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FOURIER MULTIPLIERS FOR L^2 FUNCTIONS WITH VALUES IN NONSEPARABLE HILBERT SPACES AND OPERATOR-VALUED H^p BOUNDARY FUNCTIONS

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Abstract. We extend the standard Fourier multiplier result to square integrable functions with values in (possibly nonseparable) Hilbert spaces. As a corollary, we extend the standard Hardy class boundary trace result to H^p (even Nevanlinna or bounded type) functions whose values are bounded linear operators between Hilbert spaces.

Both results have been well-known in the case that the Hilbert spaces are separable. Naturally, the results apply to functions over the unit circle/disc or over the real-line/half-plane or over other similar domains, even multidimensional in the case of the multiplier result. We briefly treat some related results, generalizations to Banach spaces and counter-examples.

1. The results

It is well-known that the operators $\mathscr{E} \in \mathscr{B}(\mathrm{L}^2(\mathbf{R}))$ (bounded and linear $\mathrm{L}^2(\mathbf{R}) \to \mathrm{L}^2(\mathbf{R})$) that *commute with the time shift* ($\tau^t \mathscr{E} = \mathscr{E} \tau^t$ for all $t \in \mathbf{R}$, where $(\tau^t f)(s) := f(s+t)$) correspond one-to-one to functions $\hat{\mathscr{E}} \in \mathrm{L}^\infty(\mathbf{R})$ through $\widehat{\mathscr{E} f} = \hat{\mathscr{E}} \widehat{f}$ (on $\mathbf{R}$), where

$$(1) \qquad (\mathscr{F} f)(r) := \widehat{f}(r) := (2\pi)^{-1/2} \int_{\mathbf{R}} f(t) \mathrm{e}^{-itr}\, dt \qquad (r \in \mathbf{R})$$

(if $f \in \mathrm{L}^1 \cap \mathrm{L}^2$; use density for general $f \in \mathrm{L}^2$) is the *Fourier(–Plancherel) transform* of f and $\mathbf{R}$ denotes the real line. (It is well-known that $\|\widehat{f}\|_2 = \|f\|_2$ and that $\mathscr{F}$ is onto.) Due to this, the set $\mathrm{L}^\infty(\mathbf{R})$ is called the set of *Fourier multipliers* for $\mathrm{L}^2(\mathbf{R})$.

In this article we shall generalize this to operators on L^2 functions with values in Hilbert spaces, i.e., to $\mathrm{TI}(X, Y)$ (TI for "time-invariant"), the set of bounded linear operators $\mathrm{L}^2(\mathbf{R}; X) \to \mathrm{L}^2(\mathbf{R}; Y)$ that commute with the time shift. We let X and Y be arbitrary complex Hilbert spaces. The case where X and Y are separable is well-known (even for unbounded closed operators, see e.g., [FS55]), but the general case is much more complicated. The set L^∞ must be replaced by $\mathrm{L}^\infty_{\mathrm{strong}}$ whenever X and Y are infinite-dimensional.

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At a first glance, the result may seem like a straightforward extension of the separable case, but the question was studied in [Tho97], which illustrates the difficulty of establishing the existence of a pointwise-defined $\mathscr{B}(X,Y)$-valued function whose restriction to each separable subspace of X has the required properties.

As a corollary, we show that for each bounded holomorphic function $\tilde{F}\colon \mathbf{C}^+ \to \mathscr{B}(X,Y)$, where $\mathbf{C}^+ := \{z \in \mathbf{C} \mid \operatorname{Im} z > 0\}$, there exists a unique boundary function (equivalence class) $\widehat{F}\colon \mathbf{R} \to \mathscr{B}(X,Y)$ such that $\tilde{F}x \to \widehat{F}x$ a.e. nontangentially (in Y), for each $x \in X$.

This is a direct generalization of the standard H^∞ boundary value function result and thus solves the problem studied in [Tho97] (where it is shown that "a.e." may depend on x). We further extend this to functions in the Nevanlinna class in Corollary 1.6. Also this result was already known in the case of separable X and Y [RR85].

An example of a nonseparable Hilbert space is the completion of the space of almost-periodic functions (it is isomorphic to $\ell^2(\mathbf{R})$). The definitions and basic properties of Bochner measurability and L^p and H^p spaces can be found in, e.g., [HP57] and [Mik02]. We prove the results of this article independently of [Mik02], to which we refer further details, extensions etc. (particularly to its Section F.1).

The natural class for our first result is the class of (equivalence classes of) strong L^∞ functions:

Definition 1.1. ($\mathrm{L}^\infty_{\text{strong}}$) A function $F\colon \mathbf{R} \to \mathscr{B}(X,Y)$ is said to be *strongly measurable*, if Fx is Bochner measurable for each $x \in X$.

We define $\mathrm{L}^\infty_{\text{strong}}(Q;\mathscr{B}(X,Y))$ to be the space[1] of (equivalence classes of) strongly measurable functions $Q \to \mathscr{B}(X,Y)$ with norm

$$\|F\|_{\mathrm{L}^\infty_{\text{strong}}} := \sup_{\|x\|\le 1} \|Fx\|_{\mathrm{L}^\infty} < \infty. \tag{2}$$

Note that F and G are identified if $\|F - G\| = 0$, i.e., if $Fx = Gx$ a.e. for each $x \in X$. If X is finite-dimensional, then, obviously, $\mathrm{L}^\infty_{\text{strong}}(\mathbf{R};\mathscr{B}(X,Y)) = \mathrm{L}^\infty(\mathbf{R};\mathscr{B}(X,Y))$, but in general we may have $\|F\|_{\mathrm{L}^\infty} = \infty$, and F need not even be Bochner measurable, even if F is a representative of $[0] \in \mathrm{L}^\infty_{\text{strong}}$ (see [Mik05] or Example 3.1.4 of [Mik02] for this and further anomalies).

Now we can state our equivalence rigorously (with uniqueness in $\mathrm{L}^\infty_{\text{strong}}$, not pointwise):

Theorem 1.2. *For each $\mathscr{E} \in \mathrm{TI}(X,Y)$ there exists a unique function (equivalence class) $\hat{\mathscr{E}} \in \mathrm{L}^\infty_{\text{strong}}(\mathbf{R};\mathscr{B}(X,Y))$ (called the* symbol *of $\mathscr{E}$) such that $\hat{\mathscr{E}}\widehat{f} = \widehat{\mathscr{E}f}$*

[1]Notes: 1. One could define the multiplication through arbitrary elements to make $\mathrm{L}^\infty_{\text{strong}}(\mathbf{R};\mathscr{B}(H))$ a Banach algebra, as shown in [Mik02], Section F.1, but we do not need this. 2. Obviously, $\mathrm{L}^\infty_{\text{strong}}$ is a subspace of $\mathscr{B}(X,\mathrm{L}^\infty(\mathbf{R};Y))$; in Theorem 2.5 we show that they are actually equal. 3. Strong (operator) measurability does not imply Bochner measurability, not even Lusin measurability to strong topology.

a.e. on $\mathbf{R}$ *for any* $f \in \mathrm{L}^2(\mathbf{R}; X)$. *Moreover,* $\|\hat{\mathscr{E}}\|_{\mathrm{L}^\infty_{\mathrm{strong}}} = \|\mathscr{E}\|_{\mathscr{B}(\mathrm{L}^2(\mathbf{R};X),\mathrm{L}^2(\mathbf{R};Y))}$, *and each* $\hat{\mathscr{E}} \in \mathrm{L}^\infty_{\mathrm{strong}}$ *is of this form.*

(This is actually an isometric B^*-algebra isomorphism; see [Mik05] or Section 3.1 of [Mik02].)

We shall prove Theorem 1.2 in Section 2, using the fact that $\mathrm{L}^\infty_{\mathrm{strong}} = \mathscr{B}(X; \mathrm{L}^\infty(\mathbf{R}; Y))$. The rest of this section will be devoted to boundary trace results. We denote by $\mathrm{H}^\infty(\mathbf{C}^+; \mathscr{B}(X,Y))$ the space of bounded holomorphic functions $\mathbf{C}^+ \to \mathscr{B}(X,Y)$ with norm $\|\widetilde{\mathscr{D}}\|_{\mathrm{H}^\infty} := \sup_{z\in\mathbf{C}^+} \|\widetilde{\mathscr{D}}(z)\|_{\mathscr{B}(X,Y)}$. By $\mathrm{TIC}(X,Y)$ we denote operators $\mathscr{D} \in \mathrm{TI}(X,Y)$ that are *causal*: $\chi_{(-\infty,0)}\mathscr{D}\chi_{(0,+\infty)}u = 0$ for all $u \in \mathrm{L}^2(\mathbf{R}; X)$ (i.e., future input does not affect past (or negative-time) output), where χ_E is the characteristic function of the set E. We set $\mathbf{R}_+ := [0, +\infty)$. The following is well known [Wei91]:

Proposition 1.3. ($\widetilde{\mathrm{TIC}} = \mathrm{H}^\infty$) *For any* $\mathscr{D} \in \mathrm{TIC}(X,Y)$ *there exists a unique function* $\widetilde{\mathscr{D}} \in \mathrm{H}^\infty(\mathbf{C}^+; \mathscr{B}(X,Y))$ *such that* $(\widetilde{\mathscr{D}f})(z) = \widetilde{\mathscr{D}}(z)\tilde{f}(z)$ *for all* $z \in \mathbf{C}^+$ *and* $f \in \mathrm{L}^2(\mathbf{R}_+; X)$.

Moreover, this identification is an isometric isomorphism of TIC *onto* H^∞.

(The extension $\tilde{f}$ of $\widehat{f}$ to $z \in \mathbf{C}^+$ (through (1)) is called the *Laplace transform* of f.)

The Fourier transform of a square integrable function is the boundary trace of its Laplace transform:

Proposition 1.4. *Assume that* $g \in \mathrm{L}^2(\mathbf{R}_+; X)$. *Then* $\widehat{g}$ *is the* nontangential boundary function *of* $\tilde{g}$ *a.e. on* $\mathbf{R}$.

This means that for a.e. $r \in \mathbf{R}$ and every $\theta > 0$ we have $\tilde{g}(z) \to \widehat{g}(r)$ as $z \to r$ in the sector $\{z \in \mathbf{C}^+ \mid |\operatorname{Re} z - r|/\operatorname{Im} z < \theta\}$. In particular, $\tilde{g}(r + is) \to \widehat{g}(r)$, as $s \to 0+$, for a.e. $r \in \mathbf{R}$.

Proof. If X is separable, then this follows from pp. 81, 85 and 90 of [RR85]. In general, the range of g is contained in a closed separable subspace of X (after redefinition on a null set, and that does not affect $\tilde{g}$ nor $\widehat{g}$). $\square$

From standard results (e.g., [RR85]) we shall derive that $\widehat{\mathscr{D}}$ is the (unique) boundary function of $\widetilde{\mathscr{D}}$ and thus obtain the following result:

Theorem 1.5. (H^∞ Boundary function) *Let* $F \in \mathrm{H}^\infty(\mathbf{C}^+; \mathscr{B}(X,Y))$. *Then there exists a unique* $F_0 \in \mathrm{L}^\infty_{\mathrm{strong}}(\mathbf{C}^+; \mathscr{B}(X,Y))$ *(the* boundary function *of* F*) for which the following holds: for each* $x \in X$, *there exists a null set* $N \subset \mathbf{R}$ *such that* $F(r + is)x \to F_0(r)x$, *as* $s \to 0+$, *for all* $r \in \mathbf{R} \setminus N$.

In fact, $Fx \to F_0x$ *nontangentially a.e., for each* $x \in X$, *and* $\|F_0\|_{\mathrm{L}^\infty_{\mathrm{strong}}} = \|F\|_{\mathrm{H}^\infty}$.

If $f \in \mathrm{H}^p(\mathbf{C}^+; X)$ *and* $G \in \mathrm{H}^\infty(\mathbf{C}^+; \mathscr{B}(Y,Z))$ $(1 \le p \le \infty)$, *where also* Z *is a Hilbert space, with boundary functions* f_0 *and* G_0, *respectively, then the boundary functions of* Ff *and* GF *equal* F_0f_0 *and* G_0F_0, *respectively.*

(We have $Ff \to F_0 f_0$ also in $\mathrm{L}^p(\mathbf{R};Y)$ if $p<\infty$; see p. 101 of [Mik02] for this and further equivalent conditions for F_0 being the boundary function of F.)

Recall that $f \in \mathrm{H}^p(\mathbf{C}^+;B)$ means that $f\colon \mathbf{C}^+ \to B$ is holomorphic and $\|f\|_{\mathrm{H}^p} := \sup_{s>0}\|f(\cdot+is)\|_{\mathrm{L}^p} < \infty$.

Proof of Theorem 1.5. 1° By Proposition 1.3, we have $F = \widetilde{\mathscr{D}}$ for some $\mathscr{D} \in \mathrm{TIC}(X,Y)$. Choose $\widehat{\mathscr{D}} =: F_0$ as in Theorem 1.2. Then

$$\|F_0\|_{\mathrm{L}^\infty_{\text{strong}}} = \|\mathscr{D}\|_{\mathscr{B}} = \|F\|_{\mathrm{H}^\infty}. \tag{3}$$

For any $g_1 \in \mathrm{L}^2(\mathbf{R}_+;X)$, we have $g := \mathscr{D}g_1 \in \mathrm{L}^2(\mathbf{R}_+;Y)$. By Proposition 1.4, the function

$$\widehat{g} = \widehat{\mathscr{D}g_1} = F_0\widehat{g}_1 \in \mathrm{L}^2(\mathbf{R};Y) \tag{4}$$

is the boundary function of

$$\tilde{g} = \widetilde{\mathscr{D}g_1} = F\tilde{g}_1 \in \mathrm{H}^2(\mathbf{C}^+;Y). \tag{5}$$

Let $\phi := \chi_{[0,1]}$. Then $\widehat{\phi}(z) = (1-\mathrm{e}^{-iz})/iz\sqrt{2\pi}$; in particular, $\widehat{\phi} \neq 0$ on $\mathbf{C}^+$ and $\widehat{\phi} \neq 0$ a.e. on $\mathbf{R}$. Let $x \in X$ be arbitrary and take $g_1 := \phi x \in \mathrm{L}^2(\mathbf{R}_+;X)$. By the above, the boundary function of $F\widehat{\phi}x \in \mathrm{H}^2(\mathbf{C}^+;Y)$ equals $F_0\widehat{\phi}x \in \mathrm{L}^2(\mathbf{R};Y)$ a.e., hence (divide by $\widehat{\phi}$) the boundary function of $Fx \in \mathrm{H}^\infty(\mathbf{C}^+;Y)$ equals $F_0 x \in \mathrm{L}^\infty(\mathbf{C}^+;Y)$ a.e. (nontangentially). Thus, we have proved the first two paragraphs of the theorem.

2° *The claim on Ff*: (The existence of f_0 follows as in Proposition 1.4.) We may replace X by the (separable) closed span of $f[\mathbf{C}^+]$. Since $F(z)[X]$ is then separable for each $z \in \mathbf{C}^+$, we may replace Y by the (separable) closed span of $\bigcup_{z\in S} F(z)[X]$, where S is a separable subset of $\mathbf{C}^+$ (and F by $P_Y F P_X^*$, where P_Y and P_X are the orthogonal projections from the old to the new Y and X, respectively); this does not affect the convergence.

Now that X and Y are separable, there exists a null set $N \subset \mathbf{R}$ such that $F \to F_0$ and $f \to f_0$ strongly and nontangentially outside N, hence

$$\begin{aligned} &\|(Ff)(z) - (F_0 f_0)(r)\|_Y \\ &\le \|F\|\|f(z)-f_0(r)\|_X + \|F(z)f_0(r) - F_0(r)f_0(r)\|_Y \to 0, \end{aligned} \tag{6}$$

as $z \to r$ nontangentially, for every $r \in \mathbf{R}\setminus N$.

3° By the above, we have $(GF)_0 f_0 = (GFf)_0 = G_0(Ff)_0 = G_0F_0f_0$ for each $f \in \mathrm{H}^p(\mathbf{C}^+;X)$. Take $p=\infty$ and let f vary over the constants $f \equiv x \in X$ to observe that $(GF)_0 = G_0F_0$ (in $\mathrm{L}^\infty_{\text{strong}}$). □

As shown on p. 92 of [RR85], $\widetilde{\mathscr{D}} = F$ need not converge to $\widehat{\mathscr{D}} = F_0$ in the operator norm (and $\widehat{\mathscr{D}} \notin \mathrm{L}^\infty$ is possible) even if X and Y are separable. Nevertheless, in the separable case $\widetilde{\mathscr{D}}$ converges to $\widehat{\mathscr{D}}$ strongly a.e., that is, a single N applies to every $x \in X$ (take a union of N's for a dense, countable subset).

However, Thomas has shown that if X and Y are nonseparable (and $X=Y$), then $\widetilde{\mathscr{D}}$ need not converge strongly anywhere and $\widehat{\mathscr{D}}$ need not be Lusin measurable to the strong topology [Tho97]. Thomas notes that $\widehat{\mathscr{D}}$ can be defined as $\mathscr{F}^{-1}\mathscr{D}\mathscr{F} \in \mathscr{B}(\mathrm{L}^2(\mathbf{R}; X))$, and that *"This seems to be about as far as one can go in the way of transfer functions in nonseparable Hilbert spaces"* (p. 133). As Theorems 1.2 and 1.5 show, one can go further and obtain a pointwise-defined function even though one cannot obtain strong convergence nor measurability with respect to strong topology (which is stronger than the strong measurability of Definition 1.1).

Next we extend Theorem 1.5 to functions of bounded type. If X is a Banach space and $\Omega \subset \mathbf{C}$ is open, then a holomorphic function $f : \Omega \to X$ is said to be of *bounded type* if $\log^+ \|f\|_X$ has a harmonic majorant on Ω. The (Nevanlinna) class of such functions is denoted by $\mathrm{N}(\mathbf{C}^+; X)$ [RR85]. From Theorem 1.5 we conclude that any Nevanlinna function has a (unique) boundary function:

Corollary 1.6. *If $F \in \mathrm{N}(\mathbf{C}^+; \mathscr{B}(X, Y))$, then there exists a strongly measurable boundary function $F_0 : \mathbf{R} \to \mathscr{B}(X, Y)$ such that $Fx \to F_0x$ nontangentially a.e. for each $x \in X$.*

Obviously, the function F_0 is unique if F_0 and G_0 are again identified when $F_0x = G_0x$ a.e. for all $x \in X$. The same result (and proof) holds with any simply connected region in place of $\mathbf{C}^+$.

Proof of Corollary 1.6. By p. 76 of [RR85], any $\mathrm{N}(\mathbf{C}^+; \mathscr{B}(X, Y))$ function can be written as $F = g/v$, where $0 < |v| \le 1$ (scalar) and and $g, v \in \mathrm{H}^\infty$. Let $g_0 \in \mathrm{L}^\infty_{\mathrm{strong}}$ and $v_0 \in \mathrm{L}^\infty$ denote the corresponding boundary functions (hence $|v_0| > 0$ a.e. on $\mathbf{R}$ [Rud87, Theorem 17.18]).

Obviously, the function $F_0 := v_0^{-1} g_0$ is strongly measurable. By Theorem 1.5, we conclude that F_0x is the boundary function of Fx a.e., for any $x \in X$: we have $gx \to g_0x$ and $1/v \to 1/v_0$ a.e., hence

$$Fx = (1/v)gx \to (1/v_0)g_0x = F_0x \tag{7}$$

a.e. on $\mathbf{R}$, nontangentially. □

Finally, from Corollary 1.6 it follows that any H^p function has a (unique) $\mathrm{L}^p_{\mathrm{strong}}$ boundary function:

Corollary 1.7. *If $F \in \mathrm{H}^p(\mathbf{C}^+; \mathscr{B}(X, Y))$ and $1 \le p < \infty$, then $F \in \mathrm{N}$, and we have $F(\cdot + is)x \to F_0x$ in $\mathrm{L}^p(\mathbf{R}; Y)$, as $s \to 0+$, for any $x \in X$.*

Proof. By p. 81 of [RR85], any H^p function is a N function, hence F_0 exists. By Theorem 4.8B, p. 90 of [RR85], Fx is the Poisson integral of F_0x for any $x \in X$, hence $Fx \to F_0x$ in L^p (by the same proof as in the scalar case; see, e.g., [Gar81], pp. 12–17). □

However, the boundary function of an arbitrary $F \in \mathrm{H}^p_{\mathrm{strong}}(\mathbf{C}^+; \mathscr{B}(X, Y))$ need not have values in $\mathscr{B}(X, Y)$, not even when $X = Y$ is separable and $p = 2$ (Example 3.3.6 of [Mik02]).

By $F \in \mathrm{H}^p_{\mathrm{strong}}(\mathbf{C}^+, \mathscr{B}(X, Y))$ we mean that $F : \mathbf{C}^+ \to \mathscr{B}(X, Y)$ is holomorphic and $Fx \in \mathrm{H}^p(\mathbf{C}^+; Y)$ for every $x \in X$. From the Closed Graph Theorem (as in the proof of Lemma 2.1) it follows that then $\|x \mapsto Fx\|_{\mathscr{B}(X,\mathrm{H}^p)} < \infty$. In particular, $\mathrm{H}^\infty_{\mathrm{strong}} = \mathrm{H}^\infty$.

Except for Corollaries 1.6 and 1.7, the results of this article are from [Mik02], which also contains further properties, extensions, proofs, examples and details, on $\mathrm{L}^\infty_{\mathrm{strong}}$ and other strongly (or weakly) measurable functions, and on H^p and $\mathrm{H}^p_{\mathrm{strong}}$, and a direct (hence longer) proof of Theorem 1.5 (see the index and Sections 3.1, D.1, F.1 and 3.3 of [Mik02]).

All above results (with essentially the same proofs) also hold with the unit disc (resp. the unit circle $\mathbf{T}$) in place of the half-plane $\mathbf{C}^+$ (resp. $\mathbf{R}$); part of them are explicitly given in [Mik05] and [Mik02] (e.g., Theorem 3.3.1(e) and Lemmata 13.1.5 and 13.1.6 of [Mik02]). The proofs can easily be modified to cover further similar domains, such as $\mathbf{R}^n$ or $\mathbf{T}^n$ (cf. [FS55]).

We finish this section by commenting briefly the extensions of the above results to Banach spaces. Allow, for a while, X and Y to be arbitrary complex Banach spaces. Then, still, any $\mathrm{TIC}(X,Y)$ operator is represented by a $\mathrm{H}^\infty(\mathbf{C}^+;\mathscr{B}(X,Y))$ function, by [Wei91], but this mapping $\mathrm{TIC} \to \mathrm{H}^\infty$ is no longer isometric, nor onto (by Example 3.3.4 of [Mik02]). Moreover, for such X and Y (e.g., for $X = \mathbf{C}$, $Y = \ell^\infty(\mathbf{N})$) Theorem 1.5 does not hold, i.e., some H^∞ functions do not have $\mathrm{L}^\infty_{\mathrm{strong}}$ boundary functions, by Example 3.3.5 of [Mik02].

A weaker form of the necessity part of Theorem 1.2 for Banach spaces X and Y is presented in Section 3.2 of [Mik02]. In Theorem 3.1.7 it is applied to show that a time-invariant operator over both $\mathrm{e}^{a\cdot}\mathrm{L}^2$ and $\mathrm{e}^{b\cdot}\mathrm{L}^2$ corresponds to a bounded holomorphic function $\{z \in \mathbf{C} \mid a < \operatorname{Im} z < b\} \to \mathscr{B}(X,Y)$. If X and Y are Hilbert spaces, then also the converse holds and this correspondence becomes an isometric isomorphism onto, by Theorem 3.1.6.

However, sufficiency results are much more popular in (Banach space) operator-valued multiplier theorems. E.g., a generalization to $\mathrm{L}^p(X) \to \mathrm{L}^p(Y)$ of the Mihlin multiplier theorem is given in [Wei01], assuming that X is an UMD space. For a survey on further results, see, e.g., [Hyt03].

Some applications of Theorem 1.5 to system theory are given in [Mik02] (e.g., Lemma 6.3.6 and its numerous applications to Riccati equations in Chapters 9 and 10, including 9.2.14–9.2.19 and 10.3.2).

2. The proof of Theorem 1.2

We start with a simple observation:

Lemma 2.1. *A strongly measurable $F\colon \mathbf{R} \to \mathscr{B}(X,Y)$ is in $\mathrm{L}^\infty_{\mathrm{strong}}$ iff $Fx \in \mathrm{L}^\infty$ for all $x \in X$.*

Proof. "Only if" is obvious. If $x_n \to 0$ in X and $Fx_n \to f$ in $\mathrm{L}^\infty(\mathbf{R};Y)$, as $n \to \infty$, then $Fx_{n_k} \to f$ a.e. for some subsequence $(n_1 < n_2 < \cdots)$, hence then $f = 0$ a.e. Thus, by the Closed Graph Theorem, $F\colon X \to \mathrm{L}^\infty$ is continuous, i.e., $F \in \mathrm{L}^\infty_{\mathrm{strong}}$. $\square$

Next we do the easy part, by showing that $\mathrm{L}^\infty_{\mathrm{strong}}$ can be identified with a subspace of $\mathscr{B}(\mathrm{L}^p(\mathbf{R};X),\mathrm{L}^p(\mathbf{R};Y))$ (isometrically):

Lemma 2.2. *Let* $F\in\mathrm{L}^\infty_{\mathrm{strong}}(\mathbf{R};\mathscr{B}(X,Y))$ *and* $f\in\mathrm{L}^p(\mathbf{R};X)$. *Then* $Ff\in\mathrm{L}^p(\mathbf{R};Y)$ *and* $\|Ff\|_p\le\|F\|_{\mathrm{L}^\infty_{\mathrm{strong}}}\|f\|_p$. *Moreover,* $\|f\mapsto Ff\|_{\mathrm{L}^p(\mathbf{R};X)\to\mathrm{L}^p(\mathbf{R};Y)}=\|F\|_{\mathrm{L}^\infty_{\mathrm{strong}}}$.

Proof. If f is simple, then the claims are rather obvious. In the general case, there exists a series $\{f_n\}$ of simple functions converging to f in $\mathrm{L}^p(\mathbf{R};X)$. It follows that also $\{Ff_n\}$ is a Cauchy-sequence, in $\mathrm{L}^p(\mathbf{R};Y)$; let g be its limit. Then

$$\|g\|_p\le\limsup_n\|F\|_{\mathrm{L}^\infty_{\mathrm{strong}}}\|f_n\|_p=\|F\|_{\mathrm{L}^\infty_{\mathrm{strong}}}\|f\|_p. \tag{8}$$

Replace $\{f_n\}$ by a subsequence twice to have $f_n\to f$ a.e. and $Ff_n\to g$ a.e. and thus observe that $Ff=g$ a.e., hence $\|Ff\|_p=\|g\|_p\le\|F\|_{\mathrm{L}^\infty_{\mathrm{strong}}}\|f\|_p$. Since $f\in\mathrm{L}^p(\mathbf{R};X)$ was arbitrary, we have $\|f\mapsto Ff\|\le\|F\|_{\mathrm{L}^\infty_{\mathrm{strong}}}$.

But if $0<M<\|F\|_{\mathrm{L}^\infty_{\mathrm{strong}}}$, then $\|Fx\|_\infty>M\|x\|$ for some $x\in X$, hence then there exists $E\subset\mathbf{R}$ such that $0<m(E)<\infty$ and $\|F(t)x\|>M\|x\|$ for all $t\in E$. Set $f:=\chi_E x$ to have $\|Ff\|_p>M\|f\|_p$. Since $M\in(0,\|F\|_{\mathrm{L}^\infty_{\mathrm{strong}}})$ was arbitrary, we have $\|f\mapsto Ff\|\ge\|F\|_{\mathrm{L}^\infty_{\mathrm{strong}}}$. □

From Lemma 2.2 we conclude that $\mathrm{L}^\infty_{\mathrm{strong}}$ is closed under pointwise multiplication:

Corollary 2.3. *If* $F\in\mathrm{L}^\infty_{\mathrm{strong}}(\mathbf{R};\mathscr{B}(X,Y))$ *and* $G\in\mathrm{L}^\infty_{\mathrm{strong}}(\mathbf{R};\mathscr{B}(Y,Z))$, *then* $GF\in\mathrm{L}^\infty_{\mathrm{strong}}(\mathbf{R};\mathscr{B}(X,Z))$ *and* $\|GF\|\le\|G\|\|F\|$.

(Indeed, GF is a function $\mathbf{R}\to\mathscr{B}(X,Z)$ and $GFx=G(Fx)\in\mathrm{L}^\infty(\mathbf{R};Z)$, $\|GFx\|_\infty\le\|G\|\|Fx\|\le\|G\|\|F\|\|x\|$ for each $x\in X$.)

By the (infinite-dimensional) Plancherel Theorem, the Fourier transform (1) is an isometric isomorphism of $\mathrm{L}^2(\mathbf{R};X)$ onto $\mathrm{L}^2(\mathbf{R};X)$. Since, obviously, $\widehat{\tau^t f}(r)=\mathrm{e}^{itr}f(r)$ for any $f\in\mathrm{L}^2$, we observe from Lemma 2.2 that given a function $F\in\mathrm{L}^\infty_{\mathrm{strong}}(\mathbf{R};\mathscr{B}(X,Y))$, we have $\mathscr{E}\in\mathrm{TI}(X,Y)$ and $\|\mathscr{E}\|=\|F\|_{\mathrm{L}^\infty_{\mathrm{strong}}}$ if we define $\mathscr{E}$ by $\widehat{\mathscr{E}f}:=F\widehat{f}$ for any $f\in\mathrm{L}^2(\mathbf{R};X)$. Thus, to prove Theorem 1.2, it only remains to be shown that each $\mathscr{E}\in\mathrm{TI}(X,Y)$ is of this form. To this end we need Lemma 2.4 below.

Given a measurable function f, we denote its equivalence class by $[f]$ (or by f when there is no risk of misinterpretation). Let B be a Banach space. By $\mathrm{L}^1_{\mathrm{loc}}(\mathbf{R};B)$ we denote the (equivalence classes of) Bochner-measurable functions $f\colon\mathbf{R}\to B$ whose restrictions to compact sets are integrable.

Our proof is based on choosing in a uniquely-determinated way representatives ("Lf") for $\mathrm{L}^1_{\mathrm{loc}}(\mathbf{R};B)$ "functions" so that these representatives have all possible Lebesgue points:

Lemma 2.4. (Lebesgue representative) *Let* $[f] \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}; B)$. *For each* $t \in \mathbf{R}$ *such that*

$$\lim_{r \to 0+} \frac{1}{2r} \int_{t-r}^{t+r} \|f(s) - x_t\|_B \, ds = 0 \tag{9}$$

for some $x_t \in B$, *we set* $(Lf)(t) := x_t$ *(if, in addition,* $x_t = f(t)$*, then* t *is called a* Lebesgue point *of* f *and we write* $t \in \mathrm{Leb}(f)$*). For other values of* t*, we set* $(Lf)(t) := 0$.

It follows that $(Lf)(t) = f(t)$ *for all* $t \in \mathrm{Leb}(f)$*, hence* $Lf = f$ *a.e. and* $\mathrm{Leb}(f) \subset \mathrm{Leb}(Lf)$. *Moreover,* $\|(Lf)(t)\| \le \|f\|_\infty$ *for all* $t \in \mathbf{R}$*, and* Lf *depends on* $[f]$ *only.*

(By Theorem 3.8.5 of [HP57], $t \in \mathrm{Leb}(f)$ for a.e. $t \in \mathbf{R}$; the rest is obvious.) We warn that $f \mapsto Lf$ and $[f] \mapsto Lf$ are not linear, nor $f \mapsto (Lf)(t)$ for any $t \in \mathbf{R}$ (but $f \mapsto [Lf]$ and $[f] \mapsto [Lf]$ are linear, since $[Lf] = [f]$).

Using the above lemma, we shall show that any element of $\mathscr{B}(X, \mathrm{L}^\infty(\mathbf{R}; Y))$ is determined by an element of $\mathrm{L}^\infty_{\mathrm{strong}}(\mathbf{R}; \mathscr{B}(X, Y))$ and that any member (representative function) of an element of $\mathrm{L}^\infty_{\mathrm{strong}}(\mathbf{R}; \mathscr{B}(X, Y))$ can be redefined so as not to exceed its norm:

Theorem 2.5. *We have* $\mathrm{L}^\infty_{\mathrm{strong}}(\mathbf{R}; \mathscr{B}(X, Y)) = \mathscr{B}(X, \mathrm{L}^\infty(\mathbf{R}; Y))$*, isometrically. In addition, for each* $T \in \mathscr{B}(X, \mathrm{L}^\infty(\mathbf{R}; Y))$*, there exists a representative* $F \colon \mathbf{R} \to \mathscr{B}(X, Y)$ *such that* $\sup_{\mathbf{R}} \|F\| = \|T\|$.

(We mention that $\mathrm{L}^p_{\mathrm{strong}}(\mathbf{R}; \mathscr{B}(X, Y)) \subsetneq \mathscr{B}(X, \mathrm{L}^p(\mathbf{R}; Y))$ and that the normed space $\mathrm{L}^p_{\mathrm{strong}}(\mathbf{R}; \mathscr{B}(X, Y))$ is incomplete when $p < \infty$ and X and Y are infinite-dimensional [Mik06, Example 4.3], [Mik02, Example F.1.10] (the case $p \neq 2$ requires a slight modification). Here $\mathbf{R}$ can be replaced, e.g., by an interval.)

Proof. It is obvious from Definition 1.1 that the space $\mathrm{L}^\infty_{\mathrm{strong}}(\mathbf{R}; \mathscr{B}(X, Y))$ is a subspace of $\mathscr{B}(X, \mathrm{L}^\infty(\mathbf{R}; Y))$, with the same norm. Assume then that $T \in \mathscr{B}(X, \mathrm{L}^\infty(\mathbf{R}; Y))$, and set $M := \|T\|$. We shall construct $F \in \mathrm{L}^\infty_{\mathrm{strong}}$ such that $Fx = Tx$ a.e. for each $x \in X$ and $\sup_{\mathbf{R}} \|F\| \le M$; this completes the proof.

For any $t \in \mathbf{R}$, the set

$$X_t := \{x \in X \mid t \in \mathrm{Leb}(LTx)\} \tag{10}$$

is a subspace of X, and $\|(LTx)(t)\| \le \|Tx\|_\infty \le M\|x\|$ for all $t \in \mathbf{R}$, $x \in X$, by Lemma 2.4.

For each $t \in \mathbf{R}$, the map $x \mapsto (LTx)(t)$ is obviously linear on X_t, hence it has a norm-preserving extension $F(t) \in \mathscr{B}(X, Y)$ (e.g., extend to $\overline{X_t}$ by density and by zero on $X_t^\perp$). Thus, $\|F(t)\| \le M$.

Let $x \in X$. Then for a.e. $t \in \mathbf{R}$ we have $x \in X_t$ and hence $(LTx)(t) = F(t)x$. But $LTx = Tx$ a.e., hence $Tx = Fx$ a.e. Consequently, $F \colon \mathbf{R} \to \mathscr{B}(X, Y)$ is strongly measurable. ☐

As one observes from the proof, in Theorem 2.5 the set $\mathbf{R}$ may be replaced by any measurable subset of $\mathbf{R}^n$ (or $\mathbf{T}^n$) and Y may be any Banach space (or X may be any Banach space if Y is finite-dimensional, see Lemma F.1.5(b) of [Mik02]). See also Theorem F.1.9 of [Mik02] for related results.

Now we are ready to prove the main result:

Proof of Theorem 1.2. Let $x \in X$. Then $\mathscr{E}x \in \mathrm{TI}(\mathbf{C}, Y)$ (by $\mathscr{E}x$ we refer to the map $\mathscr{E}x\colon f \mapsto \mathscr{E}fx$). Choose a dense countable subset $\{f_n\}$ of $\mathrm{L}^2(\mathbf{R};\mathbf{C})$. Given n, we have $g_n := \mathscr{E}f_n x \in \mathrm{L}^2(\mathbf{R};Y)$, hence there exists a separable set $Y_n \subset Y$ such that $g_n(t) \subset K_n$ for a.e. $t \in \mathbf{R}$. Let Y_x be the closed span of the union of all these sets Y_n.

Obviously, Y_x is separable and $\mathscr{E}x \in \mathrm{TI}(\mathbf{C}; Y_x)$ (because $\mathscr{E}fx \in \mathrm{L}^2(\mathbf{R};Y_x)$ for all $f \in \{f_n\}_{n=1}^\infty$, hence for all $f \in \mathrm{L}^2(\mathbf{R};\mathbf{C})$). By the separable case of Theorem 1.2 [FS55], there exists a unique

$$\tilde{T}_x \in \mathrm{L}^\infty_{\text{strong}}(\mathbf{R}; \mathscr{B}(\mathbf{C}, Y_x)) = \mathrm{L}^\infty(\mathbf{R}; Y_x) \tag{11}$$

such that $\widehat{\mathscr{E}fx} = \tilde{T}_x \widehat{f}$ for all $\widehat{f} \in \mathrm{L}^2(\mathbf{R};\mathbf{C})$. Define $T_x \in \mathrm{L}^\infty(\mathbf{R};Y)$ by $T_x(t) := \tilde{T}_x(t)$ for all $t \in \mathbf{R}$. Then $\|T_x\|_\infty = \|\tilde{T}_x\|_\infty \le \|\mathscr{E}\|_{\mathscr{B}}\|x\|_X$.

This way we obtain $T_x \in \mathrm{L}^\infty(\mathbf{R};Y)$ for any $x \in X$. Obviously, the function $T\colon x \mapsto T_x$ is linear and $\|T\|_{\mathscr{B}(X,\mathrm{L}^\infty(\mathbf{R};Y))} \le \|\mathscr{E}\|$.

By Theorem 2.5, there exists $F \in \mathrm{L}^\infty_{\text{strong}}(\mathbf{R}; \mathscr{B}(X,Y))$ such that

$$F\widehat{f}x = T\widehat{f}x = \widehat{\mathscr{E}fx} \in \mathrm{L}^2(\mathbf{R};Y) \tag{12}$$

for every $f \in \mathrm{L}^2(\mathbf{R};\mathbf{C})$ and $x \in X$. By linearity, $F\widehat{g} = \widehat{\mathscr{E}g}$ for every simple g, hence for every $g \in \mathrm{L}^2(\mathbf{R};X)$, by density and continuity (Theorem B.3.11 of [Mik02] and Lemma 2.2). $\square$

The alternative proof in [Mik02] (of its Theorem 3.1.3)[2] is based on the scalar version of Theorem 1.2 only, but it is much more difficult to follow. There the function $\hat{\mathscr{E}}$ is constructed from the formula $\hat{\mathscr{E}}x = \phi^{-1}\mathscr{F}(\mathscr{E}\phi x)$ (for a fixed $\phi \in \mathrm{L}^2(\mathbf{R};\mathbf{C})$ such that $\widehat{\phi} \ne 0$ on $\mathbf{R}$).

References

[FS55] Foures, Y., and I. E. Segal: Causality and analyticity. - Trans. Amer. Math. Soc. 78, 1955, 385–405.

[Gar81] Garnett, J. B.: Bounded analytic functions. - Academic Press, 1981.

[HP57] Hille, E., and R. S. Phillips: Functional analysis and semi-groups. - Revised edition, AMS, Providence, 1957.

[Hyt03] Hytönen, T.: Translation-invariant operators on spaces of vector-valued funcions. - Doctoral dissertation, Helsinki University of Technology, 2003, `www.math.hut.fi/reports/`.

[2]There, in (3.5), the first "$= \widehat{f}$" should be "$= \Lambda\widehat{f}$", four lines earlier, "for $t \in A_t$" should be "for $u \in A_t$", and four lines later, Ff should be $F\widehat{f}$ and, on the following line, "$Ffu = Tfu$ to $Fg = Tg$" should be "$F\widehat{f}u = \widehat{\mathscr{E}fu}$ to $F\widehat{g} = \widehat{\mathscr{E}g}$".

[Mik02] Mikkola, K. M.: Infinite-dimensional linear systems, optimal control and algebraic Riccati equations. - Doctoral dissertation, technical report A452, Institute of Mathematics, Helsinki University of Technology, Espoo, Finland, 2002, `www.math.hut.fi/~kmikkola/research/`.

[Mik05] Mikkola, K. M.: Hankel and Toeplitz operators on nonseparable Hilbert spaces. - Submitted, 2005.

[Mik06] Mikkola, K. M.: Characterization of transfer functions of Pritchard-Salamon or other realizations with a bounded input or output operator. - Integral Equations Operator Theory 54:3, 2006, 427–440.

[RR85] Rosenblum, M., and J. Rovnyak: Hardy classes and operator theory. - Oxford University Press, New York, Oxford, 1985.

[Rud87] Rudin, W.: Real and complex analysis. - Third edition, international ed., McGraw-Hill Book Company, Singapore, 1987.

[Tho97] Thomas, E. G. F.: Vector-valued integration with applications to the operator-valued H^∞ space. - IMA J. Math. Control Inform. 14, 1997, 109–136.

[Wei91] Weiss, G.: Representations of shift-invariant operators on L^2 by H^∞ transfer functions: an elementary proof, a generalization to L^p, and a counterexample for L^∞. - Math. Control Signals Systems 4, 1991, 193–203.

[Wei01] Weis, L.: Operator-valued Fourier multiplier theorems and maximal L_p-regularity. - Math. Ann. 319:4, 2001, 735–758.

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