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TOPOLOGICAL EQUIVALENCE OF METRICS IN TEICHMÜLLER SPACE

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Abstract. For d_T, d_L and d_{p_i}, $i = 1, 2$, the Teichmüller metric, the length spectrum metric and the Thurston's pseudo-metrics on Teichmüller space $T(X)$, we first give some estimations of the above (pseudo)metrics on the thick part of $T(X)$. Then we show that there exist two sequences $\{\tau_n\}_{n=1}^{\infty}$ and $\{\tilde{\tau}_n\}_{n=1}^{\infty}$ in $T(X)$, such that as $n \to \infty$, $d_L(\tau_n, \tilde{\tau}_n) \to 0$, $d_{P_1}(\tau_n, \tilde{\tau}_n) \to 0$, $d_{P_2}(\tau_n, \tilde{\tau}_n) \to 0$, while $d_T(\tau_n, \tilde{\tau}_n) \to \infty$. As an application, we give a proof that for certain topologically infinite type Riemann surface X, d_L, d_{P_1} and d_{P_2} are not topologically equivalent to d_T on $T(X)$, a result originally proved by Shiga [18]. From this we obtain a necessary condition for the topological equivalence of d_T to any one of d_L, d_{P_1} and d_{P_2} on $T(X)$.

0. Introduction

For any non-elementary Riemann surface X and any quasiconformal mapping $f\colon X \to X_0$, we denote the pair (X_0, f) a marked Riemann surface. Two marked Riemann surfaces (X_1, f_1) and (X_2, f_2) are equivalent if there is a conformal mapping $c\colon X_1 \to X_2$ which is homotopic to $f_2 \circ f_1^{-1}$. Denote $[X, f]$ to be the equivalent class of (X, f). The Teichmüller space $T(X)$ is the set of the equivalent classes $[X, f]$.

As we know, Teichmüller gave a metric on $T(X)$:

$$d_T([X_1, f_1], [X_2, f_2]) = \log\{\inf K(f_0)\},$$

where the infimum is taken over all $f_0\colon X_1 \to X_2$ in the homotopic class of $f_2 \circ f_1^{-1}$, and $K(f_0)$ is its dilatation.

Let $\Sigma_X = \{\gamma_i\}$ be the set of the representations of elements (not including the unit element) of the fundamental group $\pi_1(X)$ of surface X. Let $l_X(\gamma)$ be the shortest length under the Poincaré metric (hyperbolic length) in the freely homotopic class of closed curve γ on the Riemann surface X. $l_X(\gamma)$ is also called the Poincaré length or hyperbolic length of γ. The sequence $\{l_X(\gamma_j)\}$ corresponding to Σ_X is called the length spectrum of Riemann surface X [1]. Let Σ'_X be the set of homotopic classes of curves in Σ_X which are not homotopic to a puncture and Σ''_X

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be the set of homotopic classes of simple closed curves in Σ'_X. There is a metric on $T(X)$ using length spectrum of Riemann surface [19]:

$$d_L([X_1, f_1], [X_2, f_2]) = \log \rho([X_1, f_1], [X_2, f_2]),$$

where

$$\rho([X_1, f_1], [X_2, f_2]) = \sup_{\gamma \in \Sigma'_{X_1}} \left\{ \frac{l_{X_2}(f(\gamma))}{l_{X_1}(\gamma)}, \frac{l_{X_1}(\gamma)}{l_{X_2}(f(\gamma))} \right\},$$

and $f = f_2 \circ f_1^{-1}$. This metric is called the length spectrum metric of the Teichmüller space $T(X)$. Thurston's pseudo-metrics d_{P_1} and d_{P_2} are also defined by the length spectrum of Riemann surface as follows [21]:

$$d_{P_1}([X_1, f_1], [X_2, f_2]) = \log \sup_{\alpha \in \Sigma'_{X_1}} \frac{l_{X_2}(f(\alpha))}{l_{X_1}(\alpha)},$$

$$d_{P_2}([X_1, f_1], [X_2, f_2]) = \log \sup_{\alpha \in \Sigma'_{X_1}} \frac{l_{X_1}(\alpha)}{l_{X_2}(f(\alpha))},$$

where $f = f_2 \circ f_1^{-1}$.

By Thurston's result [21], we know that

$$d_L([X_1, f_1], [X_2, f_2]) = \log \rho'([X_1, f_1], [X_2, f_2]),$$

where

$$\rho'([X_1, f_1], [X_2, f_2]) = \sup_{\gamma \in \Sigma''_{X_1}} \left\{ \frac{l_{X_2}(f(\gamma))}{l_{X_1}(\gamma)}, \frac{l_{X_1}(\gamma)}{l_{X_2}(f(\gamma))} \right\},$$

and $f = f_2 \circ f_1^{-1}$.

$$d_{P_1}([X_1, f_1], [X_2, f_2]) = \log \sup_{\alpha \in \Sigma''_{X_1}} \frac{l_{X_2}(f(\alpha))}{l_{X_1}(\alpha)},$$

$$d_{P_2}([X_1, f_1], [X_2, f_2]) = \log \sup_{\alpha \in \Sigma''_{X_1}} \frac{l_{X_1}(\alpha)}{l_{X_2}(f(\alpha))},$$

where $f = f_2 \circ f_1^{-1}$.

In [15,16], Papadopoulos called d_{p_i}, $i = 1, 2$, Thurston's asymmetric metrics.

When X if of conformally finite type, as functions on $T(X)$, d_{P_1} and d_{P_2} satisfy:

(a) $d_{p_i}([X_1, f_1], [X_2, f_2]) = 0$ if and only if $[X_1, f_1] = [X_2, f_2]$, $i = 1, 2$.
(b) $d_{p_i}([X_1, f_1], [X_3, f_3]) \leq d_{p_i}([X_1, f_1], [X_2, f_2]) + d_{p_i}([X_2, f_2], [X_3, f_3])$, $i = 1, 2$.

But in general the equalities $d_{p_i}([X_1, f_1], [X_2, f_2]) = d_{p_i}([X_2, f_2], [X_1, f_1])$, $i = 1, 2$ are not true [21]. Therefore d_{P_1} and d_{P_2} are pseudo-metrics on $T(X)$ and are not metrics on $T(X)$ and d_{p_1} is different from d_{P_2}. We know from the definition that $d_{P_1}([X_1, f_1], [X_2, f_2]) = d_{P_2}([X_2, f_2], [X_1, f_1])$ and $d_{p_i}([X_1, f_1], [X_2, f_2]) \leq d_L([X_1, f_1], [X_2, f_2])$.

Actually d_{P_1} and d_{P_2} are not strictly pseudo-metrics. For the terminology "pseudo-metric" has the following standard meaning: a non-negative function ρ on $X \times X$ defines a pseudo-metric on X if it satisfies:

$$\rho(x,x) = 0, \quad \rho(x,y) = \rho(y,x), \text{ and}$$
$$\rho(x,z) \leq \rho(x,y) + \rho(y,z) \quad \text{for all } x, y \text{ and } z.$$

(But we may have $\rho(x,y) = 0$ and $x \neq y$.) For the sake of convenience, d_{P_1} and d_{P_2} are still called pseudo-metrics.

The pseudo-metrics d_{p_i}, $i = 1, 2$, have a close relation with mapping between surfaces. In the case of X being of topologically finite type, we know: [21] for any $\tau_1, \tau_2 \in T(X)$, there exist (not necessarily unique) measured foliations F_i ($i = 1, 2$) such that the ratio of the lengths of the measured foliations $F_i(i = 1, 2)$ on these two surfaces is equal to d_{p_i}, $i = 1, 2$, respectively. The extremal mappings that keep F_i, $i = 1, 2$, and realize the pseudo-metrics d_{p_i}, $i = 1, 2$, are Thurston stretch mappings. This is similar to the Teichmüller mapping: for any $\tau_1, \tau_2 \in T(X)$, there exists a unique measured foliation and the corresponding Teichmüller mapping that realize the Teichmüller distance. For another hand, d_{p_i}, $i = 1, 2$, are closely connected with the Lipschitz constants of mappings between surfaces.

The following lemma of Wolpert [1] is well known.

Lemma 1. *Let $f\colon X_1 \to X_2$ be a quasiconformal mapping between hyperbolic Riemann surfaces. Then*

$$\frac{l_{X_2}(f(\alpha))}{l_{X_1}(\alpha)} \leq K(f)$$

holds for all $\alpha \in \Sigma'_{X_1}$.

From this lemma, we get immediately that

$$d_L \leq d_T, \quad d_{p_i} \leq d_T, \quad i = 1, 2.$$

Let d_1 and d_2 be two (pseudo)metrics on set P.

(1) We call d_1 is topologically equivalent to d_2 if for sequence $\{t_n\}_{n=0}^{\infty} \subset P$, $\lim_{n\to\infty} d_1(t_n, t_0) = 0$ if and only if $\lim_{n\to\infty} d_2(t_n, t_0) = 0$.

(2) We call d_1 is quasi-isometric to d_2 if there exists $K > 0$ such that

$$\frac{1}{K} d_1(x,y) \leq d_2(x,y) \leq K d_1(x,y)$$

for any $x, y \in P$.

The study of the relations of various metrics or pseudo-metrics on $T(X)$ is very interesting. In 1972, Sorvali [19] defined and studied the length spectrum metric and asked the following problem: Whether the Teichmüller metric d_T is topologically equivalent to the length spectrum metric d_L for Teichmüller space of topologically finite Riemann surface? In 1975, Sorvali [20] solved this problem for tori. In 1986, Li [7] gave a positive answer to this question for the Teichmüller space of compact Riemann surface. In 1999, Liu [9] proved that the Teichmüller metric d_T is topologically equivalent to d_L for the Teichmüller space of topologically finite Riemann surface. This result gave an affirm answer to Sorvali's problem and Liu [9] asked the problem that whether d_T is topologically equivalent to d_L in the Teichmüller space of infinite topological type Riemann surface. Shiga [18] gave a negative answer to this question. In 2003, Shiga [18] constructed an example to show that in the Teichmüller space of certain Riemann surface of infinite topological type, the Teichmüller metric d_T is not topologically equivalent to the length spectrum metric d_L. And Shiga [18] gave a sufficient condition for the topological equivalence of d_T and d_L on $T(X)$. Liu [11] also showed that the metrics d_T, d_L and the pseudo-metrics d_{p_i}, $i = 1, 2$ are topologically equivalent to each other in the Teichmüller space of topologically finite Riemann surface. Recently Papadopoulos and Théret [15,16] proved the same result. Actually Papadopoulos and Théret [15,16] have obtained many results about Thurston's pseudometrics.

On the other hand, many authors studied the quasi-isometric equivalence of the above metrics and pseudo-metrics. Thurston [21] (see also [12]) showed that the Thurston's pseudo-metrics are asymmetry, that is $d_1 \neq d_2$. Liu [12] proved that d_{P_1} is not quasi-isometric to d_{P_2}. This also implies that d_L is not quasi-isometric to d_{p_i}, $i = 1, 2$. Liu [10] also showed that d_T is not quasi-isometric to d_{p_i}, $i = 1, 2$. In 2003, Li [8] proved that d_T is not quasi-isometric to d_L. Actually Li proved that there exists two sequences of points $\{\tau_n\}$ and $\{\tau_n'\}$ in $T(X)$ (X is a compact Riemann surface), such that $\lim_{n\to\infty} d_L(\tau_n, \tau_n') = 0$ while $\lim_{n\to\infty} d_T(\tau_n, \tau_n') > d_0$ where d_0 is a constant.

In this paper, we will prove the following results: In Section 1, we'll give some estimations of d_T, d_L, d_{p_i}, $i = 1, 2$, on the thick part of $T(X)$. In Section 2, we will show that (Theorem 4) for the Teichmüller space $T(X)$, here X is of finite topological type or infinite topological type, there exist two sequences $\{\tau_n\}_{n=1}^{\infty}$ and $\{\tilde{\tau}_n\}_{n=1}^{\infty}$ in $T(X)$, such that as $n \to \infty$, $d_L(\tau_n, \tilde{\tau}_n) \to 0$, $d_{P_1}(\tau_n, \tilde{\tau}_n) \to 0$, $d_{P_2}(\tau_n, \tilde{\tau}_n) \to 0$, while $d_T(\tau_n, \tilde{\tau}_n) \to \infty$. In Section 3, for Riemann surface X of infinite topological type, if there exists simple closed curves $\{\alpha_n\}$ on X with $\lim_{n\to\infty} l_X(\alpha_n) = 0$, using Theorem 4, we can take points σ_n ($n = 0, 1, 2, \ldots$) in $T(X)$ such that as $n \to \infty$, $d_L(\sigma_n, \sigma_0) \to 0$, $d_{P_1}(\sigma_n, \sigma_0) \to 0$, $d_{P_2}(\sigma_n, \sigma_0) \to 0$, while $d_T(\sigma_n, \sigma_0) \to \infty$, where $\sigma_0 \in T(X)$. This gives a new proof that in $T(X)$, d_T is not topologically equivalent to d_L which originally was proved by Shiga [18]. And from this we obtain a necessary condition for the topological equivalence of d_T and d_L in $T(X)$.

Notations. In the sequel, we use the following notations: $\sharp(\beta, \gamma)$ denotes the geometric intersection number of curves β with γ, this is exactly the least number of points of intersection of c_1 with c_2, where c_1 and c_2 represent curves in the homotopic class of β and γ, respectively. Denote $\Re z$ to be the real part of z. $\mathrm{Mod}(H(A_1, A_2, A_3, A_4))$ represents the conformal modulus of a topological quadrilateral $H(A_1, A_2, A_3, A_4)$ formed by the upper half plane H with four vertexes A_1, A_2, A_3, and A_4. All surfaces in this paper are non-elementary hyperbolic Riemann surfaces.

1. Estimations on the thick part

The Riemann surface in this section is of conformal finite type (g, m), here g is the number of genus and m is the number of punctures. Let $QD(X)$ be the space of integrable holomorphic quadratic differentials on Riemann surface X and $PQD(X)$ be the set of its projective classes [5]. As we know, the real dimension of $QD(X)$ is $6g-6+2m$ and that of $PQD(X)$ is $6g-7+2m$ [4,5]. Any $\phi \in QD(X)$ determines a pair of tranverse measured foliations. These are the horizontal trajectory together with its vertical measure and the vertical trajectory together with its horizontal measure. Let MF be the space of measured foliations of surface of genus g, and PMF be the set of its projective classes. We know that the real dimension of MF and PMF are $6g-6+2m$ and $6g-7+2m$, respectively. $PQD(X)$ and PMF may be viewed as the unit sphere in $QD(X)$ and MF, respectively. Therefore $PQD(X)$ and PMF are compact subset of $QD(X)$ and MF, respectively.

We have the following mapping [4,5],

$$H\colon QD(X) \to MF,$$

where H maps ϕ to its horizontal trajectory together with its vertical measure. We know that H is a homeomorphism.

Extremal length of simple closed curve is a very powerful tool in complex analysis. Kerckhoff [5] generalized the definition of extremal lengths of simple closed curves to measured foliations. The extremal mapping in the metric d_T between any two points in the Teichmüller space can be realized by a unique measured foliation and this measured foliation is the horizontal foliation of the corresponding Teichmüller differential together with its vertical measure.

For any $\alpha \in \Sigma''_X$, we can define its extremal length $E_X(\alpha)$ [5]. Actually any $\alpha \in \Sigma''_X$ may be viewed as a measured foliation [5]. For any $F \in MF$, $E_X(F)$ is realized by the metric determined by the holomorphic quadratic differential $H^{-1}(F)$ [4]. We know that the set of measured foliations of simple quadratic differentials are dense in the set of measured foliations. The following theorem is a natural generalization of a result of Kerckhoff [5,11,14].

Lemma 2. *For any two points $[X_1, f_1]$ and $[X_2, f_2]$ in $T(X)$, we have*

$$d_T([X_1, f_1], [X_1, f_2]) = \frac{1}{2} \log \sup_{\alpha \in \Sigma''_{X_1}} \frac{E_{X_1}(\alpha)}{E_{X_2}(f(\alpha))},$$

where $f = f_2 \circ f_1^{-1}$.

This result for compact Riemann surface was obtained by Kerckhoff [5]. Because the proof of the above theorem is the same as that of Kerckhoff, we omit the details. On the other hand, we don't know for any non-conformal-finite type Riemann surface whether the above result remains valid.

For any $F \in MF$, as a generalization of the Poincaré length of simple closed curve, we may define its Poincaré length $l_X(F)$ [15]. By the definition of extremal length and the Gauss–Bonnet theorem, we have [14]

Lemma 3. *For any $F \in MF, F \neq 0$ and any Riemann surface S, the following inequality holds.*

$$\frac{E_S(F)}{l_S^2(F)} \geq \frac{1}{2\pi|\chi(S)|},$$

where $\chi(S)$ is the Euler characteristic number of S.

Let Mod be the moduli space of Riemann surfaces of type (g, m), here g is the number of genus and m is the number of punctures and MCG be the corresponding mapping class group. For any $\varepsilon > 0$, let $M_\varepsilon \subset Mod$ be the set of Riemann surfaces with the property that the hyperbolic length of any non-trivial simple closed curve which is not homotopic to a puncture is not less than ε. By Mumford's compactness theorem we know that M_ε is a compact subset of Mod. Let $T_\varepsilon(X) \subset T(X)$ be the set of $[X_1, f_1]$ where X_1 satisfies the property that the hyperbolic length of any simple closed curve which is not homotopic to a puncture is greater than ε. We call $T_\varepsilon(X)$ the ε-thick part of the Teichmüller space $T(X)$, and $T(X) - T_\varepsilon(X)$ the ε-thin part of $T(X)$.

Similar to the discussions in [9,11]. The Poincaré length and the extremal length of measured foliation have the following relation.

Theorem 1. *For Riemann surfaces S in M_ε, there exist constants $M_1(g, m, \varepsilon)$ and $M_2(g, m, \varepsilon)$, depending only on g, m, and ε, such that for any $F \in MF, F \neq 0$, the following inequality holds:*

$$M_1(g, m, \varepsilon) \leq \frac{E_S(F)}{l_S^2(F)} \leq M_2(g, m, \varepsilon).$$

Proof. Let $G(S, F) = \frac{E_S(F)}{l_S^2(F)}$. As functions defined on MF, $E_S(F)$ and $l_S^2(F)$ are continuous, and take positive values in $MF - \{0\}$. For any $r > 0$, we have [5]

$$E_S(rF) = r^2 E_S(F), \quad l_S^2(rF) = r^2 l_S^2(F).$$

Therefore the function $G(S, F)$ is a positive continuous function on compact set $M_\varepsilon \times PMF$. So it can attain its maximum and minimum. Denote them by $M_2(g, m, \varepsilon)$ and $M_1(g, m, \varepsilon)$, respectively. This completes the proof of the theorem. Here $M_1(g, m, \varepsilon)$ and $M_2(g, m, \varepsilon)$ depend only on g, m, and ε. □

From Lemma 3 we know that $M_1(g, m, \varepsilon) \geq \frac{1}{2\pi|\chi(X)|}$.

Theorem 2. *For any $[X_1, f_1], [X_2, f_2]$ in $T_\varepsilon(X)$, there exists a constant $M(g, m, \varepsilon)$ which depends only on g, m, ε, such that*

$$d_T([X_1, f_1], [X_2, f_2]) \leq 4d_{P_2}([X_1, f_1], [X_2, f_2]) + M(g, m, \varepsilon).$$

Proof. Because any simple closed curve can be viewed as a measued foliation [5], from Theorem 1, for any $\alpha \in \Sigma''_X$,

$$\frac{E_{X_1}(f_1(\alpha))}{E_{X_2}(f_2(\alpha))} \leq \frac{M_2(g, m, \varepsilon)}{M_1(g, m, \varepsilon)} \frac{l_{X_1}^2(f_1(\alpha))}{l_{X_2}^2(f_2(\alpha))}.$$

Then by Lemma 2 and the definitions,

$$d_T([X_1,f_1],[X_2,f_2]) \le 4d_{P_2}([X_1,f_1],[X_2,f_2]) + \log\frac{M_2(g,m,\varepsilon)}{M_1(g,m,\varepsilon)}.$$

Taking $M(g,m,\varepsilon)=\log\frac{M_2(g,m,\varepsilon)}{M_1(g,m,\varepsilon)}$, we finish the proof of Theorem 2. □

Similar to the above discussion, we have

Theorem 3. *For any* $[X_1,f_1],[X_2,f_2]$ *in* $T_\varepsilon(X)$, *there exist constants* $N_i(g,m,\varepsilon)$ $(i=1,2,\dots,5)$ *which depend only on* g,m,ε, *such that*

$$\begin{aligned}
d_T([X_1,f_1],[X_2,f_2]) &\le 4d_{P_1}([X_1,f_1],[X_2,f_2]) + N_1(g,m,\varepsilon),\\
d_T([X_1,f_1],[X_2,f_2]) &\le 4d_L([X_1,f_1],[X_2,f_2]) + N_2(g,m,\varepsilon),\\
d_{P_1}([X_1,f_1],[X_2,f_2]) &\le d_{P_2}([X_1,f_1],[X_2,f_2]) + N_3(g,m,\varepsilon),\\
d_{P_2}([X_1,f_1],[X_2,f_2]) &\le d_{P_1}([X_1,f_1],[X_2,f_2]) + N_4(g,m,\varepsilon),\\
d_L([X_1,f_1],[X_2,f_2]) &\le d_{p_i}([X_1,f_1],[X_2,f_2]) + N_5(g,m,\varepsilon), \quad i=1,2.
\end{aligned}$$

From Theorem 4 in the next section, we know that $M(g,m,\varepsilon)$, $N_i(g,m,\varepsilon)$, $i=1,\dots,5$, tends to ∞ as ε tending to zero.

From Theorem 2 and Theorem 3, we can prove that d_T, d_L, d_{p_i}, $i=1,2$, are topologically equivalent to each other on $T(X)$, where X is of conformal finite type $g,m)$ [7,9,11,15,16].

2. An example in the thin part

In this section, the Riemann surface is of any non-elementary type, finite topological type or infinite topological type. We'll study examples in the thin part of Teichmüller spaces.

Theorem 4. *There exist two sequences* $\{\tau_n\}_{n=1}^\infty$ *and* $\{\tilde\tau_n\}_{n=1}^\infty$ *in* $T(X)$, *such that as* $n\to\infty$, *we have*

$$\begin{aligned}
d_L(\tau_n,\tilde\tau_n) &\to 0,\\
d_{P_1}(\tau_n,\tilde\tau_n) &\to 0,\\
d_{P_2}(\tau_n,\tilde\tau_n) &\to 0,
\end{aligned}$$

while

$$d_T(\tau_n,\tilde\tau_n)\to\infty.$$

Proof. Let $\tau_n=[X_n,f_n]$ with $l_{X_n}(f_n(\alpha))=\varepsilon_n\to 0$, as $n\to\infty$, where $\alpha\in\Sigma''_X$. Take

$$t_n=\left[\frac{\log|\log\varepsilon_n|}{\varepsilon_n}\right]+1, \quad n=1,2,\dots. \tag{1}$$

Denote $\gamma_n=f_n(\alpha)$. Let g_n be the positive t_n times Dehn twist about γ_n, $n=1,2,\dots$. Here "positive" Dehn twist means the Dehn twist with left turning.

Take $\lambda > 0$ such that a neighborhood U_n of γ_n which is defined by

$$U_n = \{z \in X_n : d_{X_n}(z, \gamma_n) < \lambda\}$$

where $d_{X_n}(\cdot,\cdot)$ is the hyperbolic distance on X_n, is conformally equivalent to an annulus. Define the Dehn twist g_n such that $g_n|U_n$ is the standard t_n times Dehn twist on the annulus and the identity on $X_n - U_n$. That is, $g_n|U_n$ is defined in terms of the polar coordinates in the annulus by

$$r\exp(i\theta) \to r\exp\left\{i\left(\theta + 2\pi t_n \frac{r-1}{R-1}\right)\right\},$$

if U_n is conformally equivalent to $\{z : 1 < |z| < R\}$.

Let β_n be a geodesic in X_n perpendicular to γ_n. We consider connected components of $\pi^{-1}(U_n)$, $\pi^{-1}(\gamma_n)$ and $\pi^{-1}(\beta_n)$ on H, where $\pi\colon H \to X_n$ is a covering map. We may further assume that the connected component of $\pi^{-1}(\gamma_n)$ is the positive half of the imaginary axis and that of $\pi^{-1}(\beta_n)$ is $\delta = \{z \in H : |z| = 1\} \cap H$. Let $\tilde{U}_n$ be the connected component of $\pi^{-1}(U_n)$ containing the positive half of the imaginary axis.

Let F_n be the lift of an extremal quasiconformal mapping in the homotopic class of g_n, normalized by $F_n(0) = 0, F_n(i) = i$ and $F_n(\infty) = \infty$. Well known that F_n can be extended to a homeomorphism of $\overline{H}$ and that the boundary mapping $F_n|_R$ depends only on the homotopic class of f_n up to an automorphism of H.

Denote $\tilde{\tau}_n = [X_n, g_n \circ f_n]$ and $g_n(\tau_n) = \tilde{\tau}_n$.

First, we consider the dilation $K(F_n)$ of F_n.

Let z_1, z_2 ($\Re z_1 < 0 < \Re z_2$) be the points of $\delta \cap \tilde{U}_n$. Then $F_n(z_1) = z_1,\ F_n(z_2) = e^{t_n\varepsilon_n} z_2$, since f_n is the positive t_n time Dehn twist. Hence $F_n(\delta \cap \tilde{U}_n)$ is an arc connecting z_1 and $e^{t_n\varepsilon_n} z_2$ in the connected component $\tilde{U}_n$. Applying a similar argument to a subarc of δ in each component of $\pi^{-1}(U_n)$, we get

$$-1 < F_n(-1) < 0 < e^{t_n\varepsilon_n} < F_n(1).$$

Therefore, for the cross ratio

$$[a, b, c, d] = \frac{(a-b)(c-d)}{(a-d)(c-b)},$$

we have

$$[-1, 0, 1, \infty] = -1, \quad \mathrm{Mod}(H(-1, 0, 1, \infty)) = 1. \tag{2}$$

Denote

$$\nu_n = [F_n(-1), F_n(0), F_n(1), F_n(\infty)] = [F_n(-1), 0, F_n(1), \infty] = \frac{F_n(-1)}{F_n(1)}.$$

Then

$$|\nu_n| = |\frac{F_n(-1)}{F_n(1)}| \leq \frac{1}{e^{t_n\varepsilon_n}} \leq \frac{1}{\log|\log\varepsilon_n|} \to 0, \quad n \to \infty.$$

Therefore, the conformal modulus

$$\mathrm{Mod}(H(F_n(-1), F_n(0), F_n(1), F_n(\infty))) \to 0, \quad \text{as } n \to \infty. \tag{3}$$

By the geometric definition of quasiconformal mappings, we get

$$\frac{1}{K(F_n)} \leq \frac{\mathrm{Mod}(H(F_n(-1), F_n(0), F_n(1), F_n(\infty)))}{\mathrm{Mod}(H(-1,0,1,\infty))}. \tag{4}$$

To sum up, (2), (3) and (4) give

$$K(F_n) \to \infty, \quad n \to \infty. \tag{5}$$

This implies that $\lim_{n\to\infty} d_T(\tau_n, \tilde{\tau}_n) = \infty$.

Let α' be any closed curve in X_n. If $\sharp(\alpha', \gamma_n) = 0$, then $l_{X_n}(f_n(\alpha')) = l_{X_n}(\alpha')$. If $\sharp(\alpha', \gamma_n) \neq 0$, then by a version of the collar Lemma [14], we have

$$l_{X_n}(\alpha') = A|\log \varepsilon_n| + B, \quad A = \sharp(\alpha', \gamma_n) \geq 1, \quad B > 0.$$

Then, by the definition of Dehn twist, one obtains

$$A|\log\varepsilon_n| + B - At_n\varepsilon_n \leq l_{X_n}(f_n(\alpha')) \leq A|\log\varepsilon_n| + B + At_n\varepsilon_n. \tag{6}$$

By (1) and (6), we have

$$\begin{aligned} \frac{l_{X_n}(f_n(\alpha'))}{l_{X_n}(\alpha')} &\leq \frac{A|\log\varepsilon_n| + B + At_n\varepsilon_n}{A|\log\varepsilon_n| + B} = 1 + \frac{At_n\varepsilon_n}{A|\log\varepsilon_n| + B} \\ &\leq 1 + \frac{\log|\log\varepsilon_n| + 2\varepsilon_n}{|\log\varepsilon_n| + \frac{B}{A}} \to 1, \quad n \to \infty, \end{aligned} \tag{7}$$

and

$$\begin{aligned} \frac{l_{X_n}(f_n(\alpha'))}{l_{X_n}(\alpha')} &\geq \frac{A|\log\varepsilon_n| + B - At_n\varepsilon_n}{A|\log\varepsilon_n| + B} = 1 - \frac{At_n\varepsilon_n}{A|\log\varepsilon_n| + B} \\ &\geq 1 - \frac{\log|\log\varepsilon_n| + 2\varepsilon_n}{|\log\varepsilon_n| + \frac{B}{A}} \to 1, \quad n \to \infty. \end{aligned} \tag{8}$$

Therefore (7) and (8) give

$$\lim_{n\to\infty} \frac{l_{X_n}(f_n(\alpha'))}{l_{X_n}(\alpha')} = 1. \tag{9}$$

Note that α' is any closed curve in X_n.

Combining with the definitions, we have

$$d_{P_1}(\tau_n, \tilde{\tau}_n) \to 0, \quad d_{P_2}(\tau_n, \tilde{\tau}_n) \to 0, \quad d_L(\tau_n, \tilde{\tau}_n) \to 0, \quad n \to \infty.$$

This completes the proof of Theorem 4. □

In [7], Li proved the following inequality

$$d_L(\tau_1, \tau_2) \leq d_T(\tau_1, \tau_2) \leq 2d_L(\tau_1, \tau_2) + C(\tau_1)$$

holds for any two points $\tau_1, \tau_2 \in T(S_0)$, where $C(\tau_1)$ is a constant depending on τ_1 and S_0 is compact Riemann surface. Li [8] proved, for compact Riemann surface X, there exist two sequences of points $\{\tau_n\}$ and $\{\tau_n'\}$ in $T(X)$, such that $\lim_{n\to\infty} d_L(\tau_n, \tau_n') = 0$ while $\lim_{n\to\infty} d_T(\tau_n, \tau_n') > d_0$, where d_0 is a constant. Theorem 4 is a slight improvement of Li's result. We remark that Theorem 4 holds for any non-elementary Riemann surface, finite or infinite topological type.

3. Topological equivalence for infinite topological type

In this section, we give an application of Theorem 4 which is about the topological equivalence of (pseudo)metrics in Teichmüller space of infinite topological type Riemann surface. The main result in this section is the following

Theorem 5. *Let X be a Riemann surface of infinite topological type such that there exists a sequence of simple closed curves α_n, $n = 1, 2, \ldots$, $\alpha_n \in \Sigma''_X$ with $\lim_{n\to\infty} l_X(\alpha_n) = 0$. Then for any point $\tau_0 \in T(X)$, there exists a sequence of points $\{\sigma_n\}_{n=1}^{\infty}$ in $T(X)$ such that*

$$d_L(\sigma_n, \tau_0) \to 0, \quad \text{as } n \to \infty,$$
$$d_{p_i}(\sigma_n, \tau_0) \to 0, \quad \text{as } n \to \infty, \quad i = 1, 2,$$

while

$$d_T(\sigma_n, \tau_0) \to \infty, \quad \text{as } n \to \infty.$$

Proof. From Lemma 1 we know that for any $\tau_0 = [X_0, f_0] \in T(X)$, there exists a sequence of simple closed curves α_n, $n = 1, 2, \ldots$, on X_0 with $lim_{n\to\infty} l_{X_0}(\alpha_n) = 0$. As in the proof of Theorem 4, let $l_{X_0}(\alpha_n) = \varepsilon_n \to 0$, as $n \to \infty$. Take $t_n = [\frac{\log|\log \varepsilon_n|}{\varepsilon_n}] + 1$, $n = 1, 2, \ldots$. Let g_n be the positive t_n times Dehn twist about α_n, $n = 1, 2, \ldots$. Denote $\sigma_n = [X_0, g_n \circ f_0]$. Then as in the proof of Theorem 4,

$$d_L(\sigma_n, \tau_0) \to 0, \quad \text{as } n \to \infty,$$
$$d_{p_i}(\sigma_n, \tau_0) \to 0, \quad \text{as } n \to \infty, \quad i = 1, 2,$$

while

$$d_T(\sigma_n, \tau_0) \to \infty, \quad \text{as } n \to \infty. \qquad \square$$

Corollary 1. *Let X be a Riemann surface of infinite topological type such that there exists a sequence of simple closed curves α_n, $n = 1, 2, \ldots$, $\alpha_n \in \Sigma''_X$ with $\lim_{n\to\infty} l_X(\alpha_n) = 0$. Then in the Teichmüller space $T(X)$, d_T is not topologically equivalent to any one of d_L, d_{p_i}, $i = 1, 2$.*

Sometimes we say two (pseudo)metrics defining the same topology if they are topologically equivalent.

Shiga [18] proved that in the Teichmüller space $T(X)$ of certain infinite topological type Riemann surface X, there exists a sequence of points $\{\sigma_n\}_{n=1}^{\infty}$ in $T(X)$ such that

$$d_L(\sigma_n, id) \to 0, \quad \text{as } n \to \infty,$$

while

$$d_T(\sigma_n, id) \to \infty, \quad \text{as } n \to \infty.$$

This implies that d_T is not topologically equivalent to d_L in $T(X)$. His proof is based on constructing some examples. He constructed a Riemann surface X such that there is a sequence of simple closed curves α_n on X with $\lim_{n\to\infty} l_X(\alpha_n) = \infty$. And for any closed curve β intersects α_n, the ratio $\frac{l_X(\beta)}{l_X(\alpha_n)}$ is very large. Then by taking a Dehn twist about α_n, he got a sequence of points τ_n in $T(X)$. Because

$l_X(\alpha_n) \to \infty$, the dilatation of the Dehn twist about α_n will tend to ∞. This implies that $d_T(\tau_0, \tau_n) \to \infty$, here $\tau_0 = [X, \mathrm{id}]$. While for any closed curve β interesting α_n, because $\frac{l_X(\beta)}{l_X(\alpha_n)}$ is relatively very large, the effect of the Dehn twist about α_n to the hyperbolic length of β is very small. From this Shiga showed that $d_L(\tau_0, \tau_n) \to 0$. Our prove is a litter different from that of Shiga. And it seems a litter more natural.

Corollary 2. *For the Teichmüller space $T(X)$, a necessary condition that d_T is topologically equivalent to any one of d_L, d_{p_i}, $i = 1, 2$, is that there exists a constant $c > 0$, such that for any $\alpha \in \Sigma''_X$, $l_X(\alpha) \geq c$.*

We call the condition in Corollary 2 lower injective radii condition. Shiga [18] proved the following result.

Lemma 4. *Let X be a Riemann surface. Assume that there exists a pants decomposition $X = \bigcup_{k=1}^{\infty} P_k$ of X satisfying the following conditions.*

(1) *Each connected component of ∂P_k is either a puncture or a simple closed geodesic of X, $k = 1, 2, \ldots$.*
(2) *There exists a constant $M > 0$ such that if α is a boundary curve of some P_k, then*
$$0 < M^{-1} < l_X(\alpha) < M$$
holds.

Then d_L defines the same topology as that of d_T on the Teichmüller space $T(X)$ of X.

Lemma 4 gives a sufficient condition for d_T and d_L define the same topology. We call it Shiga's condition.

From Theorem 5 and Lemma 4, we know that Shiga's condition implies lower injective radii condition. But from Shiga's example [18] we know that there are Riemann surfaces X of infinite topological type which satisfy lower injective radii condition, but d_T is not topologically equivalent to d_L on $T(X)$. So in this case X does not satisfy Shiga's condition.

We don't know whether Shiga's condition is also a necessary condition.

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