}$. If $K = (k_1,\dots,k_m)$ maps $\mathbf{R}^3$ in $\mathbf{R}^m$, we write

$$\operatorname{Hess} K = (\operatorname{Hess} k_1,\dots,\operatorname{Hess} k_m).$$

Consider the equation, see (11),

$$(x,y,t) = F(u,v,\tau).$$

Let us denote by $\operatorname{Hess} x$, $\operatorname{Hess} y$, $\operatorname{Hess} t$ the Euclidean Hessian matrices of $x(u,v,\tau)$, $y(u,v,\tau)$, $t(u,v,\tau)$; let $\xi=(u,v,\tau)$ and let ξ_x, ξ_y, ξ_t be the columns of the Jacobian matrix of F^{-1} computed when $\tau=0$.

Let $a=\xi_x+2y\xi_t$ and $b=\xi_y-2x\xi_t$.

Lemma 3.1. *The entries of the horizontal Hessian of δ_S satisy the following relations.*

(i) *We have*

$$|\nabla_{\mathbf{H}} f|\cdot XX\delta_S=-\Big|F_u|F_v|\operatorname{Hess} F(a,a)\Big|. \tag{18}$$

Here, $\Big|c_1|c_2|c_3\Big|$ denotes the determinant of the 3×3 matrix having columns c_j, $j=1,2,3$.

(ii) *We have*

$$|\nabla_{\mathbf{H}} f|\cdot YY\delta_S=-\Big|F_u|F_v|\operatorname{Hess} F(b,b)\Big|. \tag{19}$$

(iii) *$XY\tau$ and $YX\tau$ satisfy*

$$\begin{cases} X\delta_S\cdot XX\delta_S+Y\delta_S\cdot XY\delta_S=0,\\ X\delta_S\cdot YX\delta_S+Y\delta_S\cdot YY\delta_S=0.\end{cases} \tag{20}$$

Proof. The Inverse Function Theorem applied to the equation $(x,y,t)=F(u,v,\tau)$ gives a set of equations for the second derivatives of $\tau=\delta_S$, w.r.t. x,y,t. Let p,q be any two variables, possibly equal, chosen among x,y,t. Then

$$\begin{cases} x_u u_{pq}+x_v v_{pq}+x_\tau\tau_{pq}=-\operatorname{Hess} x(\xi_p,\xi_q),\\ y_u u_{pq}+y_v v_{pq}+y_\tau\tau_{pq}=-\operatorname{Hess} y(\xi_p,\xi_q),\\ t_u u_{pq}+t_v v_{pq}+t_\tau\tau_{pq}=-\operatorname{Hess} t(\xi_p,\xi_q).\end{cases} \tag{21}$$

We solve the linear systems with respect to the second partial derivatives of τ,

$$\tau_{pq}=\frac{\det\begin{bmatrix} x_u, & x_v, & -\operatorname{Hess} x(\xi_p,\xi_q)\\ y_u, & y_v, & -\operatorname{Hess} y(\xi_p,\xi_q)\\ t_u, & t_v, & -\operatorname{Hess} t(\xi_p,\xi_q)\end{bmatrix}}{\det\begin{bmatrix} x_u, & x_v, & x_\tau\\ y_u, & y_v, & y_\tau\\ t_u, & t_v, & t_\tau.\end{bmatrix}} \tag{22}$$

From Lemma 2.1,

$$\det\left|\begin{matrix} x_u, & x_v, & x_\tau\\ y_u, & y_v, & y_\tau\\ t_u, & t_v, & t_\tau.\end{matrix}\right|_{\tau=0}=|\nabla_{\mathbf{H}} f|$$

Since $XX=\partial_{xx}+4y\partial_{xt}+4y^2\partial_{tt}$, since the determinant is linear with respect to each column and $\operatorname{Hess} F(U,V)$ is bilinear in U and V, then (22) implies that, when

$\tau = 0$,

$$
\begin{aligned}
|\nabla_{\mathbf{H}} f| \cdot XX\delta_S &= \Big| F_u | F_v | F_\tau \Big| \cdot XX\delta_S \\
&= \Big| F_u | F_v | - [\mathrm{Hess}\, F(\xi_x, \xi_x) + 4y\, \mathrm{Hess}\, F(\xi_x, \xi_t) + 4y^2\, \mathrm{Hess}\, F(\xi_t, \xi_t)] \Big| \\
&= -\Big| F_u | F_v | \,\mathrm{Hess}\, F(a, a) \Big|
\end{aligned}
$$

and this shows (i). (ii) can be proved in the same way, starting from the expression

$$YY = \partial_{yy} - 4x\partial_{yt} + 4x^2\partial_{tt}.$$

(iii) follows taking derivatives w.r.t. X and Y of the eikonal equation (2). □

Proof of Theorem 3.1. By (i) and (ii) in Lemma 3.1, the expression of $XX\delta_S$ and $YY\delta_S$ can be computed if we know Hess$F$ restricted to $\tau = 0$ and $J(F^{-1})(x, y, t)|_{\tau=0} = (\xi_x | \xi_y | \xi_t)$. By (iii), this allows us to compute $XY\delta_S$ and $YX\delta_S$ as well.

We apply the classical Implicit Function Theorem to

$$(x, y, t) = F(u, v, \tau).$$

Recall from Lemma 2.1 that the Jacobian of F,

$$
JF(u, v, 0) = J \begin{pmatrix} x & y & t \\ u & v & \tau \end{pmatrix}_{\tau=0} = \begin{pmatrix} 1 & 0 & \frac{Xf}{|\nabla_{\mathbf{H}} f|} \\ 0 & 1 & \frac{Yf}{|\nabla_{\mathbf{H}} f|} \\ f_u & f_v & \frac{2vXf - 2uYf}{|\nabla_{\mathbf{H}} f|} \end{pmatrix}, \tag{23}
$$

whose determinant is

$$\det JF(u, v, 0) = |\, \nabla_{\mathbf{H}} f \,|\, .$$

Its inverse matrix is

$$
\begin{aligned}
J(F^{-1})(x, y, t)|_{\tau=0} &= J \begin{pmatrix} u & v & \tau \\ x & y & t \end{pmatrix}_{\tau=0} \\
&= \begin{pmatrix} 1 + \frac{Xff_u}{|\nabla_{\mathbf{H}} f|^2} & \frac{Xff_v}{|\nabla_{\mathbf{H}} f|^2} & -\frac{Xf}{|\nabla_{\mathbf{H}} f|2} \\ \frac{Yff_u}{|\nabla_{\mathbf{H}} f|^2} & 1 + \frac{Yff_v}{|\nabla_{\mathbf{H}} f|^2} & -\frac{Yf}{|\nabla_{\mathbf{H}} f|^2} \\ -\frac{f_u}{|\nabla_{\mathbf{H}} f|} & -\frac{f_v}{|\nabla_{\mathbf{H}} f|} & \frac{1}{|\nabla_{\mathbf{H}} f|}. \end{pmatrix} = (\xi_x | \xi_y | \xi_t).
\end{aligned} \tag{24}
$$

If $g(x, y, t) = t - f(x, y)$, we write $XXf = XXg$, $XYf = XYg$, and so on. The following relations will be used below.

$$
\begin{aligned}
(Xf)_u &= XXf, \quad (Yf)_u = XYf, \\
(Xf)_v &= YXf, \quad (Yf)_v = YYf.
\end{aligned} \tag{25}
$$

Recalling the derivatives of x, y, t computed in the proof of Lemma 2.1, we can compute

$$\begin{aligned}x_{uu} &= \frac{1}{4}(Xf)_{uu}\sin\alpha + \frac{1}{4}(Yf)_{uu}(1-\cos\alpha) + (\frac{1}{4}(Xf)_u\cos\alpha + \frac{1}{4}(Yf)_u\sin\alpha)\alpha_u \\ &+ \frac{1}{4}\alpha_u\sin\alpha\frac{Xf}{2}\frac{4\tau}{\mid\nabla_{\mathbf{H}}f\mid^3}\big(2Xf(Xf)_u + 2Yf(Yf)_u\big) \\ &- \frac{1}{4}\cos\alpha\big(\frac{Xf}{2}\frac{4\tau}{\mid\nabla_{\mathbf{H}}f\mid^3}\big(2Xf(Xf)_u + 2Yf(Yf)_u\big)\big)_u \\ &- \frac{1}{4}\cos\alpha\frac{Xf}{2}\frac{4\tau}{\mid\nabla_{\mathbf{H}}f\mid^3}\big(2Xf(Xf)_u + 2Yf(Yf)_u\big) \\ &- \frac{1}{4}\sin\alpha(\frac{Xf}{2}\frac{4\tau}{\mid\nabla_{\mathbf{H}}f\mid^3}\big(2Xf(Xf)_u + 2Yf(Yf)_u\big)\big)_u,\end{aligned}$$

hence

$$(x_{uu})_{|\tau=0} = 0.$$

$$\begin{aligned}x_{vu} &= \frac{1}{4}(Xf)_{vu}\sin\alpha + \frac{1}{4}(Yf)_{uv}(1-\cos\alpha) + (\frac{1}{4}(Xf)_u\cos\alpha + \frac{1}{4}(Yf)_u\sin\alpha)\alpha_v \\ &+ \frac{1}{4}\alpha_v\sin\alpha\frac{Xf}{2}\frac{4\tau}{\mid\nabla_{\mathbf{H}}f\mid^3}\big(2Xf(Xf)_u + 2Yf(Yf)_u\big) \\ &- \cos\alpha\frac{1}{4}(\frac{Xf}{2}\frac{4\tau}{\mid\nabla_{\mathbf{H}}f\mid^3}\big(2Xf(Xf)_u + 2Yf(Yf)_u\big))_v \\ &- \alpha_v\frac{1}{4}\cos\alpha\frac{Xf}{2}\frac{4\tau}{\mid\nabla_{\mathbf{H}}f\mid^3}\big(2Xf(Xf)_u + 2Yf(Yf)_u\big) \\ &- \frac{1}{4}\sin\alpha(\frac{Xf}{2}\frac{4\tau}{\mid\nabla_{\mathbf{H}}f\mid^3}\big(2Xf(Xf)_u + 2Yf(Yf)_u\big))_v,\end{aligned}$$

$$(x_{vu})_{|\tau=0} = 0.$$

$$\begin{aligned}x_{vv} &= \frac{1}{4}(Xf)_{vv}\sin\alpha + \frac{1}{4}(Yf)_{vv}(1-\cos\alpha) \\ &+ \frac{1}{4}\big((Xf)_v\cos\alpha + \frac{1}{4}(Yf)_v\sin\alpha\big) \\ &- (\frac{1}{4}\big(\cos\alpha Xf\frac{4\tau}{\mid\nabla_{\mathbf{H}}f\mid^3} + \frac{1}{4}\sin\alpha Yf\frac{4\tau}{\mid\nabla_{\mathbf{H}}f\mid^3})\big(Xf(Xf)_u + Yf(Yf)_u\big))_v,\end{aligned}$$

$$(x_{vv})_{|\tau=0} = 0.$$

$$\begin{aligned}x_{u\tau} &= \frac{1}{4}(Xf\frac{4}{\mid\nabla_{\mathbf{H}}f\mid})_u\cos\alpha - \frac{1}{4}Xf\frac{4}{\mid\nabla_{\mathbf{H}}f\mid}\sin\alpha\alpha_u \\ &+ \frac{1}{4}(Yf\frac{4}{\mid\nabla_{\mathbf{H}}f\mid})_u\sin\alpha + \frac{1}{4}Yf\alpha_u\cos\alpha\frac{4}{\mid\nabla_{\mathbf{H}}f\mid},\end{aligned}$$

$$(x_{u\tau})_{|\tau=0} = \frac{Yf}{|\nabla_{\mathbf{H}} f|^3}((Xf)_u Yf - Xf(Yf)_u)$$
$$= \frac{Yf}{|\nabla_{\mathbf{H}} f|^3}((XXf)Yf - Xf(XYf)).$$

$$x_{\tau\tau} = (Xf\frac{1}{|\nabla_{\mathbf{H}} f|})_\tau \cos\alpha - Xf\alpha_\tau \sin\alpha \frac{1}{|\nabla_{\mathbf{H}} f|}$$
$$+ (Yf\frac{1}{|\nabla_{\mathbf{H}} f|})_\tau \sin\alpha + Yf\alpha_\tau \cos\alpha \frac{1}{|\nabla_{\mathbf{H}} f|}$$
$$= -Xf\alpha_\tau \sin\alpha \frac{1}{|\nabla_{\mathbf{H}} f|} + Yf\alpha_\tau \cos\alpha \frac{1}{|\nabla_{\mathbf{H}} f|},$$

and

$$(x_{\tau\tau})_{|\tau=0} = \frac{4Yf}{|\nabla_{\mathbf{H}} f|^2}.$$

$$x_{v\tau} = (Xf\frac{1}{|\nabla_{\mathbf{H}} f|})_v \cos\alpha - Xf\alpha_v \sin\alpha \frac{1}{|\nabla_{\mathbf{H}} f|}$$
$$+ (Yf\frac{1}{|\nabla_{\mathbf{H}} f|})_v \sin\alpha + Yf\alpha_v \cos\alpha \frac{1}{|\nabla_{\mathbf{H}} f|}$$
$$= (Xf\frac{1}{|\nabla_{\mathbf{H}} f|})_v \cos\alpha + (Yf\frac{1}{|\nabla_{\mathbf{H}} f|})_v \sin\alpha,$$

and

$$(x_{v\tau})_{|\tau=0} = \frac{Yf}{|\nabla_{\mathbf{H}} f|^3}((Xf)_v Yf - Xf(Yf)_v) = \frac{Yf}{|\nabla_{\mathbf{H}} f|^3}((YXf)Yf - Xf(YYf)).$$

Moreover,

$$t_{uu} = f_{uu} + (\frac{1}{4}\big(Xf(Xf)_u + Yf(Yf)_u\big)_u(\alpha - \sin\alpha)$$
$$+ (\frac{1}{4}\big(Xf(Xf)_u - Yf(Yf)_u\big)^2(1-\cos\alpha)\frac{\tau}{|\nabla_{\mathbf{H}} f|^3}$$
$$- \frac{\tau}{2|\nabla_{\mathbf{H}} f|}\big(Xf(Xf)_u$$
$$+ Yf(Yf)_u\big)(1-\cos\alpha))_u$$
$$+ \left(\frac{\tau}{2|\nabla_{\mathbf{H}} f|}\big(Xf(Xf)_u + Yf(Yf)_u\big)\right)_u \cdot (1-\cos\alpha))$$
$$+ 2vx'_{uu} - 2uy'_{uu} - 2y'_u - 2y'_u,$$

so that

$$(t_{uu})_{\tau=0} = (f_{uu} + 2vx'_{uu} - 2uy'_{uu} - 2y'_u - 2y'_u)_{\tau=0} = f_{uu};$$

analogously

$$(t_{vu})_{\tau=0} = f_{vu},$$

and

$$(t_{vv})_{\tau=0} = f_{vv},$$

and

$$(t_{u\tau})_{\tau=0} = 2vx'_{u\tau} - 2uy'_{u\tau} - 2y'_\tau,$$

and

$$(t_{v\tau})_{\tau=0} = 2vx'_{v\tau} - 2uy'_{v\tau} + 2x'_\tau,$$

and

$$(t_{\tau\tau})_{\tau=0} = 2vx'_{\tau\tau} - 2uy'_{\tau\tau}.$$

Hence

$$\text{Hess}\, x_{\tau=0} = \frac{Yf}{\mid \nabla_{\mathbf{H}} f \mid^3} \cdot \begin{pmatrix} 0 & 0 & (XXf)Yf - Xf(XYf) \\ 0 & 0 & (YXf)Yf - Xf(YYf) \\ (XXf)Yf - Xf(XYf) & (YXf)Yf - Xf(YYf) & 4 \mid \nabla_{\mathbf{H}} f \mid \end{pmatrix}.$$

Arguing in the same way we get

$$\text{Hess}\, y_{\tau=0} = \frac{Xf}{\mid \nabla_{\mathbf{H}} f \mid^3} \cdot \begin{pmatrix} 0 & 0 & (XYf)Xf - Yf(XXf) \\ 0 & 0 & (YYf)Xf - Yf(YXf) \\ (XYf)Xf - Yf(XXf) & (YYf)Xf - Yf(YXf) & -4 \mid \nabla_{\mathbf{H}} f \mid \end{pmatrix}.$$

$$\text{Hess}\, t_{\tau=0} = \begin{pmatrix} f_{uu} & f_{vu} & 2vx'_{u\tau} - 2uy'_{u\tau} - 2y'_\tau \\ f_{vu} & f_{vv} & 2vx'_{v\tau} - 2uy'_{v\tau} + 2x'_\tau \\ 2vx'_{u\tau} - 2uy'_{u\tau} - 2y'_\tau & 2vx'_{v\tau} - 2uy'_{v\tau} + 2x'_\tau & 2vx'_{\tau\tau} - 2uy'_{\tau\tau} \end{pmatrix}.$$

The horizontal Hessian matrix can be written now directly.

A direct calculation shows that

$$a = \left(\frac{(Yf)^2}{|\nabla_{\mathbf{H}} f|^2}, -\frac{Xf \cdot Yf}{|\nabla_{\mathbf{H}} f|^2}, \frac{Xf}{|\nabla_{\mathbf{H}} f|} \right) \tag{26}$$

(viewed as a column vector) and

$$b = \left(-\frac{Xf \cdot Yf}{|\nabla_{\mathbf{H}} f|^2}, \frac{(Xf)^2}{|\nabla_{\mathbf{H}} f|^2}, \frac{Yf}{|\nabla_{\mathbf{H}} f|} \right). \tag{27}$$

We then compute

$$\text{Hess}\,|_{\tau=0} x(a,a) = 2\frac{(Yf)^2 Xf}{|\nabla_{\mathbf{H}} f|^6} Q + 4\frac{(Xf)^2 Yf}{|\nabla_{\mathbf{H}} f|^4},$$

$$\mathrm{Hess}\,|_{\tau=0}y(a,a) = -2\frac{(Xf)^2 Yf}{|\nabla_{\mathbf{H}} f|^6}Q - 4\frac{(Xf)^3}{|\nabla_{\mathbf{H}} f|^4},$$

$$\begin{aligned}\mathrm{Hess}\,|_{\tau=0}t(a,a) &= -2\frac{(Yf)^2}{|\nabla_{\mathbf{H}} f|^4}Q - 4\frac{Xf\cdot Yf}{|\nabla_{\mathbf{H}} f|^2}\\ &\quad + 8\frac{(Xf)^2}{|\nabla_{\mathbf{H}} f|^4}(vYf + uXf) + 4\frac{Xf\cdot Yf}{|\nabla_{\mathbf{H}} f|^6}(vYf+uXf)Q\end{aligned}$$

Recall now that

$$\begin{aligned}-|\nabla_{\mathbf{H}} f|\cdot XX\delta_S &= \begin{vmatrix} 1 & 0 & \mathrm{Hess}\,x|_{\tau=0}(a,a)\\ 0 & 1 & \mathrm{Hess}\,y|_{\tau=0}(a,a)\\ 2v - Xf & -2u - Yf & \mathrm{Hess}\,t|_{\tau=0}(a,a)\end{vmatrix}\\ &= -\frac{(Yf)^2}{|\nabla_{\mathbf{H}} f|^4}Q - 4\frac{Xf\cdot Yf}{|\nabla_{\mathbf{H}} f|^2}.\end{aligned}$$

Analogous calculations yield the value of $YY\delta_S$.

In order to compute the mixed derivatives, we use (20).

Let us recall now that

$$Q = XXf\ (Yf)^2 - 2(XY)^* f\ Xf\ Yf + YYf\ (Xf)^2.$$

Then, the horizontal Hessian of δ_S is that given in (14),

$$-\,\mathrm{Hess}_{\mathbf{H}}\,\delta_S = \begin{pmatrix} -\frac{(Yf)^2}{|\nabla_{\mathbf{H}} f|^5}Q - 4\frac{Xf\ Yf}{|\nabla_{\mathbf{H}} f|^3} & Yf\left[\frac{Xf}{|\nabla_{\mathbf{H}} f|^5}Q - 4\frac{Yf}{|\nabla_{\mathbf{H}} f|^3}\right]\\ Xf\left[\frac{Yf}{|\nabla_{\mathbf{H}} f|^5}Q + 4\frac{Xf}{|\nabla_{\mathbf{H}} f|^3}\right] & -\frac{(Xf)^2}{|\nabla_{\mathbf{H}} f|^5}Q + 4\frac{Xf\ Yf}{|\nabla_{\mathbf{H}} f|^3}\end{pmatrix},$$

i.e.,

$$\begin{aligned}-\,\mathrm{Hess}_{\mathbf{H}}\,\delta_S &= \frac{Q}{|\nabla_{\mathbf{H}} f|^3}\begin{pmatrix} -\frac{(Yf)^2}{|\nabla_{\mathbf{H}} f|^2} & \frac{XfYf}{|\nabla_{\mathbf{H}} f|^2}\\ \frac{XfYf}{|\nabla_{\mathbf{H}} f|^2} & -\frac{(Xf)^2}{|\nabla_{\mathbf{H}} f|^2}\end{pmatrix}\\ &\quad + \frac{4}{|\nabla_{\mathbf{H}} f|}\begin{pmatrix} -\frac{XfYf}{|\nabla_{\mathbf{H}} f|^2} & -\frac{(Yf)^2}{|\nabla_{\mathbf{H}} f|^2}\\ \frac{(Xf)^2}{|\nabla_{\mathbf{H}} f|^2} & \frac{XfYf}{|\nabla_{\mathbf{H}} f|^2}.\end{pmatrix}.\end{aligned} \tag{28}$$

Now, let

$$M = \begin{pmatrix} \frac{(Yf)^2}{|\nabla_{\mathbf{H}} f|^2} & -\frac{XfYf}{|\nabla_{\mathbf{H}} f|^2}\\ -\frac{XfYf}{|\nabla_{\mathbf{H}} f|^2} & \frac{(Xf)^2}{|\nabla_{\mathbf{H}} f|^2}\end{pmatrix}.$$

Then, if we set $p = \frac{4}{|\nabla_{\mathbf{H}} f|}$ and

$$J = \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix},$$

we have

$$\mathrm{Hess}_{\mathbf{H}}\,\delta_S = kM + pMJ = M(kI + pJ) = M\begin{pmatrix} k & p\\ -p & k\end{pmatrix}. \tag{29}$$

Let us remark that, keeping in mind (8) and that $g = t - f(x,y)$, we have

$$d(P,C) = \frac{|\nabla_{\mathbf{H}} f|}{2},$$

so that

$$p = \frac{2}{d(P,C)},$$

that represents indeed a curvature. Moreover, let us define

$$\cos\theta = \frac{Xf}{|\nabla_{\mathbf{H}} f|},$$

and

$$\sin\theta = \frac{Yf}{|\nabla_{\mathbf{H}} f|}.$$

Then

$$M = \begin{pmatrix} \sin^2\theta & -\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \cos^2\theta \end{pmatrix} = v_P S \otimes v_P S. \qquad \square$$

4. Mean curvature

In this section, we make some considerations about the notion of mean curvature for a surface in $\mathbf{H}$ (see, e.g., [16]).

Definition 4.1. Let $S = \partial\Omega$ be a smooth surface in $\mathbf{H}$, with Ω locally defined by the inequality $g(x,y,t) < 0$. The *mean curvature* of S at $P \in S$ is

$$h_S(P) = X\left(\frac{Xg(P)}{|\nabla_{\mathbf{H}} g(P)|}\right) + Y\left(\frac{Yg(P)}{|\nabla_{\mathbf{H}} g(P)|}\right). \tag{30}$$

The curvature is well defined for non characteristic points only.

Here are some characterizations of mean curvature. The first is in terms of the distance function and it is an immediate consequence of Theorem 3.1.

Corollary 4.1. *Let $S = \partial\Omega$ be the boundary of an open, C^3 set in $\mathbf{H}$. Then, for any non characteristic point P in S,*

$$\Delta_{\mathbf{H}} \delta_S(P) = h_S(P). \tag{31}$$

Monti pointed out to us that the same result can be obtained by an explicit calculation, which does not rely on the knowledge of the distance's Hessian. In [16], it is shown that the mean curvature is the trace of a certain matrix $\mathscr{M}$, which is not the Hessian of the distance function.

The second is in terms of horizontal curves on S. First, we state an immediate consequence of the existence theorem for O.D.E.'s.

Lemma 4.1. *Let S be a smooth surface in $\mathbf{H}$ and let $P \in S$ be non-characteristic. Then, modulo restrictions and reparametrizations, there exists a unique horizontal* *curve Γ through P on S. Suppose that S is defined by $g(x,y,t)=0$. A parametrization of the curve is the solution of the O.D.E.'s system*

$$\begin{cases} \dot\Gamma(\tau) = Yg(\Gamma(\tau))\cdot X_{(\Gamma(\tau))} - Xg(\Gamma(\tau))\cdot Y_{(\Gamma(\tau))}, \\ \Gamma(0)=P. \end{cases} \tag{32}$$

Before we state the theorem relating the horizontal curves Γ and the mean curvature of S, we fix some notation. We give $\mathscr{H} = \operatorname{span}\{X,Y\}$ the orientation for which the oriented angle from X to Y has amplitude $\pi/2$. Let $P \in S = \partial\Omega$. The *positive horizontal direction* V_PS tangent to S is V_P^+S, the unit vector in the direction V_PS such that the angle from N_PS to V_P^+S has amplitude $-\pi/2$, i.e.,

$$V_P^+S = dR_{-\pi/2}N_PS,$$

$dR_{-\pi/2}$ denoting the differential of $R_{-\pi/2}$. If the curve Γ of Lemma 4.1 is oriented in such a way that $\dot\Gamma$ has the same orientation of V_P^+S, we say that Γ is a *positively oriented horizontal curve on* S. Let p be the projection of $\mathbf{H}$ onto the z-plane, $p(z,t)=z$ and let $\gamma = p(\Gamma)$ be the projection of Γ. γ inherits the orientation of Γ. The *Euclidean curvature k of the oriented curve* γ at $\gamma(t_0)$ is defined (locally in t_0) by

$$\frac{d}{dt}|_{t=t_0}\,(n(\gamma(t))) = k\cdot\dot\gamma(t_0),$$

where

$$n(\gamma(t)) = \frac{dR_{\pi/2}\dot\gamma(t)}{|dR_{\pi/2}\dot\gamma(t)|}$$

is the *unit normal* to the oriented curve γ at $\gamma(t)$, which is locally well defined even if γ has self intersections.

In a picture, γ has positive curvature if it borders a convex region on its right hand side.

Theorem 4.1. *Let $S=\partial\Omega$ be a smooth surface and let $P\in S$ be non-characteristic. Let Γ be the oriented horizontal curve on S through P. Then, $h_S(P)$ is the curvature of the oriented curve $p(\Gamma)$ in $\mathbf{R}^2$ at $p(P)$.*

Proof. Suppose first that T_PS is a non-vertical plane, that is, that in a neighborhood of P, S is locally given by $S=\{f(x,y)=t\}$. The horizontal curve Γ on S through P satisfies equation (32), hence $\Gamma=(x,y,t)$ is a solution of

$$\begin{cases} \dot x = Yf(\Gamma), \\ \dot y = -Xf(\Gamma), \\ \dot t = 2yYf(\Gamma) + 2xXf(\Gamma). \end{cases}$$

Since Xf and Yf are independent of t, we solve for x and y, independently of t, and we find the differential equation for $\gamma = p(\Gamma)$,

$$\begin{cases} \dot x = Yf(\Gamma), \\ \dot y = -Xf(\Gamma). \end{cases}$$

Suppose that $Yf \neq 0$ at P (otherwise, we can let $Xf \neq 0$ at P, since P is non characteristic). Parametrizing γ as $y = \gamma(x)$, we find

$$y' = -\frac{Xf}{Yf}.$$

The curvature of γ in the Euclidean plane is defined as

$$\begin{aligned} k &= -\frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2}}\right) = -\frac{y''}{(1+(y')^2)^{3/2}} \\ &= \frac{(Yf)^3}{|\nabla_{\mathbf{H}} f|^3}\frac{d}{dx}\left(\frac{Xf}{Yf}\right) \\ &= \frac{Yf}{|\nabla_{\mathbf{H}} f|^3}\left[(\partial_x Xf + \partial_y Xf y')Yf - (\partial_x Yf + \partial_y Yf y')Xf\right] \\ &= \frac{XXf\ (Yf)^2 - 2(XY)^* f\ Xf\ Yf + YYf\ (Xf)^2}{|\nabla_{\mathbf{H}} f|^3} \\ &= \Delta_{\mathbf{H}} \delta_S = h_S(P) \end{aligned}$$

by Corollary 4.1.

The case when $T_P S$ is a vertical plane can be dealt with similarly. □

In the Heisenberg group, vectors belonging to different sections of the tangent bundle can be compared by means of left translations. Under the identification $\mathbf{H} \equiv \mathbf{R}^3$, hence of the tangent space at $P \in \mathbf{H}$ with $\mathbf{R}^3$, the differential of the left translation by $Q = (a, b, c)$, $\mathscr{L}_Q\colon P \mapsto Q \circ P$, is the matrix

$$d\mathscr{L}_Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2b & -2a & 1 \end{bmatrix}$$

which is independent of the point where the differential is calculated. By definition of Carnot–Carathéodory distance, $d\mathscr{L}_Q$ maps $\mathscr{H}_P$ in $\mathscr{H}_{Q\circ P}$, isometrically.

Fix $P \in \mathbf{H}$ and let $\phi\colon I \to \mathbf{H}$ be a smooth, horizontal curve, such that $\phi(0) = P$. Let W be a section of the horizontal bundle along ϕ: $W(s) \in \mathscr{H}_{\phi(s)}$ for $s \in I$, and W is a smooth map. We define the *derivative* of W along ϕ at P as

$$\frac{d_\phi^{\mathbf{H}} W(s)}{ds}_{|s=0} := \lim_{s\to 0} \frac{d\mathscr{L}_{(\phi(0))\circ(-\phi(s))} W(s) - W(0)}{s}.$$

We give the notion of derivative of a vector field with respect to a vector in the Heisenberg group.

Definition 4.2. Let W be a section of the horizontal bundle $\mathscr{H}$. Let $V \in \mathscr{H}_P$ and $\phi\colon I \to \mathbf{H}$ be any horizontal curve such that $\phi(0) = P$ and $\phi'(0) = V$. We set

$$\nabla_V^{\mathbf{H}} W(P) := \frac{d^{\mathbf{H}}(W \circ \phi)(s)}{ds}_{|s=0}.$$

Lemma 4.2. *Let $W = aX + bY$ be a section of the horizontal bundle $\mathscr{H}$ and $\phi\colon I \to \mathbf{H}$ be any horizontal curve such that $\phi(0) = P$ and $\phi'(0) = V$. Then*

$$\nabla^{\mathbf{H}}_V W(P) = Va(P)X(P) + Vb(P)Y(P)$$

where Va and Vb are the derivatives of the functions a and b in the direction V, respectively.

Proof. Indeed

$$\begin{aligned}
&\frac{d\mathscr{L}_{(\phi(0))\circ(-\phi(s))}W(s) - W(0)}{s}\\
&= \frac{d\mathscr{L}_{(\phi(0))\circ(-\phi(s))}\big(a(\phi(s)), b(\phi(s), 2\phi_2(s)a(\phi(s)) - 2\phi_1(s)b(\phi(s))\big) - W(\phi(0))}{s}\\
&= \frac{\big(a(\phi(s)), b(\phi(s), 2\phi_2(0)a(\phi(s)) - 2\phi_1(0)b(\phi(s))\big) - W(\phi(0))}{s}\\
&= \frac{a(\phi(s)) - a(\phi(0))}{s}X(\phi(0)) + \frac{b(\phi(s)) - b(\phi(0))}{s}Y(\phi(0))
\end{aligned}$$

and as a consequence

$$\lim_{s\to 0}\frac{d\mathscr{L}_{(\phi(0))\circ(-\phi(s))}W(\phi(s)) - W(\phi(0))}{s} = Va(P)X(P) + Vb(P)Y(P). \qquad \square$$

Let $N_S(P)$ the unit vector normal to $S = \partial\Omega$ at P, in the Heisenberg sense, and pointing outside Ω. If $g(x, y, t) = 0$ locally represents S, then

$$\nu_S(P) = \left(\frac{Xg(P)}{|\nabla_{\mathbf{H}}g(P)|}X_P + \frac{Yg(P)}{|\nabla_{\mathbf{H}}g(P)|}Y_P\right)$$

depending on the orientation of S.

Theorem 4.2. *The linear operator $M_PS\colon V \mapsto \nabla^{\mathbf{H}}_V \nu_S(P)$ maps V_PS into V_PS. M_PS acts on V_PS as multiplication times $h_S(P)$.*

Modulo a sign, the map M_PS is the Weingarten map for surfaces in the Heisenberg group.

Proof. If $W = \nu_S(P)$, we have

$$\nabla^{\mathbf{H}}_V N_S(P) \in V_PS.$$

In fact, differentiating $|\nu_S(P) \circ \phi(s)|^2 = 1$ with respect to s, $\nu_S(P) = \nu_1(P)X_P + \nu_2(P)Y_P$, we obtain

$$0 = \nu_1(\phi(s))\frac{d}{ds}\nu_1(\phi(s)) + \nu_2(\phi(s))\frac{d}{ds}\nu_2(\phi(s)).$$

For $s = 0$, this relation becomes

$$0 = \langle \nu_S(P), \nabla^{\mathbf{H}}_V \nu_S(P)\rangle_{\mathbf{H}}.$$

Since $\nu_S(P) = \frac{Xg(P)}{|\nabla_{\mathbf{H}} g(P)|} X_P + \frac{Yg(P)}{|\nabla_{\mathbf{H}} g(P)|} Y_P$ we get, for $V = \alpha_1 X(P) + \alpha_2 Y(P) \in V_P S$,

$$\nabla_V^{\mathbf{H}} \nu_S(P) = \begin{bmatrix} X(\frac{Xg(P)}{|\nabla_{\mathbf{H}} g(P)|}) & Y(\frac{Xg(P)}{|\nabla_{\mathbf{H}} g(P)|}) \\ X(\frac{Yg(P)}{|\nabla_{\mathbf{H}} g(P)|}) & Y(\frac{Yg(P)}{|\nabla_{\mathbf{H}} g(P)|}) \end{bmatrix} \cdot \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}. \tag{33}$$

The linear map $V \mapsto \nabla_V^{\mathbf{H}} N_S(P)$ is defined on the one-dimensional linear space $V_P S$, hence it acts as multiplication by a constant k. Testing (33) on $(\alpha_1, \alpha_2) = (-Yg(P), Xg(P)) = \dot{\Gamma}(t)$, if $\Gamma(t) = P$, i.e., on a basis of $V_P S$, we see that

$$k = \frac{Q}{|\nabla_{\mathbf{H}} g|^3}$$

is the mean curvature of S at P. □

Observe that multiplication by the matrix in (33) acts on $V_P S$ as multiplication by the matrix' trace.

5. Examples

In this section, we see two basic examples of surfaces in $\mathbf{H}$. First, we consider the case of the plane $\{t = 0\}$. Then, we consider the case of the ball with respect to the Heisenberg distance. In particular, we explicitely compute the Hessian of the function "distance from a point".

Calculations are easier if we exploit the symmetries of $\mathbf{H}$.

Lemma 5.1. *Let g be a smooth function on $\mathbf{H}$, which is homogeneous of degree m with respect to the dilation group at the origin. Let W_j be horizontal left invariant vector fields on $\mathbf{H}$, $j = 1, \ldots, n$. Then, $W_1 \ldots W_n g$ is homogeneous of degree $m - n$,*

$$W_1 \ldots W_n g(\delta_\lambda P) = \lambda^{m-n} g(P). \tag{34}$$

In particular, if S is a smooth surface in $\mathbf{H}$ and σ denotes the distance function from S, then σ, $\nabla_{\mathbf{H}}\sigma$ and $\operatorname{Hess}_{\mathbf{H}} \sigma$ are homogeneous of degreee 1, 0 and -1, respectively.

Recall that R_θ denotes both the rotation by θ around the t-axis in $\mathbf{H}$ and the rotation by θ in span$\{X, Y\}$. We use the same symbol for the matrix

$$R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Lemma 5.2. *Let f be a smooth function on $\mathbf{H}$, which is invariant under the rotation group at the origin,*

$$f(r_\theta P) = f(P)$$

for θ in $\mathbf{R}$. Then, $\nabla_{\mathbf{H}}$ and $\operatorname{Hess}_{\mathbf{H}}$ commute with rotations. Namely,

$$\nabla_{\mathbf{H}} f(R_\theta P) = R_\theta \nabla_{\mathbf{H}} f(P) \tag{35}$$

and

$$\operatorname{Hess}_{\mathbf{H}} f(R_\theta P) = R_\theta \operatorname{Hess}_{\mathbf{H}} f(P). \tag{36}$$

In (36), the multiplication on the right hand side is a multiplication of 2×2 matrices.

5.1. Nonvertical planes. Let Π be the plane $\{t = 0\}$. Observe that all nonvertical planes are isometric to Π. In [2] we computed the metric normal to Π at $P = (x, y, 0) = (z, 0)$,

$$\gamma_P(\sigma) = (u(\sigma),\ v(\sigma),\ s(\sigma)) = \begin{cases} \frac{x}{2}\left(1 + \cos\left(\frac{2\sigma}{|z|}\right)\right) + \frac{y}{2}\sin\left(\frac{2\sigma}{|z|}\right), \\ \frac{y}{2}\left(1 + \cos\left(\frac{2\sigma}{|z|}\right)\right) - \frac{x}{2}\sin\left(\frac{2\sigma}{|z|}\right), \\ \frac{|z|^2}{2}\left(\frac{2\sigma}{|z|} + \sin\left(\frac{2\sigma}{|z|}\right)\right). \end{cases}$$

The geodesic γ_P realizes the distance from Π when $|\sigma| \le \frac{\pi|z|}{2}$. By Lemma 5.1, δ_Π is homogeneous of degree 1, $\nabla_{\mathbf{H}}\delta_\Pi$ is homogeneous of degree 0 and $\mathrm{Hess}_{\mathbf{H}}\,\delta_\Pi$ is homogeneous of degree -1. Since δ_Π is invariant under rotations around the t-axis, $\mathrm{Hess}_{\mathbf{H}}\,\delta_\Pi$ commutes with the same group of rotations.

Let $\beta = \frac{\sigma}{|z|} \in (-\pi/2, \pi/2)$, $R = |z| \ge 0$ and $\alpha = \arg z - \frac{\sigma}{|z|} \in [0, 2\pi)$, for a suitable choice of $\arg z$. Here, we use on $\mathbf{H}$ the coordinates

$$(u, v, s) = (R\cos\beta\cos\alpha, R\cos\beta\sin\alpha, R^2(\beta + 1/2\sin(2\beta))).$$

Hence

$$J\begin{pmatrix} u, & v, & s \\ R, & \alpha, & \beta \end{pmatrix} = \begin{bmatrix} \cos\alpha\cos\beta, & -R\cos\beta\sin\alpha, & -R\sin\beta\cos\alpha \\ \cos\beta\sin\alpha, & R\cos\beta\cos\alpha, & -R\sin\beta\sin\alpha \\ 2R(\beta + \frac{1}{2}\sin(2\beta)), & 0, & 2R^2\cos^2\beta \end{bmatrix}$$

and

$$\det J\begin{pmatrix} u, & v, & s \\ R, & \alpha, & \beta \end{pmatrix} = 2R^3\cos\beta(\cos\beta + \beta\sin\beta).$$

As a consequence

$$\det J\begin{pmatrix} R, & \alpha, & \beta \\ u, & v, & s \end{pmatrix} = \begin{bmatrix} \frac{\cos^2\beta\cos\alpha}{\cos\beta+\beta\sin\beta}, & \frac{\cos^2\beta\sin\alpha}{\cos\beta+\beta\sin\beta}, & \frac{\sin\beta}{2R(\cos\beta+\beta\sin\beta)} \\ -\frac{\sin\alpha}{R\cos\beta}, & \frac{\cos\alpha}{R\cos\beta}, & 0 \\ -\frac{\cos\alpha}{R}\frac{\beta+\sin\beta\cos\beta}{\cos\beta+\beta\sin\beta}, & -\frac{\sin\alpha}{R}\frac{\beta+\sin\beta\cos\beta}{\cos\beta+\beta\sin\beta}, & \frac{\cos\beta}{2R^2(\cos\beta+\beta\sin\beta)} \end{bmatrix}$$

A long, but elementary calculation shows now that, in the new coordinates,

$$X = \frac{\cos\beta\cdot\cos(\beta-\alpha)}{\cos\beta+\beta\sin\beta}\partial_R - \frac{\sin\alpha}{R\cos\beta}\partial_\alpha - \frac{\beta\cos\alpha+\cos\beta\cdot\sin(\beta-\alpha)}{R(\cos\beta+\beta\sin\beta)}\partial_\beta \tag{37}$$

and

$$Y = -\frac{\cos\beta\cdot\sin(\beta-\alpha)}{\cos\beta+\beta\sin\beta}\partial_R + \frac{\cos\alpha}{R\cos\beta}\partial_\alpha - \frac{\beta\sin\alpha+\cos\beta\cdot\cos(\beta-\alpha)}{R(\cos\beta+\beta\sin\beta)}\partial_\beta. \tag{38}$$

Hence,

Lemma 5.3. *Let* Π *be the plane* $\{t = 0\}$. *Then*

$$X\delta_\Pi = -\sin(\beta-\alpha)$$

and

$$Y\delta_\Pi = -\cos(\beta-\alpha).$$

Proof. Recalling (37) and (38) we get $X\delta_\Pi = -\sin(\beta-\alpha)$ and $Y\delta_\Pi = -\cos(\beta-\alpha)$. □

Remark 5.1. After the computation of $X\delta_\Pi$, we can deduce the value of $Y\delta_\Pi$ keeping in mind (35).

Theorem 5.1. *Let* Π *be the plane* $\{t=0\}$*. Then*

$$\mathrm{Hess}_{\mathbf{H}}\,\delta_\Pi = \frac{1}{R\cos\beta(\cos\beta+\beta\sin\beta)}\begin{bmatrix}\cos(\alpha-\beta)A, & \cos(\alpha-\beta)B\\ \sin(\alpha-\beta)A, & \sin(\alpha-\beta)B\end{bmatrix},$$

where

$$A = -(\sin\alpha\cos\beta+\beta\cos(\alpha+\beta)+\cos^2\beta\sin(\alpha-\beta))$$

and

$$B = (\cos\alpha\cos\beta+\beta\sin(\alpha+\beta)+\cos^2\beta\cos(\alpha-\beta)).$$

Moreover, for $P\in\Pi$, $P\neq O$, *then*

$$-(\Delta\delta_\Pi)_{\sigma=0} = 0,$$

i.e., the mean curvature is 0.

Proof. Recalling Lemma 5.3 we get by straightforward computation:

$$\begin{aligned}
XX\delta_\Pi &= -\frac{\sin\alpha}{R\cos\beta}\cos(\alpha-\beta)+\frac{\beta\cos\alpha-\cos\beta\sin(\alpha-\beta)}{R(\cos\beta+\beta\sin\beta)}\cos(\alpha-\beta)\\
&= \frac{\cos(\alpha-\beta)}{R\cos\beta}A,\\
YX\delta_\Pi &= \frac{\cos\alpha}{R\cos\beta}\cos(\alpha-\beta)+\frac{\beta\sin\alpha+\cos\beta\cos(\alpha-\beta)}{R(\cos\beta+\beta\sin\beta)}\cos(\alpha-\beta)\\
&= \frac{\cos(\alpha-\beta)}{R\cos\beta}B,\\
XY\delta_\Pi &= -\frac{\sin\alpha}{R\cos\beta}\sin(\alpha-\beta)+\frac{\beta\cos\alpha-\cos\beta\sin(\alpha-\beta)}{R(\cos\beta+\beta\sin\beta)}\sin(\alpha-\beta)\\
&= \frac{\sin(\alpha-\beta)}{R\cos\beta}A,\\
YY\delta_\Pi &\frac{\cos\alpha}{R\cos\beta}\sin(\alpha-\beta)+\frac{\beta\sin\alpha+\cos\beta\cos(\beta-\alpha)}{R(\cos\beta+\beta\sin\beta)}\sin(\alpha-\beta)=\frac{\sin(\alpha-\beta)}{R\cos\beta}B.
\end{aligned}$$

In particular, we have

$$\mathrm{Trace}(\mathrm{Hess}_{\mathbf{H}}\,\delta_\Pi)_{\sigma=0} = \mathrm{Trace}\left(\frac{1}{R}\begin{bmatrix}-2\cos\alpha\sin\alpha, & 2\cos^2\alpha\\ -2\sin^2\alpha, & 2\cos\alpha\sin\alpha\end{bmatrix}\right) = 0.$$

As a consequence the mean curvature of $\Pi-O$ is 0. This fact could also be proved as follows. The punctured plane $\Pi-O$ is the union of straight half line leaving O, hence its Heisenberg curvature at a point P is the Euclidean curvature of a straight line by Theorem 4.1. □

Remark 5.2. Notice that $\mathrm{Hess}_{\mathbf{H}}\,\delta_\Pi$ is singular as $R \to 0$ and as $\beta \to \pm\pi/2$. For instance, if $\alpha = 0$, then $YY\delta_\Pi$ diverges as $\beta \to \pm\pi/2$ and $R \to 0$. Indeed

$$(\mathrm{Hess}_{\mathbf{H}}\,\delta_\Pi)_{\alpha=0} = \frac{1}{R\cos\beta(\cos\beta + \beta\sin\beta)}$$
$$\cdot \begin{bmatrix} -(\beta - \cos\beta\sin\beta)\cos^2\beta, & (\cos\beta + \beta\sin\beta + \cos^3\beta)\cos\beta \\ (\beta - \cos\beta\sin\beta)\sin\beta\cos\beta, & -(\cos\beta + \beta\sin\beta + \cos^3\beta)\sin\beta \end{bmatrix}$$

and

$$(YY\delta_\Pi)_{\alpha=0} = -\frac{\tan\beta}{R}\frac{\cos\beta + \beta\sin\beta + \cos^3\beta}{\cos\beta + \beta\sin\beta} = -\frac{\tan\beta}{R}(1 + \frac{\cos^3\beta}{\cos\beta + \beta\sin\beta}).$$

The singularity as $R \to 0$ is expected, by Theorem 3.1. The singularity as $\beta \to \pm\pi/2$ shows that the estimate at the end of Theorem 3.1 can not be extended outside the surface.

5.2. The Carnot–Charathéodory sphere and the Hessian of the distance function. Here we consider $\Sigma = \partial B(0,1)$, the sphere of unitary radius with respect to the Carnot–Charathéodory metric. The poles of Σ are $NP = (0,0,1/\pi)$ and $SP = (0,0,1/\pi)$. The calculation of the horizontal Hessian of δ_Σ will be accomplished in several steps. First, we find the equations of the horizontal curves lying on Σ. Via Theorem 4.1, this allows us to compute the mean curvature of Σ at every point. By Theorem 3.1, (14) and (16), and Corollary 4.1, the second order terms of the formula relating $\mathrm{Hess}_{\mathbf{H}}$ to the equation of Σ are all contained in the expression for the mean curvature of Σ. Hence, in order to compute $\mathrm{Hess}_{\mathbf{H}}\,\delta_\Sigma$ we are left with the calculation of some first order derivatives.

Finally, we note that the calculation of $\mathrm{Hess}_{\mathbf{H}}\,\delta_\Sigma$ immediately leads to an expression for the horizontal Hessian of the Carnot–Charathéodory distance in $\mathbf{H}$, $\mathrm{Hess}_{\mathbf{H}}\,d(O,\cdot)$.

We can parametrize Σ in several useful ways. Let $(z,t) \in \Sigma$. Then, there are $\phi \in [0,2\pi)$ and $u \in [-2\pi, 2\pi]$ such that

$$\begin{cases} z(u) = e^{i\phi} 2\frac{\sin(u/2)}{u}, \\ t(u) = 2\frac{u - \sin(u)}{u^2}. \end{cases} \tag{39}$$

The parametric representation in (39) can be easily deduced by the geodesic equations (6).

Below we consider the upper half of Σ, corresponding to $u \geq 0$, so that $B(O,1)$ is below the surface.

We see, now, how Σ is ruled by horizontal curves. Incidentally, this provides a differently useful parametrization of Σ.

Proposition 5.1. *The surface* $\Sigma - \{NP, SP\}$ *is the disjoint union of horizontal curves* $\gamma_\theta(u) = (z(u), t(u))$ $(|u| < 2\pi)$,

$$\begin{cases} z(u) = e^{i(\psi(u/2)+\theta)} 2\frac{\sin(u/2)}{u}, \\ t(u) = 2\frac{u-\sin(u)}{u^2}, \end{cases} \tag{40}$$

where ψ *satisfies the conditions* $\psi(0) = 0$ *and*

$$\psi'(p) = \cot^2 p - \frac{\cot p}{p}. \tag{41}$$

These are the only horizontal curves on Σ.

In (41), observe that ψ' is well defined for $p = 0$ as well.

Proof. First, note that, by (39), any curve parametrized as in (40) lies on Σ. Let now γ be the curve in (40) corresponding to $\theta = 0$ (by rotation invariance, $R_\theta\gamma = \gamma_\theta$ is horizontal if and only if γ is). The curve $\gamma = (x, y, t)$ is horizontal if and only if

$$\dot{t}(u) = 2\dot{x}(u)y(u) - 2x(u)\dot{y}(u). \tag{42}$$

Here,

$$\begin{cases} x(u) = \cos(\psi(u/2))\frac{\sin(u/2)}{u/2}, \\ y(u) = \sin(\psi(u/2))\frac{\sin(u/2)}{u/2}. \end{cases}$$

Let $p = u/2$ We make (42) into an explicit O.D.E. A calculation shows that

$$2\dot{x}(2p)y(2p) - 2x(2p)\dot{y}(2p) = -\psi'(p)\frac{\sin^2(p)}{p^2}$$

and that

$$\dot{t}(2p) = \cos(p)\frac{\sin(p) - p\cos(p)}{p^3}.$$

Equation (41) follows immediately. □

It can be proved that $\psi(p) \to \infty$ as $p \to \pi$, i.e., that the projection of $p(\gamma_\theta)$ onto the plane $\{t = 0\}$ spiralizes infinitely many times around the origin. It can also be shown that the Heisenberg length of γ_θ, i.e., the Euclidean length of $p(\gamma_\theta)$, is infinite.

Next, we compute the mean curvature of Σ.

Proposition 5.2. *The curvature of* Σ *at* $P = (z, t)$, *where*

$$\begin{cases} |z| = 2\frac{\sin(u/2)}{u}, \\ t = 2\frac{u-\sin(u)}{u^2} \end{cases}$$

is the number

$$h_\Sigma(P) = \frac{u\cos u - \sin u}{\frac{u}{2}\cos\left(\frac{u}{2}\right) - \sin\left(\frac{u}{2}\right)} \frac{u}{4\sin\left(\frac{u}{2}\right)}. \tag{43}$$

Proof. By Theorem 4.1, we only need to apply the formula for the computation of the mean curvature of a curve in parametric form (see, e.g., [17]) to the curves described in Proposition 5.1. □

By Proposition 5.2, in order to compute the Hessian of δ_Σ, we are left with the calculation of a number of *first order* derivatives of f.

Lemma 5.4. *Let $t = f(x, y)$ be the equation of the upper half of Σ, let (z, t) be as in (39) and $u = 2v$. Then,*

$$\begin{cases} Xf(z,t) = 2\cos\phi\frac{\cos(v)}{v} + 2\sin\phi\frac{\sin(v)}{v} = \frac{2}{v}\cos(\phi - v), \\ Yf(z,t) = 2\sin\phi\frac{\cos(v)}{v} - 2\cos\phi\frac{\sin(v)}{v} = \frac{2}{v}\sin(\phi - v). \end{cases} \tag{44}$$

Proof. Let $t = f(x, y)$. By the chain rule,

$$J\begin{pmatrix} t & \\ x & y \end{pmatrix} = J\begin{pmatrix} t & \\ u & \phi \end{pmatrix} \cdot J\begin{pmatrix} x & y \\ u & \phi \end{pmatrix}^{-1}. \tag{45}$$

By (39),

$$J\begin{pmatrix} t & \\ u & \phi \end{pmatrix} = \left(\frac{\cos(u/2)}{(u/2)^3}\left[\sin(u/2) - u/2\cos(u/2)\right], 0\right) = \left(-\frac{\cos(u/2)}{u/2}h(u/2), 0\right)$$

where

$$h(s) = \frac{s\cos(s) - \sin(s)}{s^2}$$

and

$$J\begin{pmatrix} x & y \\ u & \phi \end{pmatrix} = \begin{bmatrix} \frac{1}{2}h(u/2)\cos\phi & -\frac{\sin(u/2)}{u/2}\sin\phi \\ \frac{1}{2}h(u/2)\sin\phi & \frac{\sin(u/2)}{u/2}\cos\phi \end{bmatrix}$$

so that

$$J\begin{pmatrix} x & y \\ u & \phi \end{pmatrix}^{-1} = \frac{1}{\frac{1}{2}h(u/2)\frac{\sin(u/2)}{u/2}}\begin{bmatrix} \frac{\sin(u/2)}{u/2}\cos\phi & \frac{\sin(u/2)}{u/2}\sin\phi \\ -\frac{1}{2}h(u/2)\sin\phi & \frac{1}{2}h(u/2)\cos\phi \end{bmatrix}.$$

Using (45), we obtain

$$J\begin{pmatrix} t & \\ x & y \end{pmatrix} = -\left(\frac{2\cos\phi}{u}, \frac{2\sin\phi}{u}\right)$$

and the conclusion of the lemma follows immediately. □

We have now all the ingredients for the calculation of $\mathrm{Hess}_{\mathbf{H}}\,\delta_\Sigma$.

Theorem 5.2. *Let $P = (z, t)$ be the point of the upper half of Σ,*

$$\begin{cases} z(u) = 2\frac{\sin(u/2)}{u}, \\ t(u) = 2\frac{u - \sin(u)}{u^2}. \end{cases} \tag{46}$$

Then,

$$\mathrm{Hess}_{\mathbf{H}}\,\delta_\Sigma(P) = v_P\Sigma \otimes v_P\Sigma \cdot (h_\Sigma(P)I + p_\Sigma(P)J), \tag{47}$$

where

$$v_P\Sigma = \sin(\phi - v)\cdot X_P - \cos(\phi - v)\cdot Y_P,$$

$$h_\Sigma(P) = \frac{u\cos u - \sin u}{\frac{u}{2}\cos\left(\frac{u}{2}\right) - \sin\left(\frac{u}{2}\right)}\frac{u}{4\sin\left(\frac{u}{2}\right)},$$

and

$$p_\Sigma(P) = u.$$

Proof. The theorem follows immediately from Theorem 1.1, Proposition 5.2, Lemma 5.4, (4) and the definition of $v_P\Sigma$. □

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