Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 33, 2008, 111–119

NEW BOUNDS FOR A_∞ WEIGHTS

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Abstract. Two new constants $\tilde{G}_1(u)$ and $\tilde{A}_\infty(u)$ are studied for weights $u\colon \mathbf{R}^n \to [0,\infty)$, which are simultaneously finite exactly for A_∞ weights. The special case $v = h'$, $w = (h^{-1})'$ where $h\colon \mathbf{R} \to \mathbf{R}$ is an increasing homeomorphism induces the identity $\tilde{A}_\infty(w) = \tilde{G}_1(v)$. Other identities are established for such constants, when different measures are involved.

1. Introduction

A locally integrable weight $w\colon \mathbf{R}^n \to [0,\infty)$ belongs to the A_∞ class iff there exist constants $0 < \alpha \leqslant 1 \leqslant K$ so that

$$\frac{|F|}{|J|} \leqslant K \left(\frac{\int_F w\,dx}{\int_J w\,dx} \right)^\alpha \tag{1.1}$$

for each cube $J \subset \mathbf{R}^n$ with sides parallel to the coordinate axes and for each measurable set $F \subset J$(see [M], [CF]). In [H] it was proved that w belongs to A_∞ iff the A_∞-constant $A_\infty(w)$ of w is finite, i.e., iff

$$A_\infty(w) = \sup_J ⨍_J w\,dx \cdot \exp\left(⨍_J \log \frac{1}{w}\,dx \right) < \infty \tag{1.2}$$

where the supremum runs over all such cubes $J \subset \mathbf{R}^n$ and $⨍_J w = \frac{1}{|J|}\int_J w\,dx$ denotes the integral mean of w on J.

A somewhat "dual" definition is the following. A locally integrable weight $v\colon \mathbf{R}^n \to [0,\infty)$ belongs to the G_1 class iff there exist constants $0 < \beta \leqslant 1 \leqslant H$ so that

$$\frac{\int_E v\,dx}{\int_I v\,dx} \leqslant H \left(\frac{|E|}{|I|} \right)^\beta \tag{1.3}$$

for each cube $I \subset \mathbf{R}^n$ with sides parallel to the coordinate axes and for each measurable set $E \subset I$.

In [MS] it was proved that v belongs to G_1 iff the G_1-constant $G_1(v)$ of v is finite, i.e., iff

$$G_1(v) = \sup_I \exp\left(⨍_I \frac{v}{v_I} \log \frac{v}{v_I}\,dx \right) < \infty \tag{1.4}$$

2000 Mathematics Subject Classification: Primary 26D15, 42B25, 46E30.
Key words: Muckenhoupt weights, reverse Hölder inequalities.

where the supremum runs over all cubes $I \subset \mathbf{R}^n$ with sides parallel to the coordinate axes and v_I denotes $v_I = \fint_I v\,dx = \frac{1}{|I|}\int_I v\,dx$.

It goes back to Fefferman [F] that actually the A_∞ class and the G_1 class coincide

$$A_\infty = G_1 \tag{1.5}$$

or, in other words that

$$A_\infty(u) < \infty \quad \text{iff} \quad G_1(u) < \infty$$

for any weight $u\colon \mathbf{R}^n \to [0,\infty)$. A closer transition from A_∞-constants and G_1-constants can be achieved via suitable specifications of the weights (see [JN2], [MS], [C]).

Here we shall focus on weights $w, v\colon \mathbf{R} \to [0,\infty)$ of special structure:

$$v = h', \quad w = (h^{-1})' \tag{1.6}$$

where $h\colon \mathbf{R} \to \mathbf{R}$ is an increasing homeomorphism locally absolutely continuous with its inverse. For such a peculiar situation a number of results hold true. Typically we quote the following duality relation (see [C], [J], [JN1], [JN2])

$$A_\infty(w) = G_1(v).$$

Following [G] first we associate here other natural constants to a generic weight u (see section 2) $\tilde{A}_\infty(u)$,$\tilde{G}_1(u)$ which are simultaneously finite as $A_\infty(u)$ and $G_1(u)$ and then we prove corresponding duality identity

$$\tilde{A}_\infty(w) = \tilde{G}_1(v). \tag{1.7}$$

Further natural transitions are possible, allowing more general measures on $\mathbf{R}^n$ in the definition of $\tilde{A}_\infty$-constants and $\tilde{G}_1$-constants (see [MS] and Section 3).

2. Definitions and notations

For any A_∞ weight w satisfying (1.1) for certain constants $0 < \alpha \leqslant 1 \leqslant K$, let us define (see [G])

$$\tilde{A}_\infty(w) = \inf\left\{\frac{K}{\alpha} : 0 < \alpha \leqslant 1 \leqslant K \text{ and (1.1) holds}\right\}. \tag{2.1}$$

Similarly, for any G_1 weight v satisfying (1.3) for certain constants $0 < \beta \leqslant 1 \leqslant H$, let us define

$$\tilde{G}_1(v) = \inf\left\{\frac{H}{\beta} : 0 < \beta \leqslant 1 \leqslant H \text{ and (1.3) holds}\right\}. \tag{2.2}$$

We emphasize explicitly that a weight u belongs to the A_∞ class of Muckenhoupt if and only if $\tilde{A}_\infty(u)$ is finite or, equivalently, if and only if $\tilde{G}_1(u)$ is finite. It is immediate to check that $\tilde{A}_\infty(w) \geqslant 1$ and $\tilde{G}_1(v) \geqslant 1$. Let us prove the following

Proposition 2.1. *Let w be an A_∞ weight such that $\tilde{A}_\infty(w) = 1$, then w is constant almost everywhere.*

Proof. By the definition (2.1) of $\tilde{A}_\infty(w)$ it follows that there exist two sequences (α_j) and (K_j) such that the following conditions hold:

$$\begin{cases} 0 < \alpha_j \leqslant 1 \leqslant K_j < \infty, \\ 1 \leqslant \dfrac{K_j}{\alpha_j} < 1 + \dfrac{1}{j}, \end{cases} \tag{2.3}$$

and for $j \in \mathbf{N}$

$$\frac{|F|}{|J|} \leqslant K_j \left(\frac{\int_F w\,dx}{\int_J w\,dx} \right)^{\alpha_j} \tag{2.4}$$

for every cube J and for every measurable subset $F \subset J$. We can assume, up to a subsequence, that (α_j) converges to some $0 \leqslant \alpha_0 \leqslant 1$ and obviously also (K_j) converges to the same limit α_0. This implies $\alpha_0 = 1$. Passing to the limit as $j \to \infty$ in (2.4), we obtain

$$\frac{|F|}{|J|} \leqslant \frac{\int_F w\,dx}{\int_J w\,dx} \tag{2.5}$$

for any cube J and for any measurable set $F \subset J$. Hence, for any cube J and for almost every $x_0 \in J$

$$\fint_J w\,dx \leqslant w(x_0)$$

which immediately implies that w is a constant function. □

Remark 2.2. With a similar proof one shows that if v is an A_∞ weight such that $\tilde{G}_1(v) = 1$, then v is constant.

After these preliminaries let us introduce the A_p-classes of Muckenhoupt and the G_q-classes of Gehring ($1 < p < \infty$; $1 < q < \infty$).

A weight w belongs to the A_p-class iff

$$A_p(w) = \sup_J \fint_J w\,dx \left(\fint_J w^{-\frac{1}{p-1}}\,dx \right)^{p-1} < \infty \tag{2.6}$$

where the supremum runs over all cubes $J \subset \mathbf{R}^n$.

A weight v belongs to the G_q-class iff

$$G_q(v) = \sup_I \left[\frac{\left(\fint_I v^q\,dx \right)^{1/q}}{\fint_I v\,dx} \right]^{q'} < \infty \tag{2.7}$$

where $q' = \frac{q}{q-1}$, and the supremum runs over all cubes $I \subset \mathbf{R}^n$.

It is well known that [M]

$$A_\infty = \bigcup_{p>1} A_p = \bigcup_{q>1} G_q = G_1 \tag{2.8}$$

The following two results are very useful to illustrate the properties of A_p and G_q weights.

Theorem 2.3. ([W]) *A locally integrable weight w is in A_p, $p > 1$ if and only if there exists $1 < p_1 < p$ such that for every interval J*

$$\left(\frac{|F|}{|J|}\right)^{p_1} \leqslant A_{p_1}(w)\frac{\int_F w\,dx}{\int_J w\,dx}$$

for every measurable subset F of J.

Theorem 2.4. ([Mi]) *A locally integrable weight v is in G_q, $q > 1$, if and only if there exists $q_1 > q$ such that, for every interval I*

$$\left(\frac{\int_E v\,dx}{\int_I v\,dx}\right)^{q_1'} \leqslant G_{q_1}(v)\frac{|E|}{|I|}$$

where $q_1' = q_1/(q_1 - 1)$, for every measurable subset E of I.

Let us notice here that, by mean of the equality

$$A_p(w) = G_q(v) \quad \left(\frac{1}{p} + \frac{1}{q} = 1\right)$$

for v and w as in (1.6), for the one dimensional Dirichlet problem

$$u \to \int_a^b |u'|^r\,dx \quad (r > 1)$$

the quasiminimizing property of the inverse function of a quasiminimizer was proved in [MS] in optimal terms.

3. Main results

Suppose h is a homeomorphism from $\mathbf{R}$ onto itself which we assume increasing and such that h, h^{-1} are locally absolutely continuous with $h'(x) > 0$ a.e. and consider the weights on $\mathbf{R}$

$$v = h', \quad w = (h^{-1})'. \tag{3.1}$$

The regularity of one of the two weights relies on the other one's, as it follows from the following.

Let us consider the constants $\tilde{A}_\infty(w)$ and $\tilde{G}_1(v)$ defined by (2.1) and (2.2) respectively:

Theorem 3.1. *If h is an increasing homeomorphism from $\mathbf{R}$ into itself and v, w are the weights defined in (3.1), then*

$$\tilde{G}_1(v) = \tilde{A}_\infty(w). \tag{3.2}$$

Proof. Let $0 < \alpha \leqslant 1 \leqslant K$ be fixed constants. Then, for any interval $I \subset \mathbf{R}$, the inequality

$$\frac{\int_E v\,dx}{\int_I v\,dx} \leqslant K\left(\frac{|E|}{|I|}\right)^\alpha \tag{3.3}$$

holds for any measurable set $E \subset I$ iff, for any interval $J \subset \mathbf{R}$ the inequality

$$\frac{|F|}{|J|} \leqslant K \left(\frac{\int_F w\,dx}{\int_J w\,dx} \right)^\alpha \tag{3.4}$$

holds true for any measurable set $F \subset J$.

It is easy to check that inequality (3.3) is equivalent to

$$\frac{|h(E)|}{|h(I)|} \leqslant K \left(\frac{|E|}{|I|} \right)^\alpha \tag{3.5}$$

where $|T|$ denotes the Lebesgue measure of the measurable set $T \subset \mathbf{R}$. Similarly, inequality (3.4) is equivalent to

$$\frac{|F|}{|J|} \leqslant K \left(\frac{|h^{-1}(F)|}{|h^{-1}(J)|} \right)^\alpha \tag{3.6}$$

Hence the result follows because (3.5) holds for arbitrary measurable subset E of the arbitrary interval $I \subset \mathbf{R}$ if and only if (3.6) holds for arbitrary measurable subset F of the arbitrary interval $J \subset \mathbf{R}$. □

In order to exploit further natural transitions from condition (1.1) to condition (1.3), it will be convenient to reverse the role of the couples of measures $(w(x)\,dx, dx)$ and $(dx, v(x)\,dx)$. To this aim let us consider the more general situation of two nonnegative doubling measures $d\mu$ and $d\nu$ on $\mathbf{R}^n$ which are mutually absolutely continuous.

We will say that a weight w belongs to $A_{\infty,d\nu}$, iff there exist constants $0 < \alpha \leqslant 1 \leqslant K < \infty$ so that

$$\frac{\nu(F)}{\nu(J)} \leqslant K \left(\frac{\int_F w\,d\nu}{\int_J w\,d\nu} \right)^\alpha \tag{3.7}$$

for each cube $J \subset \mathbf{R}^n$ and for each measurable set $F \subset J$. Then (3.7) reduces to the inequality (1.1) if $d\nu(x) = dx$. Similarly, the $\tilde{G}_{1,d\mu}$ condition for the weight v involves inequalities of the type

$$\frac{\int_E v\,d\mu}{\int_I v\,d\mu} \leqslant H \left(\frac{\mu(E)}{\mu(I)} \right)^\beta \tag{3.8}$$

with $0 < \beta \leqslant 1 \leqslant H < \infty$ for any cube I with sides parallel to the axes and E measurable subset of I.

Now we define for such weights w and $v \in L^1_{\text{loc}}(\mathbf{R}^n)$

$$\tilde{A}_{\infty,d\nu}(w) = \inf \left\{ \frac{K}{\alpha} : 0 < \alpha \leqslant 1 \leqslant K \text{ and (3.7) holds} \right\} \tag{3.9}$$

and

$$\tilde{G}_{1,d\mu}(v) = \inf \left\{ \frac{H}{\beta} : 0 < \beta \leqslant 1 \leqslant H \text{ and (3.8) holds} \right\}, \tag{3.10}$$

respectively. We have the following:

Theorem 3.2. *If $d\mu$ and $d\nu$ are mutually absolutely continuous, then*

$$\tilde{G}_{1,d\mu}\left(\frac{d\nu}{d\mu}\right) = \tilde{A}_{\infty,d\nu}\left(\frac{d\mu}{d\nu}\right). \tag{3.11}$$

Proof. Let $0 < \alpha \leqslant 1 \leqslant K$ be fixed constants. Let us show that, for any cube $I \subset \mathbf{R}^n$, the inequality

$$\frac{\int_E \frac{d\nu}{d\mu}\, d\mu}{\int_I \frac{d\nu}{d\mu}\, d\mu} \leqslant K\left(\frac{\mu(E)}{\mu(I)}\right)^{\alpha} \tag{3.12}$$

holds for any measurable subset $E \subset I$ iff, for any cube $J \subset \mathbf{R}^n$ the inequality

$$\frac{\nu(F)}{\nu(J)} \leqslant K\left(\frac{\int_F \frac{d\mu}{d\nu}\, d\nu}{\int_J \frac{d\mu}{d\nu}\, d\nu}\right)^{\alpha} \tag{3.13}$$

holds true for any measurable set $F \subset J$. In fact, by the definition of Radon–Nykodym derivative $\frac{d\nu}{d\mu}$ of $d\nu$ with respect to $d\mu$, it is immediate to check that (3.12) is equivalent to

$$\frac{\nu(E)}{\nu(I)} \leqslant K\left(\frac{\mu(E)}{\mu(I)}\right)^{\alpha}. \tag{3.14}$$

Similarly, inequality (3.13) is equivalent to

$$\frac{\nu(F)}{\nu(J)} \leqslant K\left(\frac{\mu(F)}{\mu(J)}\right)^{\alpha}. \tag{3.15}$$

Hence the result holds because (3.14) holds for arbitrary measurable subset E of the arbitrary cube $I \subset \mathbf{R}^n$ if and only if (3.15) holds for arbitrary measurable subset F of the arbitrary cube $J \subset \mathbf{R}^n$. □

4. A_p and G_q bounds

Let us assume that the weight $w\colon \mathbf{R}^n \to [0,\infty)$ belongs to A_p, $p > 1$. Then it is easy to check that for any cube J and for any measurable set $F \subset J$

$$\frac{|F|}{|J|} \leqslant A_p(w)^{1/p}\left(\frac{\int_F w\,dx}{\int_J w\,dx}\right)^{1/p}. \tag{4.1}$$

In fact, Hölder inequality for $f = \chi_F$, the characteristic function of F, implies

$$\left(\frac{|F|}{|J|}\right)^p = \left(\fint_J f\,dx\right)^p \leqslant \fint_J fw\,dx \left(\fint_J w^{-\frac{1}{p-1}}\,dx\right)^{p-1} \leqslant A_p(w)\frac{\int_F w\,dx}{\int_J w\,dx}.$$

We note now that Theorem 2.3 in Section 2 focuses on an alternative description of A_p which reveals a self-improvement of exponents in (4.1).

Hence, if $A_p(w) < \infty$, then

$$\tilde{A}_\infty(w) \leqslant \inf\left\{rA_r(w)^{\frac{1}{r}} : p_1 < r \leqslant p\right\} \tag{4.2}$$

where $p_1 > 1$ is as in Theorem 2.3.

Analogous observation can be derived in the G_q-case. So, if the weight $v\colon \mathbf{R}^n \to [0,\infty)$ belongs to G_q, $q>1$ then, for any cube $I\subset \mathbf{R}^n$ and for any measurable set $E\subset I$

$$\frac{\int_E v\,dx}{\int_I v\,dx} \leqslant G_q(v)^{1/q'}\left(\frac{|E|}{|I|}\right)^{1/q'} \tag{4.3}$$

with $q' = q/(q-1)$.

Taking into account Theorem 2.4, which is an alternative description of G_q and corresponds to a self-improvement of exponents in (4.3), we conclude that, if $G_q(v)<\infty$, then

$$\tilde{G}_1(v) \leqslant \inf\left\{ r' G_r(v)^{1/r'} : q \leqslant r < q_1 \right\} \tag{4.4}$$

where $q_1>1$ is as in Theorem 2.4.

In case of dimension $n=1$ optimal results for improvement exponents p_1 in (4.1) and q_1 in (4.3) are available (see [S] for a recent account on this subject).

Proposition 4.1. *If $w\colon \mathbf{R}\to[0,\infty)$ satisfies $A_2(w)=A$, then*

$$\tilde{A}_\infty(w) \leqslant s\left[\frac{A(s-1)}{1-As(2-s)}\right]^{1/s} \tag{4.5}$$

for any $s\in \left]1+\sqrt{\frac{A-1}{A}}, 2\right]$.

Proof. From definition (2.6) it is obvious that if $A_2(w)=A<\infty$ then, for $p\geqslant 2$

$$1\leqslant A_p(w)\leqslant A_2(w).$$

Our aim here is to focus on the "self-improvement of exponents" property of the A_2 class which goes back to [M]. It consists in the fact that if $A_2(w)=A<\infty$, then there exists $1<p_1<2$ such that

$$A_s(w)<\infty$$

for $p_1<s<2$, $p_1=p_1(A)$. Sharp results are known in the one-dimensional case (see [S], [K]) because $p_1(A) = 1+\sqrt{\frac{A-1}{A}}$.

Then it is possible to prove the sharp bounds

$$A \leqslant A_s(w) \leqslant \frac{A(s-1)}{1-As(2-s)} \tag{4.6}$$

for all $p_1(A)<s\leqslant 2$. By (4.2) with $p=2$ and $p_1=p_1(A)=1+\sqrt{\frac{A-1}{A}}$ in our case and taking into account (4.6), we arrive immediately to (4.5). □

5. Some extensions and examples

In this section we will focus on the local form of A_p and G_q conditions. By this it is meant that all references to $\sup_Q$ for quantities as $A_p(w)$ and $G_q(v)$ are understood to apply to all cubes Q within a fixed cube $Q_0 \subset \mathbf{R}^n$.

Likewise, local formulations exist as well for $\tilde{A}_\infty(w)$ and $\tilde{G}_1(v)$ constants.

In the present section we will compute the various constants for power type weights on the interval $[0,1] \subset \mathbf{R}$. The following result from [K] will be useful.

Lemma 5.1. *Let $\varphi\colon [0,\infty) \to [0,\infty)$ be a convex function and let $f \in L^1([a,b])$ be non negative and such that $\varphi(f)$ belongs to $L^1([a,b])$. If $J \subset [a,b]$ is an interval such that*

$$⨍_J f(x)\,dx = ⨍_a^b f(x)\,dx,$$

then

$$⨍_J \varphi\left(f(x)\right)\,dx \leqslant ⨍_a^b \varphi\left(f(x)\right)\,dx.$$

By mean of simple calculations we then prove the following:

Proposition 5.2. *Let $r \geqslant 0$; then the weight $w(x) = x^r$ belongs to A_∞ on $[0,1]$ and*

$$A_\infty(x^r) = \sup_{J\subset[0,1]} ⨍_J x^r\,dx \exp ⨍_J \log\frac{1}{x^r}\,dx = \frac{e^r}{r+1} \tag{5.1}$$

Proof. Applying Lemma 5.1 with $\varphi(s) = \log\frac{1}{s}$ we immediately obtain

$$A_\infty(x^r) \leqslant \sup_{0<t<1} ⨍_0^t x^r \exp ⨍_0^t \log\frac{1}{x^r}\,dx$$

and (5.1) follows by elementary calculations. □

Proposition 5.3. *Let $r \geqslant 0$; then on $[0,1]$*

$$\tilde{A}_\infty(x^r) = r+1$$

Proof. A simple calculation shows that if $-1 < s \leqslant 0$, then the weight $v(y) = y^s$ for $y \in [0,1]$ satisfies the inequality

$$\frac{\int_E v(y)\,dy}{\int_I v(y)\,dy} \leqslant \left(\frac{|E|}{|I|}\right)^{s+1}$$

for any interval $I \subset [0,1]$ and any measurable set $E \subset I$. Then, on $[0,1]$

$$\tilde{G}_1(v) = \frac{1}{s+1}.$$

Since by Theorem 3.1, on $[0,1]$

$$\tilde{A}_\infty(x^r) = \tilde{G}_1(y^s)$$

for $r+1=\frac{1}{s+1}$, then

$$\tilde{A}_\infty(x^r)=r+1. \qquad \square$$

References

[CF] Coifman, R., and C. Fefferman: Weighted norm inequalities for maximal functions and singular integrals. - Studia Math. 51, 1974, 241–250.

[C] Corporente, R.: A precise relation among A_∞ and G_1 constant in one dimension. - Rend. Accad. Sci. Fis. Mat. Napoli (4) 72, 2005, 65–70.

[F] Fefferman, R.: A criterion for the absolute continuity of the harmonic measure associated with an elliptic operator. - J. Amer. Math. Soc. 2, 1989, 127–135.

[G] Gotoh, Y.: On composition operators which preserve BMO. - Pacific J. Math. 201:2, 2001, 289–307.

[H] Hruscev, S. V.: A description of weights satisfying the A_∞ condition of Muckenhoupt. - Proc. Amer. Math. Soc. 90, 1984, 253–257.

[J] Johnson, R. L.: Changes of variable and A_p-weights. - Contemp. Math. 107, 1990, 93–99.

[JN1] Johnson, R. L., and - C. J. Neugebauer: - Homeomorphisms preserving A_p. - Rev. Mat. Iberoamericana 3:2, 1987, 249–273.

[JN2] Johnson, R. L., and C. J. Neugebauer: Change of variable results for A_p and reverse Hölder RH_r-classes. - Trans. Amer. Math. Soc. 328:2, 1991, 639–666.

[K] Korenovskii, A. A.: The exact continuation of a reverse Hölder inequality and Muckenhoupt's condition. - Math. Notes 52:5-6, 1993, 1192–1201.

[MaS] Martio, O., and C. Sbordone: Quasiminimizers in one dimension: integrability of the derivative, inverse function and obstacle problems. - Ann. Mat. Pura Appl. (4) (to appear).

[Mi] Migliaccio, L.: Some characterizations of Gehring G_q-class. - Houston J. Math. 19:1, 1993, 89–95.

[MS] Moscariello, G., and C. Sbordone: A_∞ as a limit case of reverse Hölder inequalities when the exponent tends to 1. - Ricerche Mat. 44:1, 1995, 131–144.

[M] Muckenhoupt, B.: The equivalence of two conditions for weight functions. - Studia Math. 49, 1974, 101–106.

[S] Sbordone, C.: Sharp embeddings for classes of weights and applications. - Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. (5) 29:1, 2005, 339–354.

[W] Wik, I.: On Muckenhoupt's classes of weight functions. - Studia Math. 94:3, 1989, 245–255.

Received 25 September 2006