Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 33, 2008, 171–204

SUPERHARMONIC FUNCTIONS AND DIFFERENTIAL EQUATIONS INVOLVING MEASURES FOR QUASILINEAR ELLIPTIC OPERATORS WITH LOWER ORDER TERMS

Takayori Ono

Fukuyama University
Gakuencho, Fukuyama, 729-0292 Japan; ono@fuhc.fukuyama-u.ac.jp

Dedicated to Professor Yoshihiro Mizuta for his sixtieth birthday.

Abstract. We consider superharmonic functions relative to a quasi-linear second order elliptic differential operator L with lower order term and weighted structure conditions. We show that, given a nonnegative finite Radon measure ν, there is a superharmonic function u satisfying $Lu = \nu$ with weak zero boundary values. Moreover, we give a pointwise upper estimate for superharmonic functions in terms of the Wolff potential.

Introduction

Let G be an open set in $\mathbf{R}^N$ $(N \geq 2)$. In the classical potential theory, it is well known that given an ordinary superharmonic function u in G, there exists a nonnegative Radon measure ν in G such that the equation

$$-\operatorname{div}(\nabla u) = \nu \tag{1}$$

holds in the distribution sense in G. Conversely, if G is bounded and ν is a nonnegative finite Radon measure, then

$$u(x) = \int_G g(x,y)\, d\nu(y) \tag{2}$$

is superharmonic and satisfies the equation (1), where $g(x,y)$ is the Green function for the Laplace equation (for example, see [AG, Chapter 4]).

In nonlinear setting, no integral representation such as (2) is available. However, in [KM1], [KM2] and [M], relations between $\mathscr{A}$-superharmonic functions (see [HKM, Chapter 7] for the definition) and solutions for quasi-linear second order elliptic differential equations involving measures

$$-\operatorname{div} \mathscr{A}(x, \nabla u(x)) = \nu \tag{3}$$

2000 Mathematics Subject Classification: Primary 31C45; Secondary 31B05, 35J60.
Key words: $(\mathscr{A},\mathscr{B})$-superharmonic function, quasi-linear equation, involving measure, Wolff potential.

are investigated, where $\mathscr{A}(x,\xi)\colon \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}^N$ satisfies structure conditions of p-th order with $1 < p < \infty$. They showed that for every nonnegative finite Radon measure ν, there is an $\mathscr{A}$-superharmonic function satisfying the equation (3) with weak zero boundary values. Moreover, they gave a pointwise estimate for an $\mathscr{A}$-superharmonic function in terms of the Wolff potential. The existence and the uniqueness of the solution to more generally quasi-linear elliptic equations involving measures, including the equation (3), have been studied in many papers [BG], [B+5], [R] and [KX], etc.

On the other hand, in the previous papers [MO1], [MO2] and [MO3], we developed a potential theory for elliptic quasi-linear equations of the form

$$-\operatorname{div}\mathscr{A}(x,\nabla u(x)) + \mathscr{B}(x,u(x)) = 0 \tag{E}$$

on a domain Ω in $\mathbf{R}^N$ $(N \geq 2)$, where $\mathscr{A}(x,\xi)\colon \Omega \times \mathbf{R}^N \to \mathbf{R}^N$ satisfies weighted structure conditions of p-th order with weight $w(x)$ as in [HKM] and [M], and $\mathscr{B}(x,t)\colon \Omega\times\mathbf{R} \to \mathbf{R}$ is nondecreasing in t (see section 1 below for more details). We called superharmonic functions relative to the equation (E) $(\mathscr{A},\mathscr{B})$-superharmonic functions (see section 2 below for the definition).

The purpose of the present paper is to extend results in [KM1], [KM2] and [M] to those relative to the equation (E), namely, to investigate relations between $(\mathscr{A},\mathscr{B})$-superharmonic functions and solutions of the equation

$$-\operatorname{div}\mathscr{A}(x,\nabla u(x)) + \mathscr{B}(x,u(x)) = \nu \tag{E_ν}$$

with $\mathscr{A}$ and $\mathscr{B}$ as above.

We first investigate properties of $(\mathscr{A},\mathscr{B})$-superharmonic functions. Actually we show the "ess lim inf" property, the fundamental convergence theorem, and the integrability of $(\mathscr{A},\mathscr{B})$-superharmonic functions. In section 3, we show that every $(\mathscr{A},\mathscr{B})$-superharmonic function determines a nonnegative Radon measure ν by the equation (E_ν) and conversely for every nonnegative finite Radon measure ν, there is an $(\mathscr{A},\mathscr{B})$-superharmonic function u satisfying the equation (E_ν) with weak zero boundary values. In section 4, we give a pointwise upper estimate for $(\mathscr{A},\mathscr{B})$-superharmonic functions in terms of the weighted Wolff potentials, and using this estimate, we can show that an $(\mathscr{A},\mathscr{B})$-superharmonic function is finite except on $\mathscr{A}$-polar set (see [HKM, Chapter 10] for the definition). Finally, in section 5, we discuss the uniqueness of the so-called entropy solution to the equation (E_ν).

Throughout this paper, we use some standard notation without explanation. One may refer to [HKM] for most of such notation. Also, we say that ν is a Radon measure if ν is a *nonnegative*, Borel regular measure which is finite on compact sets.

1. Preliminaries

Let Ω be a domain in $\mathbf{R}^N$ ($N \geq 2$). As in [MO1], [MO2] and [MO3] we assume that $\mathscr{A}\colon \Omega \times \mathbf{R}^N \to \mathbf{R}^N$ and $\mathscr{B}\colon \Omega \times \mathbf{R} \to \mathbf{R}$ satisfy the following conditions for $1 < p < \infty$ and a *weight* w which is *p-admissible* in the sense of [HKM]:

(A.1) $x \mapsto \mathscr{A}(x,\xi)$ is measurable on Ω for every $\xi \in \mathbf{R}^N$ and $\xi \mapsto \mathscr{A}(x,\xi)$ is continuous for a.e. $x \in \Omega$;

(A.2) $\mathscr{A}(x,\xi)\cdot\xi \geq \alpha_1 w(x)|\xi|^p$ for all $\xi \in \mathbf{R}^N$ and a.e. $x \in \Omega$ with a constant $\alpha_1 > 0$;

(A.3) $|\mathscr{A}(x,\xi)| \leq \alpha_2 w(x)|\xi|^{p-1}$ for all $\xi \in \mathbf{R}^N$ and a.e. $x \in \Omega$ with a constant $\alpha_2 > 0$;

(A.4) $\big(\mathscr{A}(x,\xi_1) - \mathscr{A}(x,\xi_2)\big)\cdot\big(\xi_1-\xi_2\big) > 0$ whenever $\xi_1, \xi_2 \in \mathbf{R}^N$, $\xi_1 \neq \xi_2$, for a.e. $x \in \Omega$;

(B.1) $x \mapsto \mathscr{B}(x,t)$ is measurable on Ω for every $t \in \mathbf{R}$ and $t \mapsto \mathscr{B}(x,t)$ is continuous for a.e. $x \in \Omega$;

(B.2) For any open set $G \Subset \Omega$, there is a constant $\alpha_3(G) \geq 0$ such that $|\mathscr{B}(x,t)| \leq \alpha_3(G)w(x)(|t|^{p-1}+1)$ for all $t \in \mathbf{R}$ and a.e. $x \in G$;

(B.3) $t \mapsto \mathscr{B}(x,t)$ is nondecreasing on $\mathbf{R}$ for a.e. $x \in \Omega$.

We consider elliptic quasi-linear equations of the form

$$-\operatorname{div}\mathscr{A}(x,\,\nabla u(x)) + \mathscr{B}(x,\,u(x)) = 0 \tag{E}$$

on Ω.

For the nonnegative measure μ: $d\mu(x) = w(x)\,dx$ and an open subset G of Ω, we consider the weighted Sobolev spaces $H^{1,p}(G;\mu)$, $H_0^{1,p}(G;\mu)$ and $H_{\mathrm{loc}}^{1,p}(G;\mu)$ (see [HKM] for details).

Let G be an open subset of Ω. A function $u \in H_{\mathrm{loc}}^{1,p}(G;\mu)$ is said to be a (weak) *solution* of (E) in G if

$$\int_G \mathscr{A}(x,\nabla u)\cdot\nabla\varphi\,dx + \int_G \mathscr{B}(x,u)\varphi\,dx = 0$$

for all $\varphi \in C_0^\infty(G)$. A function $u \in H_{\mathrm{loc}}^{1,p}(G;\mu)$ is said to be a *supersolution* (resp. *subsolution*) of (E) in G if

$$\int_G \mathscr{A}(x,\nabla u)\cdot\nabla\varphi\,dx + \int_G \mathscr{B}(x,u)\varphi\,dx \geq 0 \quad (\text{resp. } \leq 0)$$

for all nonnegative $\varphi \in C_0^\infty(G)$.

Proposition 1.1. (Comparison principle) [O1, Lemma 3.6] *Let G be a bounded open set in Ω and let $u \in H^{1,p}(G;\mu)$ be a supersolution and $v \in H^{1,p}(G;\mu)$ a subsolution of* (E) *in G. If $\min(u-v,0) \in H_0^{1,p}(G;\mu)$, then $u \geq v$ a.e. in G.*

A continuous solution of (E) in an open subset G of Ω is called $(\mathscr{A},\mathscr{B})$-*harmonic* in G.

We say that an open set G in Ω is $(\mathscr{A},\mathscr{B})$-*regular*, if $G \Subset \Omega$ and for any $\theta \in H^{1,p}_{\mathrm{loc}}(\Omega;\mu)$ which is continuous at each point of ∂G, there exists a unique $h \in C(\overline{G}) \cap H^{1,p}(G;\mu)$ such that $h=\theta$ on ∂G and h is $(\mathscr{A},\mathscr{B})$-harmonic in G.

Proposition 1.2. ([MO1, Theorem 1.4] and [HKM, Theorem 6.31]) *Any ball $B \Subset \Omega$ and any polyhedron $P \Subset \Omega$ are $(\mathscr{A},\mathscr{B})$-regular.*

We recall the definition of the (p,μ)-capacity which is given in [HKM]. For a compact set K and an open set G such that $K \subset G \subset \mathbf{R}^N$, let

$$\mathrm{cap}_{p,\mu}(K,G) = \inf \int_G |\nabla u|^p \, d\mu,$$

where the infimum is taken over all $u \in C_0^\infty(G)$ with $u \geq 1$ on K. Moreover, for an open set $U \subset G$, set

$$\mathrm{cap}_{p,\mu}(U,G) = \sup_{\substack{K \subset U \\ K \text{ compact}}} \mathrm{cap}_{p,\mu}(K,G),$$

and, finally, for an arbitrary set $E \subset G$, define

$$\mathrm{cap}_{p,\mu}(E,G) = \inf_{\substack{E \subset U \subset G \\ U \text{ open}}} \mathrm{cap}_{p,\mu}(U,G),$$

and the number $\mathrm{cap}_{p,\mu}(E,G)$ is called the (p,μ)-*capacity* of (E,G).

If a set $E \subset \mathbf{R}^N$ satisfies

$$\mathrm{cap}_{p,\mu}(E \cap G, G) = 0$$

for all open sets $G \subset \mathbf{R}^N$, then we say that E is of (p,μ)-*capacity zero*, and write $\mathrm{cap}_{p,\mu} E = 0$. Also if a property holds except on a set of (p,μ)-capacity zero, we say that it holds (p,μ)-*quasieverywhere*, or simply (p,μ)-q.e.

For $E \subset \mathbf{R}^N$ and $x \in \mathbf{R}^N$, let

$$W_{p,\mu}(x,E) = \int_0^1 \left(\frac{\mathrm{cap}_{p,\mu}\big(B(x,t) \cap E, B(x,2t)\big)}{\mathrm{cap}_{p,\mu}\big(B(x,t), B(x,2t)\big)} \right)^{1/(p-1)} \frac{dt}{t}.$$

In this paper, $B(x,r)$ denotes an open ball with center x and radius r.

Proposition 1.3. ([M, Theorem 5.12], [HKM, Theorem 6.27 and Theorem 8.10]) *Suppose that G is an open set with $G \Subset \Omega$. Let $T = \big\{x \in \partial G \,|\, W_{p,\mu}(x, \complement G) < \infty\big\}$. Then* $\mathrm{cap}_{p,\mu} T = 0$.

2. Properties of $(\mathscr{A},\mathscr{B})$-superharmonic functions

In this section, we will investigate properties of $(\mathscr{A},\mathscr{B})$-superharmonic functions. Actually we will show the "ess lim inf" property, the fundamental convergence theorem, and the integrability of $(\mathscr{A},\mathscr{B})$-superharmonic functions.

Let G be an open subset in Ω. A function $u\colon G \to \mathbf{R} \cup \{\infty\}$ is said to be $(\mathscr{A},\mathscr{B})$-*superharmonic* in G if it is lower semicontinuous, finite on a dense set in G and, for each open set $U \Subset \Omega$ and for $h \in C(\overline{U})$ which is $(\mathscr{A},\mathscr{B})$-harmonic in

U, $u \geq h$ on ∂U implies $u \geq h$ in U. $(\mathscr{A}, \mathscr{B})$-*subharmonic* functions are similarly defined. Note that a continuous supersolution of (E) is $(\mathscr{A}, \mathscr{B})$-superharmonic (cf. [MO1, §2]). If u is $(\mathscr{A}, \mathscr{B})$-superharmonic in G, then so is $u+c$ for any nonnegative constant c. If u_1 and u_2 are $(\mathscr{A}, \mathscr{B})$-superharmonic in G, then so is $\min(u_1, u_2)$.

Lemma 2.1. *For any open set $U \Subset \Omega$, there exists a nonnegative bounded continuous $(\mathscr{A}, \mathscr{B})$-superharmonic function u_0 in U.*

Proof. Let V be an $(\mathscr{A}, \mathscr{B})$-regular open set such that $U \subset V \Subset \Omega$. There exists $h_0 \in C(\overline{V})$ such that it is $(\mathscr{A}, \mathscr{B})$-harmonic in V and $h_0 = 0$ on ∂V. Then h_0 is bounded, so that there exists a constant $c \geq 0$ such that $h_0 + c \geq 0$ in U. Then, $u_0 = h_0 + c$ has the required properties. □

Proposition 2.1. ([MO1, Corollary 4.1]) *Any supersolution of* (E) *has an $(\mathscr{A}, \mathscr{B})$-superharmonic representative.*

In general, an $(\mathscr{A}, \mathscr{B})$-superharmonic function is not always a supersolution (for example, see [HKM, Example 7.47] or [K, p. 108]). Using [MO1, Proposition 1.2], we can show the following proposition in the same manner as in the proof of [HKM, Theorem 7.19 and Corollary 7.20] (see [O2, Proposition 5.2.2] for details).

Proposition 2.2. *Let G be an open set in Ω and u be an $(\mathscr{A}, \mathscr{B})$-superharmonic function in G. If there is $g \in H^{1,p}_{\mathrm{loc}}(G;\mu)$ such that $u \leq g$ a.e. in G, then u is a supersolution of* (E) *in G.*

Corollary 2.1. *Let u be an $(\mathscr{A}, \mathscr{B})$-superharmonic functions in an open set $G \subset \Omega$, then $\min(u, k) \in H^{1,p}_{\mathrm{loc}}(G;\mu)$ for any $k > 0$.*

Proof. Let $U \Subset G$ and u_0 be a function as in Lemma 2.1. Then, $u_k = \min(u, u_0 + k)$ is a bounded $(\mathscr{A}, \mathscr{B})$-superharmonic function, and hence it belongs to $H^{1,p}_{\mathrm{loc}}(U;\mu)$ by the above proposition. Hence $\min(u, k) = \min(u_k, k) \in H^{1,p}_{\mathrm{loc}}(U;\mu)$. Since $U \Subset G$ is arbitrary, we have the required assertion. □

Next, we will establish the "ess lim inf" property for $(\mathscr{A}, \mathscr{B})$-superharmonic functions (Theorem 2.1). To show this property, we prepare the following lemma.

Lemma 2.2. *For each $x_0 \in \Omega$ and $\gamma \in \mathbf{R}$ there exist a ball $B(x_0, r) \Subset \Omega$ and an $(\mathscr{A}, \mathscr{B})$-harmonic function h on B such that $h(x_0) = \gamma$.*

Proof. Let $T > 0$ such that $-T \leq \gamma \leq T$. Choose $B_0 = B(x_0, r_0)$ with $\overline{B_0} \subset \Omega$. Set $b_1(x) = \mathscr{B}(x, T+1)$, $b_2(x) = \mathscr{B}(x, -T-1)$ and u_j be the continuous solution of $-\operatorname{div} \mathscr{A}(x, \nabla u) + b_j(x) = 0$ in B_0 with boundary values 0 on ∂B_0 ($j = 1, 2$). Since each u_j is continuous, there is $r > 0$ ($r \leq r_0$) such that $|u_j - u_j(x_0)| \leq 1$ on $B = B(x_0, r)$, $j = 1$, 2. Set $v_1 = u_1 - u_1(x_0) + T$ and $v_2 = u_2 - u_2(x_0) - T$ on $\overline{B}$. Since $v_1 \leq T + 1$ on B,

$$-\operatorname{div} \mathscr{A}(x, \nabla v_1(x)) + \mathscr{B}(x, v_1(x)) \leq -\operatorname{div} \mathscr{A}(x, \nabla u_1(x)) + b_1(x) = 0$$

on B. Hence, since v_1 is continuous, v_1 is $(\mathscr{A}, \mathscr{B})$-subharmonic in B. Similarly we see that v_2 is $(\mathscr{A}, \mathscr{B})$-superharmonic in B. Set $T_1 = \sup_B v_1 + 1$ and $T_2 =$

$-\inf_B v_2 + 1$. Then $T \le T_j < \infty$, $j = 1, 2$. Let h_t be the $(\mathscr{A}, \mathscr{B})$-harmonic function on B with boundary values t on ∂B. By the comparison principle, we have $h_{T_1}(x_0) \ge v_1(x_0) = T$ and $h_{-T_2}(x_0) \le v_2(x_0) = -T$. Since $t \mapsto h_t(x_0)$ is continuous (see [MO1, Corollary 3.1 and the proof of Proposition 3.1]), it follows that

$$\{h_t(x_0) \mid -T_2 \le t \le T_1\} \supset [-T, T],$$

as required. □

To show the "ess lim inf" property, we need the following proposition (see [MO1, Proposition 2.3]).

Proposition 2.3. (Poisson modification) *Let G be an open set in Ω and let $V \Subset G$ be an $(\mathscr{A}, \mathscr{B})$-regular open set. For an $(\mathscr{A}, \mathscr{B})$-superharmonic function u on G, we define*

$$u_V = \sup\{h \in C(\overline{V}) \mid h \le u \text{ on } \partial V \text{ and } h \text{ is } (\mathscr{A}, \mathscr{B})\text{-harmonic in } V\}.$$

Then

$$P(u, V) := \begin{cases} u & \text{in } G \setminus V, \\ u_V & \text{in } V \end{cases}$$

is $(\mathscr{A}, \mathscr{B})$-superharmonic in G and $(\mathscr{A}, \mathscr{B})$-harmonic in V, and $P(u, V) \le u$ in G. If $u \in H^{1,p}_{\mathrm{loc}}(G; \mu)$, then $u|_V - u_V \in H^{1,p}_0(V; \mu)$.

Theorem 2.1. (The "ess lim inf" property) *Let G be an open subset in Ω. If u is an $(\mathscr{A}, \mathscr{B})$-superharmonic function in G, then $u(x) = \operatorname{ess}\liminf_{y \to x} u(y)$ for each $x \in G$.*

Proof. Fix $x \in G$ and let $\lambda = \operatorname{ess}\liminf_{y \to x} u(y)$. Then $\lambda \ge \liminf_{y \to x} u(y) \ge u(x)$. To show the converse inequality, let $\gamma < \lambda$. By the above lemma, there is a ball $B_1 = B(x, r_1)$ and an $(\mathscr{A}, \mathscr{B})$-harmonic function h on B_1 such that $B_1 \subset G$ and $h(x) = \gamma$. Since h is continuous,

$$\operatorname{ess}\liminf_{y \to x} \{u(y) - h(y)\} = \lambda - \gamma > 0.$$

Hence there is $B = B(x, r)$ with $0 < r < r_1$ such that $u > h$ a.e. on B. Now, $\min(u, h)$ is $(\mathscr{A}, \mathscr{B})$-superharmonic on B_1 and $\min(u, h) \le h$, which assures $\min(u,h) \in H^{1,p}(B; \mu)$ by Proposition 2.2. Let $0 < \rho < r$ and $v = P(\min(u, h), B(x, \rho))$ in the notation in Proposition 2.3. Then v is a supersolution of (E) on B by Proposition 2.2, $v \le \min(u, h)$ and $\min(u, h) - v \in H^{1,p}_0(B; \mu)$. Hence, noting that $\min(u, h) = h$ a.e. on B, we have

$$\int_B \mathscr{A}(x, \nabla v) \cdot (\nabla h - \nabla v)\, dx + \int_B \mathscr{B}(x, v)(h - v)\, dx \ge 0$$

and

$$\int_B \mathscr{A}(x, \nabla h) \cdot (\nabla h - \nabla v)\, dx + \int_B \mathscr{B}(x, h)(h - v)\, dx = 0,$$

so that

$$\int_B \big[\mathscr{A}(x,\nabla h) - \mathscr{A}(x,\nabla v)\big]\cdot(\nabla h - \nabla v)\,dx + \int_B \big[\mathscr{B}(x,h) - \mathscr{B}(x,v)\big](h-v)\,dx \le 0.$$

This implies $\nabla h = \nabla v$ a.e. on B by (A.4) and (B.3). Since $v = \min(u,h) = h$ a.e. on $B \setminus B(x,\rho)$, it follows that $v = h$ a.e. on B, and hence $v = h$ everywhere on $B(x,\rho)$ by virtue of continuity of both v and h on $B(x,\rho)$. In particular, $v(x) = h(x)$. Since $v \le \min(u,h) \le h$, this implies that $\min(u(x),h(x)) = h(x)$, namely, $u(x) \ge h(x) = \gamma$. □

Corollary 2.2. *Let G be an open subset in Ω and let u and v be $(\mathscr{A},\mathscr{B})$-superharmonic functions in G. If $u \ge v$ a.e. in G, then $u \ge v$ everywhere in G.*

Next, we will show the fundamental convergence theorem (Theorem 2.2). For this, we prepare a proposition and two lemmas. The following proposition can be shown in the same manner as [HKM, Theorem 7.4] (see [O2, Proposition 5.1.4] for details).

Proposition 2.4. *Let G be an open subset in Ω. Let $\mathscr{F}$ be a family of $(\mathscr{A},\mathscr{B})$-superharmonic functions in G which is locally uniformly bounded below. Then the lower semicontinuous regularization of $\inf \mathscr{F}$ is $(\mathscr{A},\mathscr{B})$-superharmonic in G.*

Suppose that G be an open set with $G \Subset \Omega$ and $E \subset G$. Let h be a bounded $(\mathscr{A},\mathscr{B})$-harmonic function in G, u be an $(\mathscr{A},\mathscr{B})$-superharmonic function in G with $u \ge h$ in G. We define

$$\Phi_E^{u,h}(G) = \left\{ v \;\middle|\; \begin{array}{l} v \text{ is } (\mathscr{A},\mathscr{B})\text{-superharmonic in } G, \\ v \ge u \text{ on } E \text{ and } v \ge h \text{ on } G \setminus E \end{array} \right\},$$

$R_E^{u,h}(G) = \inf \Phi_E^{u,h}(G)$ and $\hat{R}_E^{u,h}(G)(x) = \lim_{r\to 0} \inf_{B(x,r)\cap G} R_E^{u,h}(G)$ for each $x \in G$. By the above proposition, $\hat{R}_E^{u,h}(G)$ is $(\mathscr{A},\mathscr{B})$-superharmonic in G.

The following lemma can be shown in the same manner as [HKM, Lemma 8.4].

Lemma 2.3. *Suppose that G is an open set with $G \Subset \Omega$ and $E \subset G$. Let h be a bounded $(\mathscr{A},\mathscr{B})$-harmonic function in G, u be an $(\mathscr{A},\mathscr{B})$-superharmonic function in G with $u \ge h$ in G. Then $\hat{R}_E^{u,h}$ is $(\mathscr{A},\mathscr{B})$-harmonic in $G \setminus \overline{E}$, $\hat{R}_E^{u,h} = R_E^{u,h}$ in $G \setminus \partial E$ and $\hat{R}_E^{u,h} = u$ in the interior of E.*

Lemma 2.4. *Suppose that G is open set with $G \Subset \Omega$ and $E \subset G$ is compact. Let h be a bounded $(\mathscr{A},\mathscr{B})$-harmonic function in G and u be an $(\mathscr{A},\mathscr{B})$-superharmonic function in G with $u \ge h$ in G. Then,*

$$\operatorname{cap}_{p,\mu}\left\{x \in G \,|\, \hat{R}_E^{u,h}(G)(x) < R_E^{u,h}(G)(x)\right\} = 0.$$

Proof. Set $S = \{x \in G \,|\, \hat{R}_E^{u,h}(G)(x) < R_E^{u,h}(G)(x)\}$. By the above lemma, $S \subset \partial E$. Let $T = \{x \in \partial E \,|\, W_{p,\mu}(x,E) < \infty\}$. Since $\operatorname{cap}_{p,\mu} T = 0$ (Proposition 1.3), the proof is complete if we show $S \subset T$.

Let U be an $(\mathscr{A},\mathscr{B})$-regular set such that $E \subset U \Subset G$. Choose an increasing sequence of nonnegative functions $\psi_i \in C_0^\infty(U)$ such that $\psi_i + h \to u$ on E. Set $\varphi_i = \psi_i + h$. For each i there exists an $(\mathscr{A},\mathscr{B})$-harmonic function s_i in $U \setminus E$ with $s_i - \varphi_i \in H_0^{1,p}(U \setminus E;\mu)$. It follows from [O1, Theorem 5.3] that $\lim_{y\to x, y\in U\setminus E} s_i(y) = \varphi_i(x)$ for $x \in \partial E \setminus T$. We shall show $R_E^{u,h}(G) \geq s_i$ in $U \setminus E$.

Choose $c > 0$ such that $h + c > 0$ on $\overline{U}$. For $\varepsilon > 0$, let $v \in \Phi_E^{u+\varepsilon,h}(G)$. Then $v_i = \min(v, h + c + \sup_U \psi_i)$ is bounded and $(\mathscr{A},\mathscr{B})$-superharmonic in U, and hence it is a supersolution of (E) in U by Proposition 2.2. Since $v \geq u + \varepsilon > \varphi_i$ on E and $\varphi_i = h$ on a complement of $\operatorname{supp}\psi_i$, $v_i \geq \varphi_i$ outside a compact set in $U \setminus E$. Thus $0 \geq \min(v_i - s_i, 0) \geq \min(v_i - \varphi_i, 0) + \min(\varphi_i - s_i, 0) \in H_0^{1,p}(U \setminus E;\mu)$, so that $\min(v_i - s_i, 0) \in H_0^{1,p}(U \setminus E;\mu)$. The comparison principle (Proposition 1.1) yields $v_i \geq s_i$ a.e. in $U \setminus E$. Since v_i is $(\mathscr{A},\mathscr{B})$-superharmonic and s_i is $(\mathscr{A},\mathscr{B})$-harmonic, by Corollary 2.2 $v_i \geq s_i$ in $U \setminus E$. Hence $v \geq s_i$, so that $R_E^{u,h}(G) + \varepsilon \geq R_E^{u+\varepsilon,h}(G) \geq s_i$ in $U \setminus E$. Letting $\varepsilon \to 0$, we have $R_E^{u,h}(G) \geq s_i$ in $U \setminus E$.

Therefore, for $x \in \partial E \setminus T$,

$$\begin{aligned}\hat{R}_E^{u,h}(G)(x) &\geq \min\left(\liminf_{y\to x, y\in U\setminus E} R_E^{u,h}(G)(y), u(x)\right)\\ &\geq \min\left(\lim_{y\to x, y\in U\setminus E} s_i(y), u(x)\right) = \min\left(\varphi_i(x), u(x)\right) = \varphi_i(x).\end{aligned}$$

Letting $i \to \infty$, we have $\hat{R}_E^{u,h}(G)(x) \geq u(x) \geq R_E^{u,h}(G)(x)$ for $x \in \partial E \setminus T$. This implies $S \subset T$. □

Now, by using the above lemmas, we can show the fundamental convergence theorem.

Theorem 2.2. (Fundamental convergence theorem) *Let G be an open subset in Ω and let $\mathscr{F}$ be a family of $(\mathscr{A},\mathscr{B})$-superharmonic functions in G which is locally uniformly bounded below. Then the lower semicontinuous regularization $\hat{s}$ of $s = \inf \mathscr{F}$ is $(\mathscr{A},\mathscr{B})$-superharmonic in G and $\hat{s} = s$ (p,μ)-q.e. in G.*

Proof. By Proposition 2.4, we only show that $\hat{s} = s$ (p,μ)-q.e. in G. In the same manner as in the proof of [HKM, Theorem 8.2], Choquet's topological lemma ([HKM, Lemma 8.3]) yields that there exists a decreasing sequence $v_i \in \mathscr{F}$ with the limit v such that the lower semicontinuous regularizations $\hat{s}$ and $\hat{v}$ coincide. Let

$$V_j = \{x \in G \mid \hat{v}(x) + \frac{1}{j} < v(x)\}.$$

Since $s \leq v$, we have $\{x \in G \mid \hat{s}(x) < s(x)\} \subset \bigcup_{j=1}^\infty V_j$. Therefore, if we can show $\operatorname{cap}_{p,\mu} V_j = 0$, the subadditivity of the capacity yields $\hat{s} = s$ (p,μ)-q.e. in G. Since V_j is a Borel set, it suffices to show that $\operatorname{cap}_{p,\mu} K = 0$ for any compact set $K \subset V_j$.

Let $G' \Subset G$ be an open neighborhood of K and h be a bounded $(\mathscr{A},\mathscr{B})$-harmonic function in G'. Since $\mathscr{F}$ is locally uniformly bounded below, there exists a constant $c \geq 0$ such that $\hat{v} + c \geq h$. Letting $u = \hat{v} + c + \frac{1}{j}$, we have $v_i + c \in \Phi_K^{u,h}(G')$ for all

i. Therefore $R_K^{u,h}(G') \le v_i + c$ in G' for all i, so that $R_K^{u,h}(G') \le v + c$ in G'. Hence $\hat{R}_K^{u,h}(G') \le \hat{v} + c$ in G'. This implies

$$\hat{R}_K^{u,h}(G') < \hat{v} + c + \frac{1}{j} = u = R_K^{u,h}(G')$$

on K. Hence by Lemma 2.4 we have $\mathrm{cap}_{p,\mu} K = 0$, so that the proof is complete. □

The rest of this section is devoted to showing the integrability of $(\mathscr{A}, \mathscr{B})$-superharmonic functions. First, following the discussion in [MZ], in which the unweighted case, namely the case $w = 1$, is treated, we will show a weak Harnack inequality for supersolutions of (E). Hereafter, c_μ denotes a constant depending only on those constants which appear in the conditions for w to be p-admissible (see [HKM, Chapter 1]).

Lemma 2.5. *Suppose that G is an open set with $G \Subset \Omega$ and $B(x, 2r) \subset G$. If u is a nonnegative supersolution of* (E) *in G, then, for any σ, $\tau \in (0, 1)$, there exists a constant $c = c(N, p, \alpha_1, \alpha_2, \alpha_3(G), r, \gamma, \sigma, \tau, c_\mu) > 0$ such that*

$$\left(\frac{1}{\mu(B(x,\sigma r))} \int_{B(x,\sigma r)} u^\gamma \, d\mu\right)^{1/\gamma} \le c \left(\operatorname*{ess\,inf}_{B(x,\tau r)} u + r\right)$$

whenever $0 < \gamma < \varkappa(p-1)$, where $\varkappa > 1$ is the exponent in the Sobolev inequality.

Proof. Fix $r > 0$ and let $\overline{u} = u + r$. Let $\beta > 0$. For a ball $B \subset G$ and a nonnegative $\eta \in C_0^\infty(B)$, set $\varphi = \overline{u}^{-\beta}\eta^p$. Then $\varphi \in H_0^{1,p}(B; \mu)$ and $\varphi \ge 0$. Since u is a supersolution of (E) and

$$\nabla\varphi = -\beta\overline{u}^{-\beta-1}\eta^p\nabla u + p\overline{u}^{-\beta}\eta^{p-1}\nabla\eta,$$

we have

$$\int_B \mathscr{A}(x, \nabla u) \cdot (-\beta\overline{u}^{-\beta-1}\eta^p\nabla u + p\overline{u}^{-\beta}\eta^{p-1}\nabla\eta)\, dx + \int_B \mathscr{B}(x, u)\overline{u}^{-\beta}\eta^p dx \ge 0.$$

From (A.2), (A.3) and (B.2) it follows that

$$\begin{aligned}\alpha_1\beta \int_B |\nabla u|^p \overline{u}^{-\beta-1}\eta^p \, d\mu \le p\alpha_2 \int_B |\nabla u|^{p-1}|\nabla\eta|\overline{u}^{-\beta}\eta^{p-1} \, d\mu \\ + \alpha_3(G) \int_B (u^{p-1} + 1)\overline{u}^{-\beta}\eta^p \, d\mu.\end{aligned} \tag{2.1}$$

By Young's inequality,

$$|\nabla u|^{p-1}|\nabla\eta|\overline{u}^{-\beta}\eta^{p-1} \le \frac{\alpha_1}{2p\alpha_2}\beta|\nabla u|^p\overline{u}^{-\beta-1}\eta^p + c\beta^{1-p}|\nabla\eta|^p\overline{u}^{p-\beta-1}$$

with $c = c(p, \alpha_1, \alpha_2) > 0$. Also, note that $u^{p-1} + 1 \le 2\max(1, r^{1-p})\overline{u}^{p-1}$. Hence, by (2.1)

$$\int_B |\nabla u|^p\overline{u}^{-\beta-1}\eta^p \, d\mu \le c\left\{\beta^{-p}\int_B |\nabla\eta|^p\overline{u}^{p-\beta-1} \, d\mu + \beta^{-1}\int_B \overline{u}^{p-1-\beta}\eta^p d\mu\right\} \tag{2.2}$$

with $c = c(p, \alpha_1, \alpha_2, \alpha_3(G), r) > 0$.

Now let $s < p-1$, $s \neq 0$, and set $v = \overline{u}^{s/p}$. Then, $|\nabla v|^p = (|s|/p)^p |\nabla u|^p \overline{u}^{s-p}$. Hence, applying (2.2) with $\beta = p-1-s$ we have

$$\begin{aligned}\int_B |\nabla v|^p \eta^p \, d\mu &\leq c \Big\{ |s|^p (p-1-s)^{-p} \int_B |\nabla \eta|^p v^p \, d\mu \\ &\qquad + |s|^p (p-1-s)^{-1} \int_B v^p \eta^p \, d\mu \Big\} \\ &\leq c\, |s|^p (1 + (p-1-s)^{-1})^p \int_B (\eta^p + |\nabla \eta|^p) v^p \, d\mu\end{aligned} \tag{2.3}$$

with $c = c(p, \alpha_1, \alpha_2, \alpha_3(G), r) > 0$. The Sobolev inequality and (2.3) yield

$$\begin{aligned}&\left(\frac{1}{\mu(B)} \int_B (\eta v)^{\varkappa p} \, d\mu \right)^{1/\varkappa p} \leq c_\mu \, \rho(B) \left(\frac{1}{\mu(B)} \int_B |\nabla(\eta v)|^p \, d\mu \right)^{1/p} \\ &\leq 2\, c_\mu \rho(B) \left(\frac{1}{\mu(B)} \int_B (\eta^p |\nabla v|^p + |\nabla \eta|^p v^p) \, d\mu \right)^{1/p} \\ &\leq c\, \rho(B)(|s|+1)(1 + (p-1-s))^{-1}) \left(\frac{1}{\mu(B)} \int_B (\eta^p + |\nabla \eta|^p) v^p \, d\mu \right)^{1/p},\end{aligned} \tag{2.4}$$

where $\rho(B)$ is the radius of B and $c = c(p, \alpha_1, \alpha_2, \alpha_3(G), r, c_\mu) > 0$.

Now, we consider the ball $B(x,r)$ as in the lemma and let $B(h) = B(x,h)$ for $h > 0$. Let $r_0 = \min(\sigma, \tau) r$. We note that $\mu(B(h)) \leq c\mu(B(r_0))$ with $c = c(\sigma, \tau, c_\mu) > 0$ for $r_0 \leq h \leq r$ by the doubling property of μ. Let $r_0 \leq h' < h \leq r$ and $\eta \in C_0^\infty(B(h))$ be chosen so that $\eta = 1$ on $B(h')$, $0 \leq \eta \leq 1$ in $B(h)$ and $|\nabla \eta| \leq 3(h-h')^{-1}$. Then, since $\eta \leq 1 \leq h(h-h')^{-1}$, (2.4) with $B = B(h)$ yields

$$\begin{aligned}&\left(\frac{1}{\mu(B(h'))} \int_{B(h')} v^{\varkappa p} \, d\mu \right)^{1/\varkappa p} \\ &\leq C_1 \, (h-h')^{-1} (1+|s|)(1 + (p-1-s)^{-1}) \left(\frac{1}{\mu(B(h))} \int_{B(h)} v^p \, d\mu \right)^{1/p}\end{aligned} \tag{2.5}$$

with $C_1 = C_1(p, \alpha_1, \alpha_2, \alpha_3(G), r, c_\mu, \sigma, \tau) > 0$.

If $s > 0$, by (2.5) we have

$$\begin{aligned}&\left(\frac{1}{\mu(B(h'))} \int_{B(h')} \overline{u}^{\varkappa s} \, d\mu \right)^{1/\varkappa s} \\ &\leq [C_1 (h-h')^{-1} (1+s)(1 + (p-1-s)^{-1})]^{p/s} \left(\frac{1}{\mu(B(h))} \int_{B(h)} \overline{u}^s \, d\mu \right)^{1/s}.\end{aligned} \tag{2.6}$$

If $s < 0$, since $(p-1-s)^{-1} < (p-1)^{-1}$, from (2.5) we obtain

$$\begin{aligned}(2.7)\qquad &\left(\frac{1}{\mu(B(h'))}\int_{B(h')}\overline{u}^{\varkappa s}\,d\mu\right)^{1/\varkappa s}\\ &\geq [C_1(h-h')^{-1}(1-s)]^{p/s}\left(\frac{1}{\mu(B(h))}\int_{B(h)}\overline{u}^{s}\,d\mu\right)^{1/s}.\end{aligned}$$

Let $0 < \gamma < \varkappa(p-1)$. Suppose $s_0 = \varkappa^{-j}\gamma$ for some integer $j \geq 2$. Set $s_i = \varkappa^i s_0$ for $i = 1, 2, ..., j-1$. Then $0 < s_i \leq \varkappa^{-1}\gamma < p-1$, and hence $p-1-s_i \geq p-1-\varkappa^{-1}\gamma$. Also, set $h_i = r\{\sigma + 2^{-i}(1-\sigma)\}$ and $h_i' = h_{i+1}$. Then $h_i - h_i' = 2^{-(i+1)}r(1-\sigma)$. Thus, by (2.6) we have

$$\left(\frac{1}{\mu(B(h_{i+1}))}\int_{B(h_{i+1})}\overline{u}^{s_{i+1}}\,d\mu\right)^{1/s_{i+1}} \leq (C_2 2^{pi})^{1/s_i}\left(\frac{1}{\mu(B(h_i))}\int_{B(h_i)}\overline{u}^{s_i}\,d\mu\right)^{1/s_i}$$

with $C_2 = C_2(p, \alpha_1, \alpha_2, \alpha_3(G), r, c_\mu, \sigma, \tau, \gamma) > 0$. Thus, since $\gamma = \varkappa^j s_0 = \varkappa s_{j-1}$, $\sigma r \leq h_j$ and $r = h_0$, we obtain by iteration

$$\begin{aligned}(2.8)\qquad &\left(\frac{1}{\mu(B(\sigma r))}\int_{B(\sigma r)}\overline{u}^{\gamma}\,d\mu\right)^{1/\gamma}\\ &\leq C_2^{\sum_{i=0}^{j-1}1/s_i}2^{p\sum_{i=0}^{j-1}i/s_i}\left(\frac{1}{\mu(B(r))}\int_{B(r)}\overline{u}^{s_0}\,d\mu\right)^{1/s_0}\\ &\leq c\left(\frac{1}{\mu(B(r))}\int_{B(r)}\overline{u}^{s_0}\,d\mu\right)^{1/s_0}\end{aligned}$$

with $c = c(p, \alpha_1, \alpha_2, \alpha_3(G), r, c_\mu, \gamma, \sigma, \tau, s_0) > 0$. Since this holds for any $s_0 = \varkappa^{-j}\gamma$, $j = 2, 3, \ldots$, by Hölder's inequality, the same inequality holds for any $s_0 > 0$.

Next, given $s_0 > 0$, set $s_i = -\varkappa^i s_0$, $h_i = r\{\tau + 2^{-i}(1-\tau)\}$ and $h_i' = h_{i+1}$. Then by (2.7) we have

$$\begin{aligned}&\left(\frac{1}{\mu(B(h_{i+1}))}\int_{B(h_{i+1})}\overline{u}^{s_{i+1}}\,d\mu\right)^{1/s_{i+1}}\\ &\geq [C_1(h_i-h_{i+1})^{-1}(1-s_i)]^{p/s_i}\left(\frac{1}{\mu(B(h_i))}\int_{B(h_i)}\overline{u}^{s_i}\,d\mu\right)^{1/s_i}.\end{aligned}$$

Since $1 - s_i = 1 + \varkappa^i s_0 \leq (1+s_0)\varkappa^i$, again by iteration we obtain

$$\begin{aligned}\left(\operatorname*{ess\,sup}_{B(\tau r)}\overline{u}^{-1}\right)^{-1} &= \lim_{i\to\infty}\left(\frac{1}{\mu(B(h_i))}\int_{B(h_i)}\overline{u}^{s_i}\,d\mu\right)^{1/s_i}\\ &\geq C_3^{\sum_{i=0}^{\infty}1/s_i}(2\varkappa)^{p\sum_{i=0}^{\infty}i/s_i}\left(\frac{1}{\mu(B(r))}\int_{B(r)}\overline{u}^{-s_0}\,d\mu\right)^{-1/s_0}\end{aligned}$$

with $C_3 = C_3(p, \alpha_1, \alpha_2, \alpha_3(G), r, c_\mu, \sigma, \tau, s_0) > 0$, that is,

$$\operatorname*{ess\,inf}_{B(\tau r)} \overline{u} \geq c \left(\frac{1}{\mu(B(r))} \int_{B(r)} \overline{u}^{-s_0} \, d\mu \right)^{-1/s_0} \tag{2.9}$$

with $c = c(p, \alpha_1, \alpha_2, \alpha_3(G), r, c_\mu, \sigma, \tau, s_0) > 0$.

Finally, we show

$$\left(\frac{1}{\mu(B(r))} \int_{B(r)} \overline{u}^{s_0} \, d\mu \right)^{1/s_0} \leq c \left(\frac{1}{\mu(B(r))} \int_{B(r)} \overline{u}^{-s_0} \, d\mu \right)^{-1/s_0} \tag{2.10}$$

for some $s_0 > 0$. Set $v = \log \bar{u}$ and let B be any ball in $B(x, r)$. Since $|\nabla v|^p = |\nabla u|^p \overline{u}^{-p}$, by (2.2) with $\beta = p - 1$ we have

$$\int_{2B} |\nabla v|^p \eta^p \, d\mu \leq c \int_{2B} (\eta^p + |\nabla \eta|^p) \, d\mu \tag{2.11}$$

with $c = c(p, \alpha_1, \alpha_2, \alpha_3(G), r) > 0$ for nonnegative $\eta \in C_0^\infty(2B)$. Choose η so that $\eta = 1$ on B, $0 \leq \eta \leq 1$ in $2B$ and $|\nabla \eta| \leq 3\rho(B)^{-1}$. Then, (2.11) yields

$$\int_B |\nabla v|^p \, d\mu \leq c\rho(B)^{-p} \mu(B)$$

with $c = c(p, \alpha_1, \alpha_2, \alpha_3(G), r) > 0$. By using Hölder's inequality and Poincaré inequality, we have

$$\frac{1}{\mu(B)} \int_B |v - v_B| \, d\mu \leq c_\mu \rho(B) \left(\frac{1}{\mu(B)} \int_B |\nabla v|^p \, d\mu \right)^{1/p} \leq C_4$$

with $C_4 = C_4(p, \alpha_1, \alpha_2, \alpha_3(G), r, c_\mu) > 0$, where $v_B = \frac{1}{\mu(B)} \int_B v \, d\mu$. Hence v satisfies the hypothesis of the John–Nirenberg lemma ([HKM, Appendix I]), so that there are positive constants s_0 and c_0 depending only on C_4, N and c_μ such that

$$\Big(\frac{1}{\mu(B(r))} \int_{B(r)} e^{s_0 v} d\mu \Big) \Big(\frac{1}{\mu(B(r))} \int_{B(r)} e^{-s_0 v} d\mu \Big) \leq c_0.$$

Hence we obtain (2.10) with $s_0 = s_0(N, p, \alpha_1, \alpha_2, \alpha_3(G), r, c_\mu) > 0$ and $c = c(N, p, \alpha_1, \alpha_2, \alpha_3(G), r, c_\mu) > 0$. Thus, by (2.8), (2.9) and (2.10) the proof is complete. □

In general, an $(\mathscr{A}, \mathscr{B})$-superharmonic function in G does not belong to $H^{1,p}_{\mathrm{loc}}(G; \mu)$. Hence, we give a definition of generalized gradient Du.

Suppose that G is an open subset in Ω. For a function u in an open set G such that $\min(u, k) \in H^{1,p}_{\mathrm{loc}}(G; \mu)$ for all $k > 0$, we define

$$Du = \lim_{k \to \infty} \nabla \min(u, k).$$

By Corollary 2.1, Du is defined for any $(\mathscr{A}, \mathscr{B})$-superharmonic function u.

Now, using the above lemma, we can show the following integrability theorem for $(\mathscr{A}, \mathscr{B})$-superharmonic functions.

Theorem 2.3. *Let G be an open subset in Ω. If u is an $(\mathscr{A},\mathscr{B})$-superharmonic function in G, then $u \in L^{\gamma}_{\mathrm{loc}}(G;\mu)$ and $Du \in L^{q(p-1)}_{\mathrm{loc}}(G;\mu)$ whenever $0 < \gamma < \varkappa(p-1)$ and*

$$0 < q < \frac{\varkappa p}{\varkappa(p-1)+1}. \tag{2.12}$$

Proof. Let $G' \Subset G$. Since u is bounded below on G', by adding a positive constant we may assume that u is nonnegative. By Lemma 2.1, there is a nonnegative bounded continuous $(\mathscr{A},\mathscr{B})$-superharmonic function u_0 in G'. For $k > 0$, let $u_k = \min(u, u_0 + k)$. Then, u_k is a supersolution of (E) in G'.

Let $B = B(x,r)$ be a ball with $2B \subset G'$. By the above lemma, we have

$$\left(\int_B u_k^{\gamma}\, d\mu\right)^{1/\gamma} \le c\, \left(\operatorname*{ess\,inf}_B u_k + r\right) \le c\, \left(\operatorname*{ess\,inf}_B u + r\right) < \infty$$

whenever $0 < \gamma < \varkappa(p-1)$ with a constant c independent of k. Hence, letting $k \to \infty$, we have $\int_B u^{\gamma}\, d\mu < \infty$.

Next, we show the integrability of Du. Let q satisfy (2.12). Since $h_0 \ge 0$, $\min(u,k) = u = u_k$ on $\{u \le k\}$, so that $\nabla \min(u,k) = \nabla u_k$ a.e. on $\{u \le k\}$. Hence

$$\begin{aligned}\int_B |\nabla \min(u,k)|^{q(p-1)}\, d\mu &= \int_{B\cap\{u\le k\}} |\nabla \min(u,k)|^{q(p-1)}\, d\mu \\ &= \int_{B\cap\{u\le k\}} |\nabla u_k|^{q(p-1)}\, d\mu \le \int_B |\nabla u_k|^{q(p-1)}\, d\mu.\end{aligned}$$

Set $\overline{u_k} = u_k + r$. If $\varepsilon > 0$, by Hölder's inequality and (2.2) in Lemma 2.5 we have

$$\begin{aligned}&\int_B |\nabla u_k|^{q(p-1)}\, d\mu = \int_B |\nabla u_k|^{q(p-1)} \overline{u}_k^{-(1+\varepsilon)(p-1)q/p} \overline{u}_k^{(1+\varepsilon)(p-1)q/p}\, d\mu \\ &\le \left(\int_B |\nabla u_k|^p \overline{u}_k^{-1-\varepsilon} d\mu\right)^{(p-1)q/p} \left(\int_B \overline{u}_k^{(1+\varepsilon)(p-1)q/\{p-q(p-1)\}}\, d\mu\right)^{\{p-(p-1)q\}/p} \\ &\le c\left(\int_{2B} \overline{u}_k^{p-1-\varepsilon} d\mu\right)^{(p-1)q/p} \left(\int_B \overline{u}_k^{(1+\varepsilon)(p-1)q/\{p-q(p-1)\}}\, d\mu\right)^{\{p-(p-1)q\}/p} \\ &\le c\left(\int_{2B} (u+r)^{p-1-\varepsilon} d\mu\right)^{(p-1)q/p} \left(\int_B (u+r)^{(1+\varepsilon)(p-1)q/\{p-q(p-1)\}}\, d\mu\right)^{\{p-(p-1)q\}/p}.\end{aligned}$$

Now choose ε so that $0 < \varepsilon < p-1$ and

$$\frac{(1+\varepsilon)(p-1)q}{p-q(p-1)} < \varkappa(p-1).$$

Thus, the integrability of u implies the integrability of Du. □

3. Existence of $(\mathscr{A},\mathscr{B})$-superharmonic solutions

In this section, we investigate relations between $(\mathscr{A},\mathscr{B})$-superharmonic functions and solutions for the equation (E_ν) with weak zero boundary values.

We define

$$Lu = -\operatorname{div}\mathscr{A}(x,\,\nabla u(x)) + \mathscr{B}(x,\,u(x)).$$

Let G be an open subset in Ω. If u is a supersolution of (E) in G, then $u \in H^{1,p}_{\mathrm{loc}}(G;\mu)$, and hence by Riesz representation theorem it is clear that Lu is a Radon measure in G. In general, an $(\mathscr{A},\mathscr{B})$-superharmonic function in G does not always belong to $H^{1,p}_{\mathrm{loc}}(G;\mu)$ (see section 2). However, by the integrability of $(\mathscr{A},\mathscr{B})$-superharmonic functions the following theorem holds.

Theorem 3.1. *Let G be an open subset in Ω and u be an $(\mathscr{A},\mathscr{B})$-superharmonic function u in G. Then there is a Radon measure ν on G such that*

$$\int_G \mathscr{A}(x,Du)\cdot\nabla\varphi\,dx + \int_G \mathscr{B}(x,u)\varphi\,dx = \int_G \varphi\,d\nu$$

for all $\varphi \in C_0^\infty(G)$.

Proof. Let $\varphi \in C_0^\infty(G)$ be nonnegative, U be an open set with $\mathrm{spt}\varphi \subset U \Subset G$ and u_0 be a bounded nonnegative $(\mathscr{A},\mathscr{B})$-superharmonic function in U (see Lemma 2.1). Set $u_k = \min(u, u_0 + k)$. Then $\nabla u_k \to Du$ a.e. in U. Hence, by (A.1)

$$\mathscr{A}(x,\nabla u_k)\cdot\nabla\varphi \to \mathscr{A}(x,Du)\cdot\nabla\varphi$$

a.e. $x \in U$. Moreover, by Theorem 2.3, $|Du|^{p-1} \in L^1(U)$, so that,

$$|\mathscr{A}(x,\nabla u_k)\cdot\nabla\varphi| \le \alpha_2|\nabla u_k|^{p-1}|\nabla\varphi| \le 2^{p-1}\alpha_2(|Du|^{p-1} + |\nabla h_0|^{p-1})|\nabla\varphi| \in L^1(U).$$

Again, by Theorem 2.3, $|u|^{p-1} \in L^1(U)$, so that,

$$|\mathscr{B}(x,u_k)\varphi| \le \alpha_3(U)(|u_k|^{p-1}+1)|\varphi| \le \alpha_3(U)(|u|^{p-1}+1)|\varphi| \in L^1(U).$$

Hence, by Lebesgue's convergence theorem we have

$$\begin{aligned}&\int_G \mathscr{A}(x,Du)\cdot\nabla\varphi\,dx + \int_G \mathscr{B}(x,u)\,\varphi\,dx\\ &\quad= \lim_{k\to\infty}\left(\int_U \mathscr{A}(x,\nabla u_k)\cdot\nabla\varphi\,dx + \int_U \mathscr{B}(x,u_k)\,\varphi\,dx\right) \ge 0.\end{aligned}$$

Therefore, from the Riesz representation theorem we obtain the claim of this theorem. □

Remark 3.1. By the proof of Theorem 3.1 we can see: if u is an $(\mathscr{A},\mathscr{B})$-superharmonic function, $\{u_k\}$ is the sequence of functions as in the proof of Theorem 3.1, $\nu = Lu$ and $\nu_k = Lu_k$ in G, then $\nu_k \to \nu$ weakly in G, namely,

$$\lim_{n\to\infty}\int_G \varphi\,d\nu_n = \int_G \varphi\,d\nu$$

for all $\varphi \in C_0^\infty(G)$.

Next, we will show that given a nonnegative Radon measure ν, there is an $(\mathscr{A},\mathscr{B})$-superharmonic function which satisfies the equation (E_ν) with weak zero boundary values. We use the notation X^* as the dual space of X.

Let G be an open set with $G \Subset \Omega$. We can regard L as an operator $H_0^{1,p}(G;\mu) \to (H_0^{1,p}(G;\mu))^*$ by

$$(Lu, v) = \int_G \mathscr{A}(x, \nabla u)\cdot \nabla v\, dx + \int_G \mathscr{B}(x,u)v\, dx.$$

In fact, by (A.3) and (B.2),

$$\left|\int_G \mathscr{A}(x,\nabla u)\cdot\nabla v\,dx\right| \le \alpha_2\Big(\int_G |\nabla u|^p\,d\mu\Big)^{(p-1)/p}\Big(\int_G |\nabla v|^p\,d\mu\Big)^{1/p}$$
$$\left|\int_G \mathscr{B}(x,u)\,v\,dx\right| \le 2\alpha_3(G)\Big(\int_G \big(|u|+1\big)^p\,d\mu\Big)^{(p-1)/p}\Big(\int_G |v|^p\,d\mu\Big)^{1/p},$$

so that, L is a bounded operator. Moreover, in the same manner as [O1, Lemma 3.3], we can show that L is demicontinuous and coercive. Thus, if $\nu \in (H_0^{1,p}(G;\mu))^*$, then it follows from [M, Lemma 2.6] that there exists a solution $u \in H_0^{1,p}(G;\mu)$ which satisfies (E_ν). Then, u is a supersolution of (E), so that u can be chosen to be $(\mathscr{A},\mathscr{B})$-superharmonic in G by Proposition 2.1. Further, by Lemma 3.1 below, u is unique. Namely, the following theorem holds.

Theorem 3.2. *Suppose that G is an open set with $G \Subset \Omega$ and $\nu \in (H_0^{1,p}(G;\mu))^*$ is a Radon measure in G. Then there is a unique $(\mathscr{A},\mathscr{B})$-superharmonic function u in G which satisfies (E_ν) and belongs to $H_0^{1,p}(G;\mu)$.*

Lemma 3.1. *Suppose that G is an open set with $G \Subset \Omega$ and $u_1,\ u_2 \in H_0^{1,p}(G;\mu)$ are $(\mathscr{A},\mathscr{B})$-superharmonic functions in G with $Lu_i = \nu_i$ for $i = 1,\ 2$. If $\nu_1 \le \nu_2$, then $u_1 \le u_2$ in G.*

Proof. Let $\eta = \min(u_2 - u_1, 0)$. Since $\eta \in H_0^{1,p}(G;\mu)$ and $\eta \le 0$, we have by (A.4) and (B.3)

$$\begin{aligned}
0 &\ge \int_G \eta\, d\nu_2 - \int_G \eta\, d\nu_1 \\
&= \int_G \mathscr{A}(x,\nabla u_2)\cdot\nabla\eta\,dx + \int_G \mathscr{B}(x,u_2)\,\eta\,dx \\
&\quad - \Big(\int_G \mathscr{A}(x,\nabla u_1)\cdot\nabla\eta\,dx + \int_G \mathscr{B}(x,u_1)\,\eta\,dx\Big) \\
&= \int_{\{u_1>u_2\}} (\mathscr{A}(x,\nabla u_2) - \mathscr{A}(x,\nabla u_1))\cdot\nabla\eta\,dx \\
&\quad + \int_{\{u_1>u_2\}} (\mathscr{B}(x,u_2) - \mathscr{B}(x,u_1))\,\eta\,dx \ge 0.
\end{aligned}$$

Hence,

$$\int_{\{u_1>u_2\}} (\mathscr{A}(x,\nabla u_2) - \mathscr{A}(x,\nabla u_1)) \cdot (\nabla u_1 - \nabla u_2)\, dx = 0.$$

Again from (A.4), we obtain $\nabla u_1 - \nabla u_2 = 0$ a.e. in $\{u_1 > u_2\}$, and hence $\nabla \eta = 0$ a.e. in G. Since $\eta \in H_0^{1,p}(G;\mu)$, we have $\eta = 0$ a.e. in G. Therefore, we conclude that $u_1 \le u_2$ a.e. in G. By Corollary 2.2 we see that $u_1 \le u_2$ in G. Hence the proof is complete. □

In order to show the existence of $(\mathscr{A},\mathscr{B})$-superharmonic solutions of (E_ν) with weak zero boundary values for general finite Radon measures, we prepare some lemmas.

Lemma 3.2. ([M, Lemma 2.12]) *If G is a bounded open set in Ω and ν is a finite Radon measure in G, then there is a sequence of Radon measures $\nu_n \in (H_0^{1,p}(G;\mu))^*$ such that $\nu_n(G) \le \nu(G)$ for all $n = 1, 2, \ldots$ and $\nu_n \to \nu$ weakly in G.*

Lemma 3.3. ([M, Theorem 2.14]) *Suppose that G is an open set with $G \Subset \Omega$. If $\{u_n\}$ is a bounded sequence in $H_0^{1,p}(G;\mu)$, then there is a subsequence $\{u_{n_i}\}$ and a function $u \in H_0^{1,p}(G;\mu)$ such that $u_{n_i} \to u$ in $L^s(G;\mu)$ for all $1 \le s < \varkappa p$.*

Suppose that G is an open set in Ω. A function u is said to be $(\mathscr{A},\mathscr{B})$-*hyperharmonic* in G if it is lower semicontinuous, and for each open set $U \Subset G$ and for $h \in C(\overline{U})$ which is $(\mathscr{A},\mathscr{B})$-harmonic in U, $u \ge h$ on ∂U implies $u \ge h$ in U. Note that Du is defined for every $(\mathscr{A},\mathscr{B})$-hyperharmonic function u in G, since $\min(u,k) \in H^{1,p}_{\mathrm{loc}}(G;\mu)$ for any $k > 0$ by Corollary 2.1.

Lemma 3.4. *Suppose that G is an open set in Ω. If $\{u_n\}$ is a sequence of $(\mathscr{A},\mathscr{B})$-superharmonic functions in G which is locally uniformly bounded below, then there is a subsequence $\{u_{n_i}\}$ and an $(\mathscr{A},\mathscr{B})$-hyperharmonic function u in G such that $u_{n_i} \to u$ a.e. in G and $Du_{n_i} \to Du$ a.e. in the set $\{u < \infty\}$.*

Proof. First, let $U \Subset G$, $U \Subset G' \Subset G$ and we assume that there is a constant $M \ge 0$ such that $u_n \le M$ in G' for all n. Then, by Proposition 2.2, $u_n \in H^{1,p}_{\mathrm{loc}}(G';\mu)$ is a supersolution of (E) in G'. Let $U \Subset U' \Subset G'$. Choose $\eta \in C_0^\infty(G')$ with $0 \le \eta \le 1$ in G', $\eta = 1$ in U'. Then since $(M - u_n)\eta^p \in H_0^{1,p}(G';\mu)$ and $(M - u_n)\eta^p \ge 0$ we have

$$\int_{G'} \mathscr{A}(x,\nabla u_n) \cdot \nabla[(M - u_n)\eta^p]\, dx + \int_{G'} \mathscr{B}(x,u_n)\,(M - u_n)\eta^p\, dx \ge 0.$$

Hence,

$$\int_{G'} [\mathscr{A}(x,\nabla u_n) \cdot \nabla u_n]\eta^p\, dx \le p \int_{G'} [\mathscr{A}(x,\nabla u_n) \cdot \nabla \eta](M - u_n)\eta^{p-1}\, dx \\ + \int_{G'} \mathscr{B}(x,u_n)\,(M - u_n)\eta^p\, dx.$$

We may assume that $u_n \geq -m$ for any n in G' $(m \geq 0)$. From the structure condition and the inequality $\mathscr{B}(x,u_n)\,(M-u_n) \leq |\mathscr{B}(x,M)|\,(M+m)$ we obtain

$$\begin{aligned}\alpha_1 \int_{G'} |\nabla u_n|^p \eta^p \, d\mu &\leq p\alpha_2 \int_{G'} |\nabla u_n|^{p-1} |\nabla \eta| (M+m) \eta^{p-1} \, d\mu \\ &\quad + \alpha_3(G') \int_{G'} (M^{p-1}+1)\,(M+m)\, d\mu \\ &\leq p\alpha_2 (M+m) \left(\int_{G'} |\nabla u_n|^p \eta^p \, d\mu\right)^{(p-1)/p} \left(\int_{G'} |\nabla \eta|^p \, d\mu\right)^{1/p} \\ &\quad + \alpha_3(G')\,(M^{p-1}+1)\,(M+m)\,\mu(G').\end{aligned}$$

An application of Young's inequality yields that $X \leq AX^{(p-1)/p} + C$ implies $X \leq A^p + pC$ for $X \geq 0$, $A \geq 0$ and $C \geq 0$. Therefore, $\{\int_{G'} |\nabla u_n|^p \eta^p \, d\mu\}$ is bounded. Moreover, since $\{\int_{G'} |u_n|^p |\nabla \eta|^p \, d\mu\}$ is bounded, $\{\eta u_n\}$ is bounded in $H_0^{1,p}(G';\mu)$. By Lemma 3.3, there is a subsequence $\{\eta u_{n_i}\}$ and a function $u_{U'} \in H_0^{1,p}(G';\mu)$ such that $\eta u_{n_i} \to u_{U'}$ in $L^s(G';\mu)$ for all $1 \leq s < \varkappa p$, especially $u_{n_i} \to u_{U'}$ a.e. in U'. It follows from [HKM, Theorem 1.32] that $\nabla u_{n_i} \to \nabla u_{U'}$ weakly in $L^p(U';\mu)$. We write this subsequence u_{n_i} by u_n.

Now we will show that $u_{U'}$ has an $(\mathscr{A},\mathscr{B})$-superharmonic representative. Set $v_i = \inf_{n \geq i} u_n$ and $\hat{v}_i(x) = \liminf_{y \to x} v_i(x)$ $(i = 1, 2, \ldots)$. Then, the fundamental convergence theorem yields that $\hat{v}_i$ is $(\mathscr{A},\mathscr{B})$-superharmonic in U' and $\hat{v}_i = v_i$ (p,μ)-q.e., and hence a.e. in U'. Moreover, since $\{\hat{v}_i\}$ is an increasing sequence of bounded $(\mathscr{A},\mathscr{B})$-superharmonic functions, $\hat{v} = \lim_{i\to\infty} \hat{v}_i$ is $(\mathscr{A},\mathscr{B})$-superharmonic in U' ([MO1, Proposition 2.2]). Moreover, we have

$$u_{U'}(x) = \lim_{n\to\infty} u_n(x) = \lim_{i\to\infty} v_i(x) = \lim_{i\to\infty} \hat{v}_i(x) = \hat{v}(x)$$

for a.e. $x \in U'$. Thus $u_{U'}$ has an $(\mathscr{A},\mathscr{B})$-superharmonic representative.

Next, we will show that $\nabla u_n \to \nabla u_{U'}$ a.e. in U. Fix $\varepsilon > 0$. Let

$$\begin{aligned}&E_{n,\varepsilon} := \{x \in U \,|\, (\mathscr{A}(x,\nabla u_n) - \mathscr{A}(x,\nabla u_{U'})) \cdot (\nabla u_n - \nabla u_{U'}) \geq \varepsilon\}, \\ &E^1_{n,\varepsilon} := \{x \in E_{n,\varepsilon} \,|\, |u_n - u_{U'}| \geq \varepsilon^2\} \quad \text{and} \quad E^2_{n,\varepsilon} := E_{n,\varepsilon} \setminus E^1_{n,\varepsilon}.\end{aligned}$$

Since $u_n \to u_{U'}$ in $L^p(U;\mu)$, $|E^1_{n,\varepsilon}| \to 0$ as $n \to \infty$. On the other hand,

$$|E^2_{n,\varepsilon}| \leq \frac{1}{\varepsilon} \int_{E^2_{n,\varepsilon}} (\mathscr{A}(x,\nabla u_n) - \mathscr{A}(x,\nabla u_{U'})) \cdot (\nabla u_n - \nabla u_{U'}) \, dx.$$

Let $\eta \in C_0^\infty(U')$ with $0 \leq \eta \leq 1$ in U' and $\eta = 1$ in U, and $v_n = \min\{\max(u_n - u_{U'} + \varepsilon^2, 0), 2\varepsilon^2\}$. Then since $u_{U'}$ is a supersolution of (E) in U' and $\eta v_n \in H_0^{1,p}(U';\mu)$ is nonnegative,

$$
\begin{aligned}
0 &\le \int_{U'} \mathscr{A}(x, \nabla u_{U'}) \cdot \nabla(\eta v_n)\, dx + \int_{U'} \mathscr{B}(x, u_{U'})\, \eta v_n\, dx \\
&\le \int_{U'} \mathscr{A}(x, \nabla u_{U'}) \cdot (v_n \nabla \eta)\, dx + \int_{U' \cap \{|u_n - u| < \varepsilon^2\}} \mathscr{A}(x, \nabla u_{U'}) \cdot (\eta \nabla(u_n - u_{U'}))\, dx \\
&\quad + 2\varepsilon^2 \int_{U'} |\mathscr{B}(x, u_{U'})|\, \eta\, dx.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\int_{U' \cap \{|u_n - u_{U'}| < \varepsilon^2\}} \mathscr{A}(x, \nabla u_{U'}) \cdot (\eta \nabla(u_{U'} - u_n))\, dx \\
&\le \int_{U'} \mathscr{A}(x, \nabla u_{U'}) \cdot (v_n \nabla \eta)\, dx + 2\varepsilon^2 \int_{U'} |\mathscr{B}(x, u_{U'})|\, dx \\
&\le \alpha_2 \varepsilon^2 \int_{U'} |\nabla u_{U'}|^{p-1} |\nabla \eta|\, d\mu + 2\varepsilon^2 \alpha_3(G') \int_{U'} (|u_{U'}|^{p-1} + 1)|\, d\mu \\
&\le c\varepsilon^2 \left(\int_{U'} |\nabla u_{U'}|^p\, d\mu \right)^{(p-1)/p} \left(\int_{U'} |\nabla \eta|^p\, d\mu \right)^{1/p} + c\varepsilon^2 \le c\varepsilon^2
\end{aligned}
$$

with $c > 0$ independent of ε and n. Similarly, considering $\tilde{v_n} = \min\{\max(u_{U'} - u_n + \varepsilon^2, 0), 2\varepsilon^2\}$, we have

$$
\int_{U' \cap \{|u_n - u_{U'}| < \varepsilon^2\}} \mathscr{A}(x, \nabla u_n) \cdot (\eta \nabla(u_n - u_{U'}))\, dx \le c\varepsilon^2
$$

with the same c. Thus

$$
|E^2_{n,\varepsilon}| \le \frac{1}{\varepsilon} \int_{E^2_{n,\varepsilon}} (\mathscr{A}(x, \nabla u_n) - \mathscr{A}(x, \nabla u_{U'})) \cdot (\nabla u_n - \nabla u_{U'})\, dx \le 2c\varepsilon,
$$

so that, for $n \ge n_\varepsilon$,

$$
|E_{n,\varepsilon}| = |E^1_{n,\varepsilon}| + |E^2_{n,\varepsilon}| \le (c+1)\varepsilon, \tag{3.1}
$$

where c does not depend on n and ε. To obtain the claim that $\nabla u_n \to \nabla u_{U'}$ a.e. in U, we will show that for any $\lambda > 0$

$$
|\{x \in U |\, |\nabla u_n - \nabla u_{U'}| \ge \lambda\}| \to 0 \tag{3.2}
$$

as $n \to \infty$. To the contrary, we assume that there exist $\lambda > 0$, $a > 0$ and the subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that

$$
|\{x \in U |\, |\nabla u_{n_i} - \nabla u_{U'}| \ge \lambda\}| \ge a \tag{3.3}
$$

for any i. Since $u_{U'} \in H^{1,p}(U; \mu)$, we have $|\nabla u_{U'}| < \infty$ a.e. in U, so that there exists a constant $R > 0$ such that

$$
|\{x \in U |\, |\nabla u_{U'}| > R\}| \le \frac{a}{3}. \tag{3.4}
$$

Set $\mathscr{A}_x(\xi,\eta) = (\mathscr{A}(x,\xi) - \mathscr{A}(x,\eta)) \cdot (\xi - \eta) \quad (\xi,\eta \in \mathbf{R}^N)$. If $|\eta| \le R$, then

$$\begin{aligned}\mathscr{A}_x(\xi,\eta) &= \mathscr{A}(x,\xi)\cdot\xi - \mathscr{A}(x,\xi)\cdot\eta - \mathscr{A}(x,\eta)\cdot\xi + \mathscr{A}(x,\eta)\cdot\eta \\ &\ge w(x)(-\alpha_2|\xi|^{p-1}|\eta| - \alpha_2|\xi||\eta|^{p-1} + \alpha_1|\xi|^p) \\ &\ge w(x)(-\alpha_2|\xi|^{p-1}R - \alpha_2|\xi|R^{p-1} + \alpha_1|\xi|^p).\end{aligned}$$

There exists a constant $R' > 0$ such that

$$-\alpha_2|\xi|^{p-1}R - \alpha_2|\xi|R^{p-1} + \alpha_1|\xi|^p \ge 1$$

if $|\xi| \ge R'$. It follows that $\mathscr{A}_x(\xi,\eta) \ge w(x)$ for a.e. $x \in U$ if $|\xi| \ge R'$ and $|\eta| \le R$. Since $\mathscr{A}_x(\xi,\eta)$ is continuous in (ξ,η) and $\mathscr{A}_x(\xi,\eta) > 0$ for a.e. $x \in U$ whenever $\xi,\eta \in \mathbf{R}^N$, $\xi \ne \eta$, we have

$$\delta(x) := \inf\{\mathscr{A}_x(\xi,\eta) \,\big|\, |\xi| \le R', |\eta| \le R \text{ and } |\xi-\eta| \ge \lambda\} > 0$$

for a.e. $x \in U$. Therefore, if $|\eta| \le R$ and $|\xi - \eta| \ge \lambda$, then

$$\mathscr{A}_x(\xi,\eta) \ge \min(w(x),\delta(x)) > 0 \tag{3.5}$$

for a.e. $x \in U$. Setting

$$F_{n_i} = \{x \in U \,\big|\, |\nabla u_{n_i} - \nabla u_{U'}| \ge \lambda \text{ and } |\nabla u_{U'}| \le R\},$$

we have by (3.3) and (3.4)

$$|F_{n_i}| \ge a - \frac{a}{3} = \frac{2a}{3}. \tag{3.6}$$

Since $\min(w(x),\delta(x)) > 0$ for a.e. $x \in U$, there exists $\alpha > 0$ such that

$$|\{x \in U \mid \min(w(x),\delta(x)) < \alpha\}| \le \frac{a}{3}. \tag{3.7}$$

Then from (3.5), (3.6) and (3.7) we obtain

$$\begin{aligned}|\{x \in U|\, \mathscr{A}_x(\nabla u_{n_i}, \nabla u_n) \ge \alpha\}| &= |E_{n_i,\alpha}| \ge |E_{n_i,\alpha} \cap F_{n_i}| \\ &= |F_{n_i}| - |F_{n_i} \cap \{x \in U|\, \mathscr{A}_x(\nabla u_{n_i}, \nabla u_n) < \alpha\}| \\ &\ge |F_{n_i}| - |\{x \in U|\, \min(w(x),\delta(x)) < \alpha\}| \\ &\ge \frac{2a}{3} - \frac{a}{3} = \frac{a}{3}.\end{aligned}$$

Choosing $\varepsilon > 0$ such that $\varepsilon < \min\left(\dfrac{a}{3(c+1)}, \alpha\right)$ with c in (3.1), we have

$$(c+1)\varepsilon \ge |E_{n_i,\varepsilon}| \ge |E_{n_i,\alpha}| \ge \frac{a}{3} \ge (c+1)\varepsilon,$$

which is a contradiction. Consequently, (3.2) is established.

Secondly, we relax the assumption that $\{u_n\}$ is uniformly bounded. Let U be an open set with $U \Subset G$, U' be a regular set with $U \Subset U' \Subset G$ and h_0 be the continuous solution of (E) in U' with boundary values 0 on $\partial U'$. By the above argument there exist a subsequence $\{u_n^{(1)}\}$ of $\{u_n\}$ and an $(\mathscr{A},\mathscr{B})$-superharmonic function $u^{(1)} \in H^{1,p}(U;\mu)$ such that

$$\min(u_n^{(1)}, h_0+1) \to u^{(1)} \text{ and } \nabla \min(u_n^{(1)}, h_0+1) \to \nabla u^{(1)}$$

a.e. in U. Inductively we define a subsequence $\{u_n^{(k)}\}$ of $\{u_n^{(k-1)}\}$ and an $(\mathscr{A},\mathscr{B})$-superharmonic function $u^{(k)} \in H^{1,p}(U;\mu)$ such that

$$\min(u_n^{(k)}, h_0+k) \to u^{(k)} \text{ and } \nabla \min(u_n^{(k)}, h_0+k) \to \nabla u^{(k)}$$

a.e. in U. Then $\{u^{(k)}\}$ is a increasing sequence, so that $u_U := \lim_{k\to\infty} u^{(k)}$ is $(\mathscr{A},\mathscr{B})$-hyperharmonic in U ([MO1, Proposition 2.2]). Since $u^{(k)} = \min(u_U, h_0+k)$, for any $k = 1, 2, \dots$ it follows from the diagonal method that

$$\min(u_n^{(n)}, h_0+k) \to \min(u_U, h_0+k) \text{ and } \nabla \min(u_n^{(n)}, h_0+k) \to \nabla \min(u_U, h_0+k)$$

a.e. in U. Since $\min(u_n^{(n)}, h_0+k) \to u_n^{(n)}$ $(k\to\infty)$, we have $u_n^{(n)} \to u_U$ a.e. in U and $Du_n^{(n)} \to Du_U$ a.e. in $\{x \in U \,|\, u_U(x) < \infty\}$.

Finally, we show the assertion in G. Let U_k be an open set such that $U_k \Subset U_{k+1} \Subset G$ and $G = \cup_k U_k$. There exist a subsequence $\{u_{1,n}\}$ of $\{u_n\}$ and an $(\mathscr{A},\mathscr{B})$-hyperharmonic function u_{U_1} in U_1 such that $u_{1,n} \to u_{U_1}$ a.e. in U_1 and $Du_{1,n} \to Du_{U_1}$ a.e. in $\{x \in U_1 \,|\, u_{U_1}(x) < \infty\}$. Inductively we define a subsequence $\{u_{k+1,n}\}$ of $\{u_{k,n}\}$ and an $(\mathscr{A},\mathscr{B})$-hyperharmonic function $u_{U_{k+1}}$ in U_{k+1} such that $u_{k+1,n} \to u_{U_{k+1}}$ a.e. in U_{k+1} and $Du_{k+1,n} \to Du_{U_{k+1}}$ a.e. in $\{x \in U_{k+1} \,|\, u_{U_{k+1}}(x) < \infty\}$. Thus $u_{k+1,n} = u_{k,n}$ a.e. in U_{k+1}, and hence Corollary 2.2 yields $u_{k+1,n} = u_{k,n}$ in U_{k+1}. Setting $u = u_{U_k}$ in U_k, u is $(\mathscr{A},\mathscr{B})$-hyperharmonic in G. Again, it follows from the diagonal method that $u_{k,k} \to u$ a.e. in G and $Du_{k,k} \to Du$ a.e. in $\{x \in G \,|\, u(x) < \infty\}$. Hence the proof is complete. $\square$

Now we will show the existence of $(\mathscr{A},\mathscr{B})$-superharmonic solutions of (E_ν) with weak zero boundary values.

Theorem 3.3. *Suppose that G is an open set with $G \Subset \Omega$ and ν is a finite Radon measure in G. Then there is an $(\mathscr{A},\mathscr{B})$-superharmonic function u in G satisfying (E_ν) with $\min(u,k) \in H_0^{1,p}(G;\mu)$ for all $k > 0$.*

Proof. By Lemma 3.2, there is a sequence of Radon measures $\nu_n \in (H_0^{1,p}(G;\mu))^*$ such that $\nu_n(G) \le \nu(G)$ for all $n = 1, 2, \dots$ and $\nu_n \to \nu$ weakly in G. Let G' be a regular set such that $G \Subset G' \Subset \Omega$. Then by Proposition 1.2 there is a bounded $(\mathscr{A},\mathscr{B})$-harmonic function h_0 in G' with $h_0 \in H_0^{1,p}(G';\mu)$ and by Theorem 3.2 there is a unique $(\mathscr{A},\mathscr{B})$-superharmonic function u_n in G satisfying (E_{ν_n}) with $u_n \in H_0^{1,p}(G;\mu)$. Since h_0 is bounded, there exists $c_0 \ge 0$ such that $h_0 - c_0 \le 0$ in $\overline{G}$. Therefore, comparison principle yields $u_n \ge h_0 - c_0$ in G for all n. By Lemma 3.4 there is a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ and an $(\mathscr{A},\mathscr{B})$-hyperharmonic function u in G such that $u_{n_i} \to u$ a.e. in G and $\nabla u_{n_i} \to Du$ a.e. in the set $\{u < \infty\}$. On the other hand, since $\min(u_n,k) \in H_0^{1,p}(G;\mu)$ and $0 \le (\mathscr{B}(x,u_n) - \mathscr{B}(x,0)) \min(u_n,k)$, we have

$$
\begin{aligned}
\int_G |\nabla \min(u_n,k)|^p \,d\mu &\le \alpha^{-1}\int_G \mathscr{A}(x,\nabla u_n)\cdot\nabla\min(u_n,k)\,dx\\
&\le \alpha^{-1}\int_G \mathscr{A}(x,\nabla u_n)\cdot\nabla\min(u_n,k)\,dx\\
&\quad+\alpha^{-1}\int_G (\mathscr{B}(x,u_n)-\mathscr{B}(x,0))\min(u_n,k)\,dx\\
&=\alpha^{-1}\int_G \min(u_n,k)\,d\nu_n-\alpha^{-1}\int_G \mathscr{B}(x,0)\min(u_n,k)\,dx\\
&\le \alpha^{-1}\nu(G)k+\alpha^{-1}\alpha_3(G)\mu(G)k
\end{aligned}
\tag{3.8}
$$

for $k=1,2,\dots$. Hence, in the same manner as in the proof of [HKM, Lemma 7.43], for fixed $0<s<\varkappa(p-1)$, there exists $c>0$ such that

$$
\int_G \max(u_n,0)^s\,d\mu < c,
$$

where c does not depend on n. On the other hand, $\min(u_n,0)\ge h_0-c_0$ in G for all n. Therefore

$$
\int_G |u|^s\,d\mu<\infty,
\tag{3.9}
$$

so that $u<\infty$ a.e. in G. Hence u is $(\mathscr{A},\mathscr{B})$-superharmonic in G. Moreover, since $\{\min(u_n,k)\}$ is bounded in $H_0^{1,p}(G;\mu)$ and $\min(u_{n_i},k)\to\min(u,k)$ a.e. in G, we have $u_k:=\min(u,k)\in H_0^{1,p}(G;\mu)$ for fixed $k>0$.

Theorem 3.1 yields that there exists a Radon measure $\tilde\nu$ in G such that

$$
\int_G \mathscr{A}(x,Du)\cdot\nabla\varphi\,dx+\int_G \mathscr{B}(x,u)\varphi\,dx=\int_G \varphi\,d\tilde\nu
$$

for all $\varphi\in C_0^\infty(G)$. To obtain that $\nu=\tilde\nu$, we will show $\nu_n\to\tilde\nu$ weakly in G. Fix $1<q<\frac{\varkappa p}{\varkappa(p-1)+1}$. Again, in the same manner as in the proof of [HKM, Lemma 7.43], by (3.8) there exists $c>0$ such that

$$
\int_G |\nabla u_n|^{q(p-1)}\,d\mu\le c,
\tag{3.10}
$$

where c does not depend n. Hence

$$
\begin{aligned}
\int_G |\mathscr{A}(x,\nabla u_n)w^{-1+\frac1q}|^q\,dx &\le c\int_G \left(|\nabla u_n|^{p-1}\right)^q w^q w^{-q+1}\,dx\\
&= c\int_G |\nabla u_n|^{q(p-1)}\,d\mu\le c
\end{aligned}
$$

for all n. Moreover, since $\nabla u_{n_i}\to Du$ a.e. in G,

$$
\mathscr{A}(x,\nabla u_{n_i})w^{-1+\frac1q}\to\mathscr{A}(x,Du)w^{-1+\frac1q}
$$

weakly in $L^q(G;dx)$. On the other hand, by Theorem 2.3, for any $U \Subset G$

$$\int_U |\mathscr{B}(x,u_n)w^{-1+\frac{1}{q}}|^q\,dx \le \alpha_3(U)\int_U \left(|u_n|^{p-1}+1\right)^q w^q w^{-q+1}\,d\mu \le c\int_U \left(|u_n|^{q(p-1)}+1\right)\,d\mu \le c.$$

Since $u_{n_i} \to u$ a.e. in G, we have

$$\mathscr{B}(x,u_{n_i})w^{-1+\frac{1}{q}} \to \mathscr{B}(x,u)w^{-1+\frac{1}{q}}$$

weakly in $L^q(U;dx)$. Let $\varphi \in C_0^\infty(G)$ and U be an open set in G with spt $\varphi \subset U$. Since $w^{1-\frac{1}{q}}\nabla\varphi \in L^{q/(q-1)}(G;dx)$ and $w^{1-\frac{1}{q}}\varphi \in L^{q/(q-1)}(U;dx)$, we have

$$\begin{aligned}
&\lim_{i\to\infty}\int_G \varphi\,d\nu_{n_i}\\
&=\lim_{i\to\infty}\left(\int_G \mathscr{A}(x,\nabla u_{n_i})w^{-1+\frac{1}{q}}w^{1-\frac{1}{q}}\cdot\nabla\varphi\,dx+\int_G \mathscr{B}(x,u_{n_i})w^{-1+\frac{1}{q}}w^{1-\frac{1}{q}}\varphi\,dx\right)\\
&=\int_U \mathscr{A}(x,Du)w^{-1+\frac{1}{q}}w^{1-\frac{1}{q}}\cdot\nabla\varphi\,dx+\int_U \mathscr{B}(x,u)w^{-1+\frac{1}{q}}w^{1-\frac{1}{q}}\varphi\,dx\\
&=\int_G \mathscr{A}(x,Du)\cdot\nabla\varphi\,dx+\int_G \mathscr{B}(x,u)\varphi\,dx=\int_G \varphi\,d\tilde{\nu}.
\end{aligned}$$

Hence the proof is complete. □

4. Upper estimate of $(\mathscr{A},\mathscr{B})$-superharmonic functions

In this section, we give a pointwise upper estimate for an $(\mathscr{A},\mathscr{B})$-superharmonic function in terms of the (weighted) Wolff potential (see below for the definition). Also, using this estimate, we obtain that an $(\mathscr{A},\mathscr{B})$-superharmonic function is finite (p,μ)-q.e.

As before, we define

$$Lu = -\operatorname{div}\mathscr{A}(x,\,\nabla u(x)) + \mathscr{B}(x,\,u(x)).$$

In order to show the pointwise upper estimate for an $(\mathscr{A},\mathscr{B})$-superharmonic function, we prepare following two lemmas.

Lemma 4.1. *Suppose that G is an open set in Ω, u is a supersolution of (E) in G and $\nu = Lu$ in G. If $G' \Subset G$, then*

$$\int_{G'} \mathscr{A}(x,\nabla u)\cdot\nabla\varphi\,dx + \int_{G'} \mathscr{B}(x,u)\varphi\,dx = \int_{G'} \varphi\,d\nu$$

for all bounded (p,μ)-quasicontinuous $\varphi \in H_0^{1,p}(G';\mu)$.

Proof. Let $\varphi \in H_0^{1,p}(G';\mu)$ be bounded (p,μ)-quasicontinuous in G'. Choose a sequence of functions $\varphi_n \in C_0^\infty(G')$ such that $\{\varphi_n\}$ is uniformly bounded, $\varphi_n \to \varphi$ in $H^{1,p}(G';\mu)$ and $\varphi_n \to \varphi$ (p,μ)-q.e. in G'. Then, since $\varphi_n \to \varphi$ ν-a.e. in G' (note that $\nu \in (H_0^{1,p}(G';\mu))^*)$ and $\nu(G') < \infty$, by Lebesgue's convergence theorem we have

$$\lim_{n\to\infty}\int_{G'}\varphi_n\,d\nu = \int_{G'}\varphi\,d\nu.$$

Also, from (A.3) and (B.2), we obtain

$$\begin{aligned}
&\Big|\int_{G'}\mathscr{A}(x,\nabla u)\cdot\nabla\varphi\,dx + \int_{G'}\mathscr{B}(x,u)\varphi\,dx \\
&\quad - \Big(\int_{G'}\mathscr{A}(x,\nabla u)\cdot\nabla\varphi_n\,dx + \int_{G'}\mathscr{B}(x,u)\varphi_n\,dx\Big)\Big| \\
&\le \alpha_2\int_{G'}|\nabla u|^{p-1}|\nabla\varphi-\nabla\varphi_n|\,d\mu + \alpha_3(G')\int_{G'}\big(|u|^{p-1}+1\big)|\varphi-\varphi_n|\,d\mu \\
&\le \alpha_2\Big(\int_{G'}|\nabla u|^p\,d\mu\Big)^{(p-1)/p}\Big(\int_{G'}|\nabla\varphi-\nabla\varphi_n|^p\,d\mu\Big)^{1/p} \\
&\quad + 2\alpha_3(G')\Big(\int_{G'}(|u|+1)^p\,d\mu\Big)^{(p-1)/p}\Big(\int_{G'}|\varphi-\varphi_n|^p\,d\mu\Big)^{1/p},
\end{aligned}$$

where in the last inequality we have used Hölder's inequality. Because the last integral tends to zero as $n\to\infty$, we have

$$\begin{aligned}
&\int_{G'}\mathscr{A}(x,\nabla u)\cdot\nabla\varphi\,dx + \int_{G'}\mathscr{B}(x,u)\varphi\,dx \\
&= \lim_{n\to\infty}\Big(\int_{G'}\mathscr{A}(x,\nabla u)\cdot\nabla\varphi_n\,dx + \int_{G'}\mathscr{B}(x,u)\varphi_n\,dx\Big) \\
&= \lim_{n\to\infty}\int_{G'}\varphi_n\,d\nu = \int_{G'}\varphi\,d\nu,
\end{aligned}$$

and the lemma follows. □

In the following lemma, we use the notation $u_+ = \max(u,0)$.

Lemma 4.2. *Suppose that G is an open set with $G \Subset \Omega$, u is an $(\mathscr{A},\mathscr{B})$-superharmonic function in G, $\nu = Lu$ in G, $2B = B(x_0,2R) \subset G$ and $p-1 < \gamma < \frac{\varkappa p(p-1)}{\varkappa+p-1}$. Then there exists a constant $c = c(p,\alpha_1,\alpha_2,\alpha_3(G),c_\mu,\gamma) > 0$ such that, for every $l \in \mathbf{R}$,*

$$\begin{aligned}
\Big(\frac{1}{\mu(B)}\int_B(u-l)_+^\gamma\,d\mu\Big)^{1/\gamma} &\le c\,A^{\frac{1}{\gamma}(1-\frac{1}{\varkappa})}\Big(\frac{1}{\mu(2B)}\int_{2B}(u-l)_+^\gamma\,d\mu\Big)^{1/\gamma} \\
&+ c\,R^{\frac{p}{p-1}}\,A^{\frac{1}{p-1}-\frac{1}{\varkappa(p-1)}+\frac{1}{\gamma}}\,(|l|^{p-1}+1)^{1/(p-1)} + c\,A^{\frac{1}{\gamma}-\frac{1}{\varkappa(p-1)}}\Big(R^p\,\frac{\nu(2B)}{\mu(2B)}\Big)^{1/(p-1)},
\end{aligned}$$

where

$$A = \frac{\mu\,(2B\cap\{u>l\})}{\mu\,(2B)}.$$

Proof. First, we assume $u \in H^{1,p}_{\mathrm{loc}}(G;\mu)$, i.e. u is a supersolution of (E) in G. Let $\delta > 0$. Set $\tau = \frac{\gamma}{p-1}$,

$$\Phi(t) = \begin{cases} \left(1 + \frac{t-l}{\delta}\right)^{-\tau} & \text{if } t > l, \\ 0 & \text{if } t \le l \end{cases}$$

and

$$\Psi(t) = \int_l^t \Phi(s)\,ds.$$

Then $\tau > 1$ and $\Psi(t) \le \frac{\delta}{\tau-1}$. Let $2B_{l^+} = \{x \in 2B \,|\, u(x) > l\}$ and $\eta \in C_0^\infty(2B)$ with $0 \le \eta \le 1$, $\eta = 1$ on B and $|\nabla\eta| \le 2/R$. Since $\varphi(x) = \Psi(u(x))\eta^p(x) \in H_0^{1,p}(G;\mu)$, we may assume that φ is (p,μ)-quasicontinuous and $\nabla\varphi = \eta^p\Phi(u)\nabla u + p\Psi(u)\eta^{p-1}\nabla\eta$, by Lemma 4.1 we have

$$\begin{aligned} &\int_{2B} [\mathscr{A}(x,\nabla u)\cdot\nabla u]\Phi(u)\eta^p\,dx + p\int_{2B}[\mathscr{A}(x,\nabla u)\cdot\nabla\eta]\Psi(u)\eta^{p-1}\,dx \\ &\quad + \int_{2B}\mathscr{B}(x,u)\Psi(u)\eta^p\,dx = \int_{2B}\Psi(u)\eta^p\,d\nu. \end{aligned}$$

From (A.2), (A.3) and (B.2) it follows that

$$\tag{4.1} \begin{aligned} \alpha_1\int_{2B_{l^+}} |\nabla u|^p\Phi(u)\eta^p\,d\mu \le p\alpha_2 &\int_{2B_{l^+}} |\nabla u|^{p-1}\Psi(u)|\nabla\eta|\eta^{p-1}\,d\mu \\ &+ \alpha_3(G)\int_{2B_{l^+}} (|l|^{p-1}+1)\Psi(u)\eta^p\,d\mu \\ &+ \int_{2B_{l^+}}\Psi(u)\eta^p\,d\nu, \end{aligned}$$

where we have used $-\mathscr{B}(x,u) \le -\mathscr{B}(x,l) \le \alpha_3(G)w(x)(|l|^{p-1}+1)$ on $2B_{l^+}$. Setting $v = \frac{(u-l)_+}{\delta}$, from (4.1) we obtain

$$\tag{4.2} \begin{aligned} \alpha_1\int_{2B_{l^+}} |\nabla u|^p(1+v)^{-\tau}\eta^p\,d\mu \le \frac{\delta}{\tau-1}\Bigg(p\alpha_2&\int_{2B_{l^+}}|\nabla u|^{p-1}|\nabla\eta|\eta^{p-1}\,d\mu \\ + \alpha_3(G)\int_{2B_{l^+}}(|l|^{p-1}+1)\eta^p\,d\mu &+ \int_{2B_{l^+}}\eta^p\,d\nu\Bigg). \end{aligned}$$

Young's inequality yields that, for any $\varepsilon > 0$,

$$\begin{aligned} |\nabla u|^{p-1}|\nabla\eta|\eta^{p-1} &= |\nabla u|^{p-1}\eta^{p-1}(1+v)^{-\tau\frac{p-1}{p}}(1+v)^{\tau\frac{p-1}{p}}|\nabla\eta| \\ &\le \frac{p-1}{p}\varepsilon|\nabla u|^p(1+v)^{-\tau}\eta^p + \frac{1}{p}\varepsilon^{1-p}(1+v)^{\gamma}|\nabla\eta|^p. \end{aligned}$$

It follows from (4.2) that

$$
\begin{aligned}
&\alpha_1 \int_{2B_{l+}} |\nabla u|^p (1+v)^{-\tau} \eta^p \, d\mu \\
&\le \frac{\delta}{\tau-1} \left(\alpha_2 (p-1) \varepsilon \int_{2B_{l+}} |\nabla u|^p (1+v)^{-\tau} \eta^p \, d\mu \right. \\
&\left. + \alpha_2 \varepsilon^{1-p} \int_{2B_{l+}} (1+v)^{\gamma} |\nabla \eta|^p \, d\mu \right) \\
&+ \frac{\delta}{\tau-1} \left(\alpha_3(G) \int_{2B_{l+}} (|l|^{p-1}+1) \eta^p \, d\mu + \int_{2B_{l+}} \eta^p \, d\nu \right).
\end{aligned}
\tag{4.3}
$$

Setting $\frac{\alpha_2 \delta (p-1)}{\tau-1} \varepsilon = \frac{\alpha_1}{2}$, that is $\varepsilon = \frac{\alpha_1(\tau-1)}{2\alpha_2 \delta (p-1)}$, we have

$$
\begin{aligned}
&\frac{\alpha_1}{2} \int_{2B_{l+}} |\nabla u|^p (1+v)^{-\tau} \eta^p \, d\mu \\
&\le c \left(\delta^p \int_{2B_{l+}} (1+v)^{\gamma} |\nabla \eta|^p \, d\mu + \delta (|l|^{p-1}+1) \int_{2B_{l+}} \eta^p \, d\mu + \delta \int_{2B_{l+}} \eta^p \, d\nu \right),
\end{aligned}
\tag{4.4}
$$

where $c = c(p, \alpha_1, \alpha_2, \alpha_3(G), \gamma) > 0$. Set $g = (1+v)^{1-\frac{\tau}{p}} - 1$. Then, we have $g \in H^{1,p}_{\mathrm{loc}}(G;\mu)$, so that $\eta g \in H^{1,p}_0(2B;\mu)$. It follows from the Sobolev inequality that

$$
\begin{aligned}
&\left(\frac{1}{\mu(2B)} \int_{2B} |\eta g|^{\varkappa p} \, d\mu \right)^{1/\varkappa p} \le c\, R \left(\frac{1}{\mu(2B)} \int_{2B} |\nabla (\eta g)|^p \, d\mu \right)^{1/p} \\
&\le c\, R \left\{ \left(\frac{1}{\mu(2B)} \int_{2B} |\nabla \eta|^p g^p \, d\mu \right)^{1/p} + \left(\frac{1}{\mu(2B)} \int_{2B} |\nabla g|^p \eta^p \, d\mu \right)^{1/p} \right\},
\end{aligned}
$$

so that

$$
\begin{aligned}
&\left(\frac{1}{\mu(2B)} \int_{2B} |\eta g|^{\varkappa p} \, d\mu \right)^{1/\varkappa} \\
&\le \frac{c\, R^p}{\mu(2B)} \left(\int_{2B} |\nabla \eta|^p g^p \, d\mu + \int_{2B} |\nabla g|^p \eta^p \, d\mu \right).
\end{aligned}
\tag{4.5}
$$

Since

$$
|\nabla g|^p = \left| \left(1 - \frac{\tau}{p}\right) (1+v)^{-\frac{\tau}{p}} \nabla v \right|^p = \left| 1 - \frac{\tau}{p} \right|^p (1+v)^{-\tau} |\nabla u|^p \delta^{-p} \chi_{2B_{l+}},
$$

where $\chi_{2B_{l^+}}$ is a characteristic function on $2B_{l^+}$, from (4.4) we obtain

$$\begin{aligned}\int_{2B}|\nabla g|^p\eta^p\,d\mu &= c\delta^{-p}\int_{2B_{l^+}}|\nabla u|^p(1+v)^{-\tau}\eta^p\,d\mu\\ &\le c\biggl(\int_{2B_{l^+}}(1+v)^\gamma|\nabla\eta|^p\,d\mu+\delta^{1-p}(|l|^{p-1}+1)\\ &\quad\cdot\int_{2B_{l^+}}\eta^p\,d\mu+\delta^{1-p}\int_{2B_{l^+}}\eta^p\,d\nu\biggr).\end{aligned}\tag{4.6}$$

Also, since $p-1<\gamma$, we have $p-\tau<\gamma$, so that

$$g^p\le(1+v)^{p-\tau}\le(1+v)^\gamma\tag{4.7}$$

on $2B_{l^+}$ and $g=0$ on $2B\setminus 2B_{l^+}$. It follows from (4.5), (4.6) and (4.7) that

$$\begin{aligned}&\left(\frac{1}{\mu(2B)}\int_{2B}|\eta g|^{\varkappa p}\,d\mu\right)^{1/\varkappa}\\ &\le\frac{c\,R^p}{\mu(2B)}\biggl(\int_{2B_{l^+}}(1+v)^\gamma|\nabla\eta|^p\,d\mu+\delta^{1-p}(|l|^{p-1}+1)\\ &\quad\cdot\int_{2B_{l^+}}\eta^p\,d\mu+\delta^{1-p}\int_{2B_{l^+}}\eta^p\,d\nu\biggr)\\ &\le c\,R^p\left(\frac{R^{-p}}{\mu(2B)}\int_{2B_{l^+}}(1+v)^\gamma\,d\mu+A\,\delta^{1-p}\,(|l|^{p-1}+1)+\delta^{1-p}\,\frac{\nu(\operatorname{supp}\eta)}{\mu(2B)}\right),\end{aligned}\tag{4.8}$$

where

$$A=\frac{\mu\;(2B\cap\{u>l\})}{\mu\;(2B)}.$$

Since $\gamma<\varkappa p-\frac{\gamma\varkappa}{p-1}=\varkappa p(1-\frac{\tau}{p})$, we have $v^\gamma\le v^{\varkappa p(1-\frac{\tau}{p})}\le c\,g^{\varkappa p}$ on $\{v\ge 1\}$. Hence (4.8) yields

$$\begin{aligned}&\left(\frac{1}{\mu(2B)}\int_{2B}\eta^{\varkappa p}v^\gamma\,d\mu\right)^{1/\varkappa}\\ &\le\left(\frac{\mu(2B\cap\{0<\eta^{\varkappa p}v^\gamma<1\})}{\mu(2B)}\right)^{1/\varkappa}+\left(\frac{1}{\mu(2B)}\int_{2B\cap\{\eta^{\varkappa p}v^\gamma\ge1\}}\eta^{\varkappa p}v^\gamma\,d\mu\right)^{1/\varkappa}\\ &\le A^{1/\varkappa}+c\left(\frac{1}{\mu(2B)}\int_{2B}\eta^{\varkappa p}g^{\varkappa p}\,d\mu\right)^{1/\varkappa}\\ &\le A^{1/\varkappa}\\ &\quad+c\,R^p\left(\frac{R^{-p}}{\mu(2B)}\int_{2B_{l^+}}(1+v)^\gamma\,d\mu+A\,\delta^{1-p}(|l|^{p-1}+1)+\delta^{1-p}\,\frac{\nu(\operatorname{supp}\eta)}{\mu(2B)}\right),\end{aligned}$$

so that

$$
\begin{aligned}
&\left(\frac{\delta^{-\gamma}}{\mu(2B)}\int_{2B}\eta^{\varkappa p}(u-l)_+^\gamma\,d\mu\right)^{1/\varkappa}\\
&\le A^{1/\varkappa}+c\,\delta^{-\gamma}\left(\frac{1}{\mu(2B)}\int_{2B_{l+}}(u-l)_+^\gamma\,d\mu\right)\\
&\quad+c\,R^p\,\delta^{1-p}\left(A\,(|l|^{p-1}+1)+\frac{\nu(\operatorname{supp}\eta)}{\mu(2B)}\right)+c_1A,
\end{aligned}
\tag{4.9}
$$

where $c_1=c_1(p,\alpha_1,\alpha_2,\alpha_3(G),\gamma)>0$. Setting

$$
\left(\frac{\delta^{-\gamma}}{\mu(2B)}\int_{2B}\eta^{\varkappa p}(u-l)_+^\gamma\,d\mu\right)^{1/\varkappa}=(2+c_1)\,A^{1/\varkappa},
$$

that is,

$$
\delta=(2+c_1)^{-\varkappa/\gamma}\,A^{-1/\gamma}\left(\frac{1}{\mu(2B)}\int_{2B}\eta^{\varkappa p}(u-l)_+^\gamma\,d\mu\right)^{1/\gamma},
$$

from (4.9) we obtain

$$
\begin{aligned}
A^{1/\varkappa}\le{}& c\,A\left(\frac{1}{\mu(2B)}\int_{2B}\eta^{\varkappa p}(u-l)_+^\gamma\,d\mu\right)^{-1}\frac{1}{\mu(2B)}\int_{2B_{l+}}(u-l)_+^\gamma\,d\mu\\
&+c\,R^p\,A\,(|l|^{p-1}+1)\,A^{(p-1)/\gamma}\left(\frac{1}{\mu(2B)}\int_{2B}\eta^{\varkappa p}(u-l)_+^\gamma\,d\mu\right)^{-(p-1)/\gamma}\\
&+c\,R^p\,A^{(p-1)/\gamma}\left(\frac{1}{\mu(2B)}\int_{2B}\eta^{\varkappa p}(u-l)_+^\gamma\,d\mu\right)^{-(p-1)/\gamma}\frac{\nu(\operatorname{supp}\eta)}{\mu(2B)},
\end{aligned}
$$

where we have used $A\le A^{1/\varkappa}$. It follows that either

$$
\frac{A^{1/\varkappa}}{2}\le c\,A\left(\frac{1}{\mu(2B)}\int_{2B}\eta^{\varkappa p}(u-l)_+^\gamma\,d\mu\right)^{-1}\frac{1}{\mu(2B)}\int_{2B_{l+}}(u-l)_+^\gamma\,d\mu
$$

or

$$
\begin{aligned}
\frac{A^{1/\varkappa}}{2}\le{}& c\,R^p\,A\,(|l|^{p-1}+1)\,A^{(p-1)/\gamma}\left(\frac{1}{\mu(2B)}\int_{2B}\eta^{\varkappa p}(u-l)_+^\gamma\,d\mu\right)^{-(p-1)/\gamma}\\
&+c\,R^p\,A^{(p-1)/\gamma}\left(\frac{1}{\mu(2B)}\int_{2B}\eta^{\varkappa p}(u-l)_+^\gamma\,d\mu\right)^{-(p-1)/\gamma}\frac{\nu(\operatorname{supp}\eta)}{\mu(2B)}.
\end{aligned}
$$

Therefore, either

$$
\begin{aligned}
&\left(\frac{1}{\mu(2B)}\int_{2B}\eta^{\varkappa p}(u-l)_+^\gamma\,d\mu\right)^{1/\gamma}\\
&\le c\,A^{\frac{1}{\gamma}(1-\frac{1}{\varkappa})}\left(\frac{1}{\mu(2B)}\int_{2B_{l+}}(u-l)_+^\gamma\,d\mu\right)^{1/\gamma}
\end{aligned}
\tag{4.10}
$$

or

$$(4.11)\quad \begin{aligned} &\left(\frac{1}{\mu(2B)}\int_{2B}\eta^{\varkappa p}(u-l)_+^\gamma\,d\mu\right)^{1/\gamma}\\ &\le cR^{p/(p-1)}A^{\frac{1}{p-1}(1-\frac{1}{\varkappa})+\frac{1}{\gamma}}(|l|^{p-1}+1)^{1/(p-1)}\\ &\quad+cA^{\frac{1}{\gamma}-\frac{1}{\varkappa(p-1)}}\left(R^p\frac{\nu(\operatorname{supp}\eta)}{\mu(2B)}\right)^{1/(p-1)}. \end{aligned}$$

Therefore the doubling property, (4.10) and (4.11) yield

$$\begin{aligned} &\left(\frac{1}{\mu(B)}\int_B(u-l)_+^\gamma\,d\mu\right)^{1/\gamma}\le c\left(\frac{1}{\mu(2B)}\int_{2B}\eta^{\varkappa p}(u-l)_+^\gamma\,d\mu\right)^{1/\gamma}\\ &\le cA^{\frac{1}{\gamma}(1-\frac{1}{\varkappa})}\left(\frac{1}{\mu(2B)}\int_{2B}(u-l)_+^\gamma\,d\mu\right)^{1/\gamma}\\ &\quad+cR^{p/(p-1)}A^{\frac{1}{p-1}-\frac{1}{\varkappa(p-1)}+\frac{1}{\gamma}}(|l|^{p-1}+1)^{1/(p-1)}+cA^{\frac{1}{\gamma}-\frac{1}{\varkappa(p-1)}}\left(R^p\frac{\nu(\operatorname{supp}\eta)}{\mu(2B)}\right)^{1/(p-1)}. \end{aligned}$$

Hence the required inequality holds with $\nu(B)$ replaced by $\nu(\operatorname{supp}\eta)$ in the case $u\in H^{1,p}_{\mathrm{loc}}(G;\mu)$.

To conclude the proof, let u_0 be a nonnegative bounded $(\mathscr{A},\mathscr{B})$-superharmonic function in G (see Lemma 2.1) and let $u_k=\min(u,u_0+k)$ for $k>0$. Then, $u_k\in H^{1,p}_{\mathrm{loc}}(G;\mu)$. Letting $\nu_k=Lu_k$, we have $\nu_k\to\nu$ weakly in G by Remark 3.1. Therefore, we obtain from [M, Lemma 2.11] that

$$\limsup_{k\to\infty}\nu_k(\operatorname{supp}\eta)\le\nu(\operatorname{supp}\eta)$$

in G. Hence Lebesgue's convergence theorem yields the claim of this lemma. □

For $x_0\in\Omega$ and $R>0$, we define

$$W^\nu_{p,\mu}(x_0,R)=\int_0^R\left(t^p\,\frac{\nu(B(x_0,t))}{\mu(B(x_0,t))}\right)^{\frac{1}{p-1}}\,\frac{1}{t}\,dt,$$

and $W^\nu_{p,\mu}$ is said to be the (weighted) Wolff potential of ν (cf. [M, §3]).

Using Lemma 4.2, we can show the following theorem.

Theorem 4.1. *Suppose that $0<R$, G is an open set with $G\Subset\Omega$, $2B=B(x_0,2R)\subset G$, u is an $(\mathscr{A},\mathscr{B})$-superharmonic function in G and $\nu=L(u)$. Then for any γ with $p-1<\gamma$, there exists a constant $c=c(p,\alpha_1,\alpha_2,\alpha_3(G),c_\mu,\gamma)>0$ such that*

$$u_+(x_0)\le c\,\left(\frac{1}{\mu(B)}\int_B u_+^\gamma\,d\mu\right)^{1/\gamma}+c\,W^\nu_{p,\mu}(x_0,2R)+c\,R^{p/(p-1)}.$$

Proof. By Hölder's inequality, we may only show the case $p-1<\gamma<\frac{\varkappa p(p-1)}{\varkappa+p-1}$. Let $R_j=2^{1-j}R$, $B_j=B(x_0,R_j)$,

$$M_j=\left(R_j^p\,\frac{\nu(B_j)}{\mu(B_j)}\right)^{\frac{1}{p-1}}$$

and $\lambda>0$ be a real number. We define a sequence $\{l_j\}$ inductively. Let $l_0=0$ and

$$l_{j+1}=l_j+\lambda^{-1}\left(\frac{1}{\mu(B_{j+1})}\int_{B_{j+1}}(u-l_j)_+^\gamma\,d\mu\right)^{1/\gamma}.$$

Set

$$A_j=\frac{\mu\,(B_j\cap\{u>l_j\})}{\mu\,(B_j)}.$$

Then since

$$\begin{aligned}\mu(B_j\cap\{u>l_j\})&\le(l_j-l_{j-1})^{-\gamma}\int_{B_j\cap\{u>l_j\}}(u-l_{j-1})_+^\gamma\,d\mu\\&\le(l_j-l_{j-1})^{-\gamma}\int_{B_j}(u-l_{j-1})_+^\gamma\,d\mu=\lambda^\gamma\mu(B_j),\end{aligned}\tag{4.12}$$

we have $A_j\le\lambda^\gamma$. This inequality and Lemma 4.2 yield

$$\begin{aligned}l_{j+1}-l_j&=\lambda^{-1}\left(\frac{1}{\mu(B_{j+1})}\int_{B_{j+1}}(u-l_j)_+^\gamma\,d\mu\right)^{1/\gamma}\\&\le c\,\lambda^{-1}\,A_j^{\frac{1}{\gamma}(1-\frac{1}{\varkappa})}\left(\frac{1}{\mu(B_j)}\int_{B_j}(u-l_j)_+^\gamma\,d\mu\right)^{1/\gamma}\\&\quad+c\,\lambda^{-1}\,R_j^{p/(p-1)}\,A_j^{\frac{1}{p-1}-\frac{1}{\varkappa(p-1)}+\frac{1}{\gamma}}\,(l_j^{p-1}+1)^{1/(p-1)}+\,c\,\lambda^{-1}\,A_j^{\frac{1}{\gamma}-\frac{1}{\varkappa(p-1)}}\,M_j\\&\le c\,A_j^{\frac{1}{\gamma}(1-\frac{1}{\varkappa})}\,(l_j-l_{j-1})+c\,\lambda^{-1}\,R_j^{p/(p-1)}\,A_j^{\frac{1}{p-1}-\frac{1}{\varkappa(p-1)}+\frac{1}{\gamma}}\,(l_j^{p-1}+1)^{1/(p-1)}\\&\quad+\,c\,\lambda^{-1}\,A_j^{\frac{1}{\gamma}-\frac{1}{\varkappa(p-1)}}\,M_j\\&\le c\,\lambda^{1-\frac{1}{\varkappa}}\,(l_j-l_{j-1})+c\,R_j^{p/(p-1)}\,\lambda^{\frac{\gamma}{p-1}-\frac{\gamma}{\varkappa(p-1)}}\,(l_j^{p-1}+1)^{1/(p-1)}+\,c\,\lambda^{-\frac{\gamma}{\varkappa(p-1)}}\,M_j.\end{aligned}$$

It follows that

$$
\begin{aligned}
l_k - l_1 \le l_{k+1} - l_1 &= \sum_{j=1}^{k} (l_{j+1} - l_j) \\
&\le c\,\lambda^{1-\frac{1}{\varkappa}} \sum_{j=1}^{k} (l_j - l_{j-1}) + c\,\lambda^{\frac{\gamma}{p-1} - \frac{\gamma}{\varkappa(p-1)}} \sum_{j=1}^{k} R_j^{p/(p-1)} (l_j^{p-1} + 1)^{1/(p-1)} \\
&\quad + c\,\lambda^{-\frac{\gamma}{\varkappa(p-1)}} \sum_{j=1}^{k} M_j \\
&\le c\,\lambda^{1-\frac{1}{\varkappa}} l_k + c\,\lambda^{\frac{\gamma}{p-1} - \frac{\gamma}{\varkappa(p-1)}} (l_k^{p-1} + 1)^{1/(p-1)} \sum_{j=1}^{k} R_j^{p/(p-1)} + c\,\lambda^{-\frac{\gamma}{\varkappa(p-1)}} \sum_{j=1}^{k} M_j,
\end{aligned}
$$

in the last inequality we have used $l_0 = 0$. Choosing λ small enough, we can obtain

$$
l_k \le c\,l_1 + c \sum_{j=1}^{\infty} M_j + c \sum_{j=1}^{\infty} R_j^{p/(p-1)}, \tag{4.13}
$$

where $c = c(p, \alpha_1, \alpha_2, \alpha_3(G), c_\mu, \gamma) > 0$. Also, letting $\lambda < 1$, by the definition of l_j we have

$$
l_j - l_{j-1} \ge \inf_{B_j} (u - l_{j-1})_+ \ge \inf_{B_j} u_+ - l_{j-1},
$$

so that

$$
\inf_{B_j} u_+ \le l_j. \tag{4.14}
$$

Also,

$$
\sum_{j=1}^{\infty} M_j \le W^{\nu}_{p,\mu}(x_0, 2R). \tag{4.15}
$$

Hence from the lower semicontinuity, (4.13), (4.14) and (4.15) we obtain

$$
\begin{aligned}
u_+(x_0) &\le \lim_{k\to\infty} \inf_{B_k} u_+ \le \lim_{k\to\infty} l_k \\
&\le c \left(\frac{1}{\mu(B)} \int_B u_+^{\gamma}\, d\mu \right)^{1/\gamma} + c\,W^{\nu}_{p,\mu}(x_0, 2R) + c\,R^{p/(p-1)},
\end{aligned}
$$

as required. □

Let G be an open subset in Ω and $E = \{x \in G \mid W^{\nu}_{p,\mu}(x, r) = \infty \text{ for some } r > 0\}$. Then, it is known that $\operatorname{cap}_{p,\mu} E = 0$ (for example, see [M, Theorem 3.1] and [HKM, Theorem 10.1]). Hence, from the above theorem we obtain the following corollary which will be used to show the uniqueness result of $(\mathscr{A}, \mathscr{B})$-superharmonic solutions of (E_ν) with weak zero boundary values in next section.

Corollary 4.1. *An $(\mathscr{A}, \mathscr{B})$-superharmonic function is finite (p, μ)-q.e.*

5. Uniqueness of $(\mathscr{A}, \mathscr{B})$-superharmonic solutions

In this section, we discuss uniqueness of $(\mathscr{A}, \mathscr{B})$-superharmonic solutions of (E_ν) with boundary conditions $\min(u, k) \in H_0^{1,p}(\Omega; \mu)$ for all $k > 0$.

If $\nu(E) = 0$ whenever $\mathrm{cap}_{p,\mu} E = 0$, then we say that ν is absolutely continuous with respect to (p, μ)-capacity.

Proposition 5.1. ([M, Corollary 6.5]) *If G is an open set with $G \Subset \Omega$ and ν is a finite Radon measure in G which is absolutely continuous with respect to (p, μ)-capacity, then there is a nondecreasing sequence of Radon measures $\nu_n \in (H_0^{1,p}(G; \mu))^*$ such that $\nu_n(G) \le \nu(G)$ for all $n = 1, 2, \dots$ and*

$$\lim_{n\to\infty} \int_G \varphi \, d\nu_n = \int_G \varphi \, d\nu$$

for any bounded Borel measurable function φ on G.

Hereafter, we shall always assume that functions in $H^{1,p}_{\mathrm{loc}}(\Omega; \mu)$ are (p, μ)-quasicontinuous. (see [HKM, Theorem 4.4]).

Let G be an open set with $G \Subset \Omega$. If an $(\mathscr{A}, \mathscr{B})$-superharmonic solution u of (E_ν) in G satisfies $u \in L^{p-1}(G; dx)$, $|\nabla T_k^\sigma(u)| \in L^{p-1}(G; \mu)$ and for $\sigma \in \{+, -\}$

$$\int_G \mathscr{A}(x, Du) \cdot \nabla T_k^\sigma(u - \varphi) \, dx + \int_G \mathscr{B}(x, u) T_k^\sigma(u - \varphi) \, dx = \int_G T_k^\sigma(u - \varphi) \, d\nu$$

for all bounded $\varphi \in H_0^{1,p}(G; \mu)$ and $k > 0$, then we call u an entropy solution of (E_ν) in G. Here,

$$T_k^+(t) = \max\{\min(t, k), 0\} \quad \text{and} \quad T_k^-(t) = \min\{\max(t, -k), 0\}.$$

Then, there exists an $(\mathscr{A}, \mathscr{B})$-superharmonic entropy solutions of (E_ν) with weak boundary values zero.

Theorem 5.1. *Suppose that G is an open set with $G \Subset \Omega$, ν is a finite Radon measures in G which is absolutely continuous with respect to (p, μ)-capacity. Then, there exists an $(\mathscr{A}, \mathscr{B})$-superharmonic entropy solution u of (E_ν) in G with $\min(u, k) \in H_0^{1,p}(G; \mu)$ for all $k > 0$.*

Proof. By Proposition 5.1, we can choose Radon measures $\nu_n \in (H_0^{1,p}(G; \mu))^*$ such that $\nu_n \le \nu_{n+1} \le \nu$ for all $n = 1, 2, \dots$ and $\nu_n \to \nu$ weakly in G. Then, Theorem 3.2 yields that there exists an $(\mathscr{A}, \mathscr{B})$-superharmonic function $u_n \in H_0^{1,p}(G; \mu)$ such that $Lu_n = \nu_n$. By Lemma 3.1, $u_n \le u_{n+1}$. As in the proof of Theorem 3.3, we can choose a subsequence $\{u_{n_i}\}$ and an $(\mathscr{A}, \mathscr{B})$-superharmonic function u in G such that $u_{n_i} \to u$ a.e. in G, $\nabla u_{n_i} \to Du$ a.e. in G and $Lu = \nu$ with $\min(u, k) \in H_0^{1,p}(G; \mu)$ for $k = 1, 2, \dots$.

By (3.8) in the proof of Theorem 3.3, we see that $\{\int_G |\nabla \min(u_{n_i}, k)|^p \, d\mu\}$ is bounded, so that $\{\mathscr{A}(x, \nabla \min(u_{n_i}, k)) w^{-1/p}\}$ is bounded in $L^{p/(p-1)}(G; dx)$. Since $\nabla u_{n_i} \to Du$ a.e. in G, it follows that

$$\mathscr{A}(x, \nabla \min(u_{n_i}, k)) w^{-1/p} \to \mathscr{A}(x, \nabla \min(u, k)) w^{-1/p}$$

weakly in $L^{p/(p-1)}(G;dx)$ for any $k>0$. Moreover, since u_n increases to u a.e. in G, $\mathscr{B}(x,u_{n_i})\to\mathscr{B}(x,u)$ a.e. in G and $\mathscr{B}(x,u_1)\le\mathscr{B}(x,u_{n_i})\le\mathscr{B}(x,u)$ a.e. in G. Choosing $s=p-1$ in (3.9) in the proof of Theorem 3.3, we see that $\mathscr{B}(x,u)\in L^1(G;dx)$.

Let $\varphi\in H_0^{1,p}(G;\mu)$ be bounded and let $|\varphi|\le M$. Since $u_n\le u\le k+M$ whenever $u-\varphi\le k$ and $|\nabla T_k^\sigma(u-\varphi)|w^{1/p}\in L^p(G;dx)$, we have

$$
\begin{aligned}
&\int_G T_k^\sigma(u-\varphi)\,d\nu=\lim_{i\to\infty}\int_G T_k^\sigma(u-\varphi)\,d\nu_{n_i}\\
&=\lim_{i\to\infty}\left(\int_G \mathscr{A}(x,\nabla u_{n_i})\cdot\nabla T_k^\sigma(u-\varphi)\,dx+\int_G \mathscr{B}(x,u_{n_i})T_k^\sigma(u-\varphi)\,dx\right)\\
&=\lim_{i\to\infty}\left(\int_G \mathscr{A}(x,\nabla\min(u_{n_i},k+M))w^{-1/p}\cdot\nabla T_k^\sigma(u-\varphi)w^{1/p}\,dx\right.\\
&\quad\left.+\int_G \mathscr{B}(x,u_{n_i})T_k^\sigma(u-\varphi)\,dx\right)\\
&=\int_G \mathscr{A}(x,\nabla\min(u,k+M))w^{-1/p}\cdot\nabla T_k^\sigma(u-\varphi)w^{1/p}\,dx\\
&\quad+\int_G \mathscr{B}(x,u)T_k^\sigma(u-\varphi)\,dx\\
&=\int_G \mathscr{A}(x,Du)\cdot\nabla T_k^\sigma(u-\varphi)\,dx+\int_G \mathscr{B}(x,u)T_k^\sigma(u-\varphi)\,dx.
\end{aligned}
$$

Hence the proof is complete. □

In the same manner as [KX, Lemma 2.3], we obtain the following lemma.

Lemma 5.1. *Suppose that G is an open set with $G\Subset\Omega$, ν is a finite Radon measure in G which is absolutely continuous with respect to (p,μ)-capacity, and u is an $(\mathscr{A},\mathscr{B})$-superharmonic entropy solution of (E_ν) in G. Then for any $M>0$ and $k>0$,*

$$
\begin{aligned}
&\alpha_1\int_{\{x\in G\,|\,k\le u(x)\le k+M\}}|Du|^p\,d\mu\\
&\le M\nu(\{x\in G\,|\,u(x)>k\})+M\int_{\{x\in G\,|\,u(x)>k\}}|\mathscr{B}(x,u)|\,dx.
\end{aligned}
$$

By the above lemma and Corollary 4.1, we have the following corollary.

Corollary 5.1 *Suppose that M is a positive constant, G is an open subset in Ω, ν is a finite Radon measure in G which is absolutely continuous with respect to (p,μ)-capacity, and u is an entropy solution of (E_ν) in G. Then*

$$
\lim_{k\to\infty}\int_{\{x\in G\,|\,k\le u(x)\le k+M\}}|Du|^p\,d\mu=0.
$$

Using the above corollary, as in the proof of [KX, Theorem 2.5], we can show the following uniqueness result of $(\mathscr{A},\mathscr{B})$-superharmonic solutions of (E_ν) with weak zero boundary values. (Note that we use Corollary 2.2 to show that the inequality $u_1 \le u_2$ holds everywhere in G.)

Theorem 5.2. *Suppose that G is an open set with $G \Subset \Omega$, ν_1 and ν_2 are finite Radon measures in G that are absolutely continuous with respect to (p,μ)-capacity and u_i is an $(\mathscr{A},\mathscr{B})$-superharmonic entropy solution in G of (E_{ν_i}) with $\min(u_i,k) \in H_0^{1,p}(G;\mu)$ for all $k>0$ for $i=1,2$. If $\nu_1 \le \nu_2$, then $u_1 \le u_2$ in G.*

Acknowledgments. The author is grateful to Professor Fumi-Yuki Maeda for letting me include his proof of Theorem 2.1 and for giving many valuable comments.

This research has been supported by a Grant-in-Aid for Scientific Research in Japan.

References

[AG] Armitage, D. H., and S. J. Gardiner: Classical potential theory. - Springer, 2001.

[B+5] Bénilan, L., L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, and J. Vazques: An L^1-theory of existence and uniqueness of solutions nonlinear elliptic equations. - Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) 22, 1995, 241–274.

[BG] Boccardo, L., and T. Gallouët: Nonlinear elliptic and parabolic equations involving measure data. - J. Funct. Anal. 87, 1989, 149–169.

[HK] Heinonen, J., and T. Kilpeläinen: Polar sets for supersolutions of degenerate elliptic equations. - Math. Scand. 63, 1988, 136–150.

[HKM] Heinonen, J., T. Kilpeläinen, and O. Martio : Nonlinear potential theory of degenerate elliptic equations. - Clarendon Press, 1993.

[K] Kilpeläinen, T.: Nonlinear potential theory and PDEs. - Potential Anal. 3, 1994, 107–118.

[KM1] Kilpeläinen, T., and J. Malý: Degenerate elliptic equations with measure data nonlinear potentials. - Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) 19, 1992, 591–613.

[KM2] Kilpeläinen, T., and J. Malý: The Wiener test and potential estimates for quasilinear elliptic equations. - Acta Math. 172, 1994, 137–161.

[KX] Kilpeläinen, T., and X. Xu: On the uniqueness problem for quasilinear elliptic equations involving measures. - Rev. Mat. Iberoamericana 12, 1996, 461–475.

[MO1] Maeda, F.-Y., and T. Ono: Resolutivity of ideal boundary for nonlinear Dirichlet problems. - J. Math. Soc. Japan 52, 2000, 561–581.

[MO2] Maeda, F.-Y., and T. Ono: Properties of harmonic boundary in nonlinear potential theory. - Hiroshima Math. J. 30, 2000, 513–523.

[MO3] Maeda, F.-Y., and T. Ono: Perturbation theory for nonlinear Dirichlet problems. - Ann. Acad. Sci. Fenn. Math. 28, 2003, 207–222.

[MZ] Malý, J., and W. P. Ziemer: Fine regularity of solutions of elliptic partial differential equations. - Math. Surveys Monogr. 51, American Mathematical Society, 1997.

[M] Mikkonen, P.: On the Wolff potential and quasilinear elliptic equations involving measures. - Ann. Acad. Sci. Fenn. Math. Diss. 104, 1996, 1–71.

[O1] Ono, T.: On solutions of quasi-linear partial differential equations $-\mathrm{div}\mathscr{A}(x, \nabla u) + \mathscr{B}(x, u) = 0$. - RIMS Kokyuroku 1016, 1997, 146–165.

[O2] Ono, T.: Potential theory for quasi-linear partial differential equations $-\mathrm{div}\mathscr{A}(x, \nabla u) + \mathscr{B}(x, u) = 0$. - Doctoral Thesis, Hiroshima University, 2000.

[R] Rakotoson, J. M.: Uniqueness of renormalized solutions in a T-set for the L^1-data problem and the link between various formulations. - Indiana Univ. Math. J. 43, 1994, 685–702.

Received 1 February 2007