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EUCLIDEAN QUASICONVEXITY

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Abstract. We exhibit compact totally disconnected sets in $\mathbf{R}^n$, with Hausdorff dimension $n-1$, whose complements fail to be quasiconvex, and similar sets with positive n-measure whose complements are quasiconvex. We characterize the finitely connected quasiconvex plane domains. We present related results for bounded turning.

1. Introduction

A rectifiable path is *c-quasiconvex*, $c \geq 1$, if its length is at most c times the distance between its endpoints. A metric space is *c-quasiconvex* if each pair of points can be joined by a c-quasiconvex path. That is, for all points x, y there exists a rectfiable path γ joining x, y and satisfying

$$\ell(\gamma) \leq c\,|x-y|\,.$$

Quasiconvex spaces are precisely the spaces which are bilipschitz equivalent to length spaces. The notion of quasiconvexity plays a prominent role in the theory of analysis in the metric space setting; especially, there are strong connections with the so-called John and uniform spaces. For example, a John disk is a quasidisk if and only if it is quasiconvex. See [Geh82], [Geh87], [Väi88], [NV91] and the references mentioned therein. Also, the quasiconvexity of a bounded simply connected plane domain is closely related to Hölder continuity properties of the associated Riemann map and/or its inverse, as explained in [NP83]. Other examples of quasiconvex spaces are upper regular Loewner spaces and doubling metric measure spaces which support a $(1,p)$-Poincaré inequality; this list includes Carnot groups and certain Riemannian manifolds with non-negative Ricci curvature; see [HK98, 3.13, 3.18, §6].

The primary purpose of this article is to investigate Euclidean quasiconvexity. In particular, we seek to understand which closed sets in Euclidean n-dimensional space $\mathbf{R}^n$ $(n \geq 2)$ have quasiconvex complements. An example is provided by any closed set whose projections onto each coordinate $(n-1)$-plane have $(n-1)$-measure

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zero. To prove this, use the ACL characterization of Sobolev functions to see that such a set is removable for Sobolev functions, hence a null set for capacity and therefore a so-called null set for extremal distance; a 'modulus of curves argument' now gives quasiconvexity as explained in [GM85, 2.7, 2.9]. Thus the complement of a closed set is quasiconvex if, e.g., the set itself has zero $(n-1)$-dimensional Hausdorff measure.

Geometric reasoning reveals the result stated below. Note that the proof of this provides an alternative elementary argument for the aforementioned result. As an application we see that, by taking products of closed totally disconnected (i.e., nowhere dense) subsets of $\mathbf{R}$, we can construct closed totally disconnected sets in $\mathbf{R}^n$—even with positive n-measure—whose complements are quasiconvex. In particular, there are quasiconvex domains in $\mathbf{R}^n$ whose boundaries have positive n-measure.

Theorem A. *Let A be a closed set in $\mathbf{R}^n$. Suppose each projection of A onto a coordinate $(n-1)$-plane is nowhere dense. Then $\mathbf{R}^n \setminus A$ is quasiconvex.*

In light of the above, the following result[†] completes the picture regarding the relations, or lack thereof, between the Hausdorff dimension of a closed set and quasiconvexity of its complement. This result should be contrasted with Proposition 4.1.

Theorm B. *There exists a compact totally disconnected set in $\mathbf{R}^n$ that has Hausdorff dimension $n-1$ and a non-quasiconvex complement.*

Combining Theorem B with Theorem A, and the comments preceding it, we obtain the following information concerning the quasiconvexity of the complement of a totally disconnected closed set versus its Hausdorff dimension or measure.

Corollary C. *If a closed set in $\mathbf{R}^n$ has $(n-1)$-measure zero, then its complement is quasiconvex. On the other hand, for each $d \in [n-1, n]$, there exist compact totally disconnected sets A_d and B_d, each having Hausdorff dimension d and with A_d^c not quasiconvex while B_d^c is quasiconvex. Moreover, we can select B_d so that it is has positive finite Hausdorff d-measure. For $d \in (n-1, n]$, the same holds for A_d.*

Indeed, we can take $B_d = C_d^n$ where $C_d \subset [0,1]$ is an appropriate Cantor type set. Similarly, we can take $A_d := A_{n-1} \cup C_d^n$ where A_{n-1} is the set given by Theorem B.

Our original interest in studying totally disconnected sets was due to the fact that one knows exactly what the complements of quasiconvex plane domains 'look like'. For this discussion, it is convenient to introduce the terminology *Jordan curve domain* for an open connected plane region each of whose boundary components is either a single point or a Jordan curve. The reader may consult Figures 1, 2, and 3 for several illuminating examples of Jordan curve domains.

Quasiconvex plane domains enjoy a number of nice properties.

Theorem D. *Suppose $D \subsetneq \mathbf{R}^2$ is a c-quasiconvex domain. Then:*

[†]We would like to thank the many participants at the recent Workshop in Geometric Function Theory (May 8–12, 2006 at the University of Michigan) with whom we discussed this issue.

(1) *D is a Jordan curve domain*[‡],
(2) *∂D has at most $\pi/\arcsin(1/c)$ unbounded components, and*
(3) *for any $b > c$, each pair of points $\xi, \eta \in \bar{D}$ can be joined by a b-quasiconvex path in $D \cup \{\xi, \eta\}$; in particular, each point of ∂D is rectifiably accessible.*

Note that condition (3) above is best possible since there may be boundary points which cannot be joined by c-quasiconvex paths. Also, using (2) with $c = 1$ we see that any convex plane domain can have at most two unbounded boundary components. In fact (2) is sharp: given an integer n with $n \geq 2$, there exists a simply connected c-quasiconvex plane domain with $c = 1/\sin(\pi/n)$ and having n unbounded boundary components; see Example 2.8.

The above necessary conditions are also sufficient for plane domains which have finitely many boundary components.

Theorem E. *Let $D \subsetneq \mathbf{R}^2$ be a Jordan curve domain with ∂D having finitely many components. Suppose $c \geq 1$ and each pair of rectifiably accessible points $\xi, \eta \in \partial D$ can be joined by a c-quasiconvex path in $D \cup \{\xi, \eta\}$. When $c > 1$, D is c-quasiconvex; if $c = 1$, then $D = G \setminus F$ where G is strictly convex and F is a finite set.*

We mention that if E is any closed set of points lying on some strictly convex curve, then the complement of E satisfies all the hypotheses of the above with $c = 1$, but clearly it is not convex. We can weaken the hypothesis that there be 'finitely many boundary components' if instead we require that all boundary points be joinable by quasiconvex paths. We use this alternative to characterize finitely connected quasiconvex plane domains. We note that there are simply connected Jordan curve domains having infinitely many unbounded boundary components; see Figure 3.

Corollary F. *Let $D \subsetneq \mathbf{R}^2$ be a finitely connected domain. Then D is c-quasiconvex if and only if*
(1) *D is a Jordan curve domain, and*
(2) *each pair of points $\xi, \eta \in \partial D$ can be joined by a b-quasiconvex path in $D \cup \{\xi, \eta\}$.*

For the necessity, we can take any $b > c$; for the sufficiency, $c = b$ works (provided $b > 1$).

For a simply connected quasiconvex domain D we find that either ∂D is a Jordan loop (which occurs when D is bounded), or a union of finitely many Jordan lines (when D is unbounded).

Thus we know precisely when a finitely connected plane region will be quasiconvex. Perhaps the simplest non-finitely connected domains are complements of closed totally disconnected sets, hence our interest in these regions. (Note that the complement of a closed totally disconnected set in $\mathbf{R}^n$ is rectifiably connected; see

[‡]In fact, each non-degenerate component of D^c is a closed b-John disk with $b = b(c)$; see Corollary 3.4.

§4.B for even more information.) It would be worthwhile to have criteria describing when such a region is quasiconvex and so we ask the following.

1.1. Question. Suppose $A \subset \mathbf{R}^n$ is compact and totally disconnected with Hausdorff dimension in $[n-1, n]$. When will A^c be quasiconvex?

Of course, Theorem A and the comments just preceding it provide topologic and measure-theoretic sufficient conditions for A^c to be quasiconvex.

Looking carefully at the proof of Theorem B we see that the set constructed there has infinite $(n-1)$-dimensional Hausdorff measure. Thus it is natural to ask the following.

1.2. Question. Does there exists a compact totally disconnected set in $\mathbf{R}^n$ with a non-quasiconvex complement and *finite* $(n-1)$-dimensional Hausdorff measure?

This document is organized as follows: Section 2 contains preliminary information including basic definitions and terminology as well as elementary examples. In Section 3 we examine quasiconvexity and bounded turning of plane domains and corroborate Theorems D, E and Corollary F; see 3.9, 3.10 and 3.11 respectively. We establish Theorems A and B in Section 4.

We thank the two referees for their thoughtful suggestions which improved our paper. We especially thank the referee who drew our attention to Fact 3.5, which in turn provided simplifications to our original proofs of Propositions 3.6 and 3.8, and who also recommended that we examine bounded turning and suggested the argument for Corollary 4.2.

2. General metric spaces

Here we set forth our (relatively standard) notation and terminology, provide fundamental definitions, present basic information, and exhibit elementary examples. Throughout this section (X, d) denotes a general metric space which we usually refer to as just X. In this setting, all topological notions refer to the metric topology. We write $\bar{X}_d$ and $\partial_d X := \bar{X}_d \setminus X$ to denote the metric completion and metric boundary, respectively, of (X, d).

2.A. Basic definitions. We write the distance between points $x, y \in X$ as $|x-y| = d(x,y)$. The open ball (sphere) of radius r centered at the point x is $B(x;r) := \{y : |x-y| < r\}$ $(S(x;r) := \{y : |x-y| = r\})$.

Points of X are *separated* by a closed set F if they lie in different components of $X \setminus F$.

A *continuum* is a non-empty compact connected space that we always assume is *non-degenerate* which means that it contains more than a single point. Points are *joined* by a continuum if they belong to it. We require the following result which can be found in [Kur68, p. 172] or [HY88, p. 47].

2.1. Fact. If A is a closed subspace of a continuum K and C is a component of A, then $C \cap \overline{(K \setminus A)} \neq \emptyset$.

A *path* (*arc*) is a continuous (homeomorphic) map of an interval; intervals are assumed to be compact unless explicitly indicated otherwise. We use the notation $|\gamma|$ for the trajectory (i.e., image) of a path γ. However, for points $x, y \in \mathbf{R}^n$, we write $[x, y]$ both for the line segment joining x and y as well as the affine path $[0, 1] \ni t \mapsto x + t(y - x)$. For paths and arcs, the phrase *joins x to y* is also meant to describe an orientation, and when x, y are points on an arc α, we write $\alpha[x, y], \alpha(x, y), \alpha[x, y)$ for the various (closed, open, etc.) subarcs of α joining x to y. We mention that every path contains an arc which joins its endpoints; see [Väi94].

When α and β are paths which join x to y and y to z respectively, we write $\alpha\star\beta$ for the concatenation of α and β; so $\alpha\star\beta$ joins x to z. Of course, $|\alpha\star\beta| = |\alpha| \cup |\beta|$.

The length of a path $\gamma : [0, 1] \to X$ is defined in the usual way by

$$\ell(\gamma) := \sup\{\sum_{i=1}^{n} |\gamma(t_i) - \gamma(t_{i-1})| \,:\, 0 = t_0 < t_1 < \cdots < t_n = 1\}\,.$$

We call γ *rectifiable* when $\ell(\gamma) < \infty$, and X is *rectifiably connected* provided each pair of points in X can be joined by a rectifiable path. Every such metric space X admits a natural (or *intrinsic*) metric, its so-called *length distance* given by

$$l(x, y) := \inf\{\ell(\gamma) : \gamma \text{ a rectifiable path joining } x, y \text{ in } X\}\,.$$

A metric space (X, d) is a *length* space provided $d(x, y) = l(x, y)$ for all points $x, y \in X$; it is also common to call such a d an *intrinsic* distance function. If A is the trajectory $|\alpha|$ of some arc α, we also write $\ell(A) := \ell(\alpha)$.

2.B. Quasiconvexity & bounded turning. Here we examine these concepts for a metric space which possesses no additional presumed properties. A metric space satisfies the *bounded turning* condition if points can be joined by continua whose diameters are no larger than a fixed constant times the distance between the original points. To be precise, given a constant $a \geq 1$, we say that X has the *a-bounded turning* property if each pair of points $x, y \in X$ can be joined by a continuum K satisfying $\operatorname{diam} K \leq a\,|x - y|$; we abbreviate this by declaring that X is a-BT. The bounded turning condition has a venerable position in quasiconformal analysis; see the references in [Geh82], [NV91], [Tuk96].

There are related notions where one replaces 'joined by a continuum' with 'joined by a connected set' or 'joined by a path'; cf. [NP83], [Tuk96]. Below we consider the related condition obtained by replacing 'joined by a continuum' with 'joined by a rectifiable path' and using arc length in place of diameter. We remark that, in an ambient length space, for each $\varepsilon > 0$ one can always replace a continuum K, which joins two points in some open set, by a path γ, which joins the same two points in the same open set, with $\operatorname{diam}|\gamma| \leq (1 + \varepsilon) \operatorname{diam} K$. Tukia established a far more interesting result in [Tuk96].

A rectifiable path γ with endpoints x, y is a *c-quasiconvex path*, $c \geq 1$, if $\ell(\gamma) \leq c\,|x - y|$. A metric space is *$c$-quasiconvex* if each pair of points can be joined by a c-quasiconvex path. (Note that in general, the trajectory of a quasiconvex path need not be quasiconvex.) Thus a metric space is a length space if and only if it is c-quasiconvex for each $c > 1$. A 1-quasiconvex metric space is usually called a *geodesic* space. By cutting out any loops, we can always replace a c-quasiconvex path with a c-quasiconvex arc having the same endpoints; see [Väi94].

Since $|x - y| \le l(x, y)$ for all x, y, the identity map $(X, l) \xrightarrow{\text{id}} (X, d)$ is always Lipschitz continuous. Evidently, X is quasiconvex if and only if this identity map is bilipschitz. In particular, if X is c-quasiconvex, then this map is c-bilipschitz. Since Lipschitz maps are *a fortiori* uniformly continuous, id would have a c-bilipschitz extension between the completions $\bar{X}_l$ and $\bar{X}_d$; in particular, for quasiconvex spaces, $\partial_l X = \partial_d X$ as sets. Also, $\text{id}\colon (X, l) \to (X, d)$ being c-bilipschitz implies that $\bar{X}_d$ is b-quasiconvex for each $b > c$. Thus the metric completion of a quasiconvex space is quasiconvex. In fact, slightly more is true; e.g., all boundary points of a quasiconvex space are rectifiably accessible.

2.2. Lemma. *Fix $b > c \ge 1$ and let $\xi, \eta \in \partial_d X$. If X is c-quasiconvex (or c-BT), then $X \cup \{\xi, \eta\}$ is b-quasiconvex (or b-BT, respectively); in particular, $\bar{X}_d$ is b-quasiconvex (or b-BT, respectively).*

This result is sharp in that it may not be possible to join boundary points by c-quasiconvex paths; see Example 2.7.

Proof. This is established for quasiconvex domains in Euclidean space in [HK91, 2.7]. The same argument works in the general metric space setting. Minor modifications yield the corresponding result for bounded turning. $\square$

In [New51, Theorem 3.3, p. 78] we find the following fact:

> *If X is a connected subspace of a connected space Z and C is a component of $Z \setminus X$, then $Z \setminus C$ is connected.*

In Lemmas 2.4 and 2.5 we present analogs of this for the quasiconvex, bounded turning and locally connected settings. Roughly speaking, we can always assume that the complement of an open quasiconvex (or BT) subspace of a quasiconvex (or BT) space is connected. Our proofs use the following fact.

2.3. Lemma. *Let X be an open subspace of Z, let C be a component of $Z \setminus X$, and put $Y := Z \setminus C$. Suppose $K \subset Z$ is a continuum with $K \cap Y \ne \emptyset \ne K \cap C$. Then for each $y \in K \cap Y$ there is a subcontinuum $K_y \subset K$ that satisfies $y \in K_y \subset Y$ and $K_y \cap X \ne \emptyset$.*

Proof. Fix a point $y \in K \cap Y$. The assertion is not hard to check when $y \in X$, so assume $y \notin X$. This means that y lies in some component, say C_y, of $Z \setminus X$ and $C_y \cap C = \emptyset$. Since $y \notin C$, $d := \text{dist}(y, C) > 0$. For each $\varepsilon \in (0, d)$, let $N_\varepsilon = \bigcup_{z \in C} B(z; \varepsilon)$ and let K_ε be the component of $K \setminus N_\varepsilon$ which contains y. According to Fact 2.1, $K_\varepsilon \cap \bar{N}_\varepsilon \ne \emptyset$.

We claim that there is an $\varepsilon \in (0, d)$ with $K_\varepsilon \cap X \ne \emptyset$. For if this were false, then the set $S := \bigcup_{\varepsilon \in (0,d)} K_\varepsilon$ would satisfy $\bar{S} \subset C_y$, but as $\bar{S} \cap C \ne \emptyset$ this would yield a contradiction. Pick such an ε; then $K_y := K_\varepsilon$ has the asserted properties. $\square$

2.4. Lemma. *Let X be an open subspace of Z. Let C be a component of $Z \setminus X$ and put $Y := Z \setminus C$. If X is c-quasiconvex (or c-BT) and Z is b-quasiconvex (or b-BT), then Y is bc-quasiconvex (or $b(c+1)$-BT, respectively).*

Note that when Z is geodesic, this asserts that $Z \setminus C$ is c-quasiconvex (or $(c+1)$-BT).

Proof. The two proofs are quite similar, so we only sketch the argument for the quasiconvex case. Assume X is c-quasiconvex and Z is b-quasiconvex.

Fix $x, y \in Y$ and let α be a b-quasiconvex arc joining x to y in Z. If α stays in Y we are done, so assume otherwise; thus α meets C. We claim there are points $u, v \in |\alpha| \cap X$ with $|\alpha[x, u]|, |\alpha[y, v]| \subset Y$. Given this, we select a c-quasiconvex path β joining u, v in X; then the concatenation $\alpha[x, u] \star \beta \star \alpha[v, y]$ is a bc-quasiconvex path joining x, y in Y.

Now assume X is c-BT and Z is b-BT. Fix $x, y \in Y$ and let $K \subset Z$ be a continuum joining x, y with $\operatorname{diam} K \le b|x - y|$. If $K \subset Y$ we are done, so assume otherwise; thus $K \cap C \ne \emptyset$. Choose subcontinuua $K_x, K_y \subset K \cap Y$, as provided by Lemma 2.3, which contain x, y respectively and which both meet X. Pick $u \in K_x \cap X$, $v \in K_y \cap X$ and select a continuum $B \subset X$ joining u, v with $\operatorname{diam} B \le c|u - v|$. Now we easily check that $A = K_x \cup B \cup K_y$ is a continuum joining x, y in Y with $\operatorname{diam} A \le b(c + 1)|x - y|$. $\square$

2.C. Local connectivity. Recall that X is *locally connected at $x \in X$* provided for all $t > 0$ there is an $r > 0$ such that points in $B(x; r)$ can be joined by a connected set in $B(x; t)$; that is, $B(x; r)$ lies in a component of $B(x; t)$. We call X *uniformly locally connected* if such an r can be chosen independently of x. It is not hard to see that quasiconvex and bounded turning spaces are uniformly locally connected.

A subspace $A \subset X$ is *locally connected at $x \in X$* provided for all $t > 0$ there is an $r > 0$ such that $A \cap B(x; r)$ lies in a component of $A \cap B(x; t)$; cf. [New51, VI.13, p. 159]. This is only an interesting notion for points on ∂A; we say that A is *locally connected along its boundary* when A is locally connected at each point of ∂A. A subspace $A \subset X$ is *finitely connected at $x \in X$* provided for all $t > 0$ there is an $r > 0$ such that $A \cap B(x; r)$ lies in finitely many components of $A \cap B(x; t)$. A standard reference for these notions is the classic text [New51, §4 in Chapters IV & VI]. See also [Näk70].

For future reference, we note that every set which is bounded turning or quasiconvex is locally connected along its boundary.

Open subspaces of BT or quasiconvex spaces which are locally connected along their boundaries can always be assumed to have a connected complement.

2.5. Lemma. *Let X be an open connected subspace of some a-BT space Z. Let C be a component of $Z \setminus X$ and put $Y := Z \setminus C$. Suppose X is locally connected along its boundary. Then Y is locally connected along its boundary.*

Proof. First we note that $\partial Y = \partial C \subset \partial X$. The equality statement is trivial; that $\partial C \subset \partial X$ follows from Lemma 2.3 in conjunction with Z being BT.

Now let $\zeta \in \partial Y$, $t > 0$, and choose $s > 0$ so that $X \cap B(\zeta; s)$ lies in a component of $X \cap B(\zeta; t)$. Put $r = s/(2a+1)$. We show that $Y \cap B(\zeta; r)$ lies in a component of $Y \cap B(\zeta; t)$.

Let $x, y \in Y \cap B(\zeta; r)$. Select a continuum K joining x, y in Z with $\operatorname{diam} K \le a\,|x-y|$. For each $z \in K$,

$$|z-\zeta| \le |z-x| + |x-\zeta| < \operatorname{diam} K + r \le (2a+1)r$$

so $K \subset B(\zeta; s)$. Therefore, if $K \subset Y$, we are done; assume otherwise, so $K \cap C \ne \emptyset$.

An appeal to Lemma 2.3 produces subcontinua $K_x, K_y \subset K \cap Y$ containing x, y respectively and both meeting X. Pick points $u \in K_x \cap X$, $v \in K_y \cap X$ and select a connected set $A \subset X \cap B(\zeta; t)$ joining u, v. We readily check that $K_x \cup A \cup K_y$ is a connected set joining x, y in $Y \cap B(\zeta; t)$. $\square$

2.D. Examples. We finish this section with a few simple, but illustrative examples. Using the Law of Cosines, it is easy to establish the following handy estimate.

2.6. Fact. If the angle 2φ between $x, y \in \mathbf{R}^n$ satisfies $0 \le 2\theta \le 2\varphi \le \pi$, then $|x-y| \ge \sin\theta\,(|x|+|y|)$.

We identify $\mathbf{R}^2$ with the complex number field $\mathbf{C}$ and use complex variables notation. In particular, given $\theta \in (0, \pi/2]$ we let C_θ and D_θ be the closed convex sector and open concave sector pictured in Figure 1 and defined by $C_\theta = \{z \in \mathbf{C} : |\operatorname{Arg}(z)| \le \theta\}$ and $D_\theta = \mathbf{R}^2 \setminus C_\theta$.

First we present an example which provides (among other things), for each $c \ge 1$, a c-quasiconvex plane domain having exactly one unbounded boundary component.

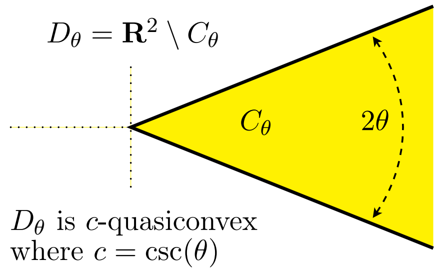
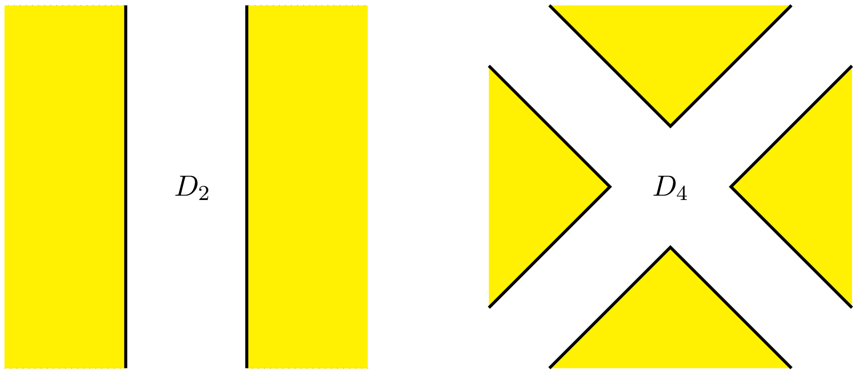


Figure 1. Sectors.

2.7. Example. Fix $0 < \theta \le \pi/2$. The concave sector $D_\theta = \mathbf{R}^2 \setminus C_\theta$ is c-quasiconvex and a-BT with $c = \csc\theta$ and $a = \csc 2\theta$ for $\theta \in (0, \pi/4]$, $a = 1$ for $\theta \in [\pi/4, \pi/2]$. However, there exist points $\xi, \eta \in \partial D_\theta$ such that any rectifiable path α joining ξ and ζ in $D_\theta \cup \{\xi, \eta\}$ has $\ell(\alpha) > c\,|\xi - \eta|$.

We leave the straightforward proof to the reader but make a few comments. When x, y are points in D_θ lying in different half-planes, we can select $0 < \varepsilon < (1/2)(c\,|x-y| - |x| - |y|)$ and use Fact 2.6 to check that the path $[x, -\varepsilon] \cup [-\varepsilon, y]$ is c-quasiconvex. For the last claim, choose boundary points ξ, η with $\{\xi, \eta\} = \partial D_\theta \cap S(0; 1)$.

Next we exhibit a quasiconvex plane domain which has a maximal number of unbounded boundary components.



A convex domain A $\sqrt{2}$-quasiconvex domain

Figure 2. Complements of closed convex sectors.

2.8. Example. Let $n \in \mathbf{N}$ with $n \geq 2$. Put $\theta = \pi/n$ and $c = 1/\sin\theta$. For each $1 \leq k \leq n$, define $\zeta_k = e^{2ki\theta}$, $C_k = \zeta_k C_\theta$, and $B_k = C_k + \zeta_k$. Thus C_k is a closed convex sector obtained by rotating C_θ and B_k is a translation of C_k. Then $D_n = \mathbf{R}^2 \setminus \cup_{k=1}^n B_k$—see Figure 2—is a simply connected c-quasiconvex plane domain with n unbounded boundary components.

Again we leave the justification of this to the reader.

3. Plane domains

Here we focus our attention on quasiconvex domains in the Euclidean plane $\mathbf{R}^2$. After providing certain preliminary information, we establish Theorems D, E and Corollary F.

As mentioned in the Introduction, all Euclidean toplogy is with respect to $\mathbf{R}^n$; we add a hat to indicate notions relative to the extended space $\hat{\mathbf{R}}^n = \mathbf{R}^n \cup \{\infty\}$. In particular, given $A \subset \mathbf{R}^n$, we write $A^c := \mathbf{R}^n \setminus A$, $\bar{A}$, ∂A for the complement, closure, boundary of A in $\mathbf{R}^n$, whereas $\hat{A}$ and $\hat{\partial} A$ denote the ĉlosure and b̂oundary of A in $\hat{\mathbf{R}}^n$; e.g., $\hat{A} = \bar{A}$ when A is bounded and $\hat{A} = \bar{A} \cup \{\infty\}$ when A is unbounded.

A path λ in $\mathbf{R}^n$ is called *piecewise linear*, abbreviated PL, if its trajectory consists of finitely many straight line segments; that is, λ is a so-called 'broken-line-segment path'.

A *Jordan loop* is the homeomorphic image of a round circle, and thus always compact. We use the phrase *Jordan line* for the trajectory of an arc $\lambda : \mathbf{R} \to \mathbf{R}^n$ which has the property that $\lambda(t) \to \infty$ (in $\hat{\mathbf{R}}^n$) as $t \to \pm\infty$; every Jordan line in $\mathbf{R}^n$ corresponds to a Jordan loop in $\hat{\mathbf{R}}^n$. We call C a *Jordan curve* in $\mathbf{R}^n$ if it is a Jordan loop in $\mathbf{R}^n$ or a Jordan line in $\mathbf{R}^n$.

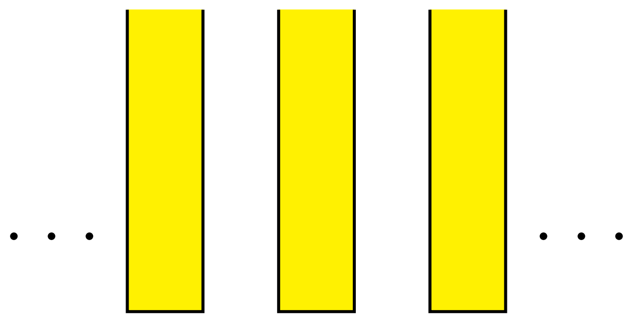
3.A. Plane topology. First we state a useful result which provides a one-to-one correspondence between the boundary components of a plane domain and its complementary components. This can be found in the proof of [New51, Theorem VI.16.3, p. 168].

3.1. Fact. Let D be a Euclidean plane domain, B a component of ∂D, C the component of D^c which contains B, and put $G = C^c$. Then G is a domain containing D with $\partial G = \partial C = B$.

Recall from the Introduction that a plane domain with the property that each boundary component is either a point or a Jordan curve is termed a *Jordan curve domain*. Figure 3 displays a simply connected Jordan curve domain having infinitely many non-degenerate unbounded boundary components. A *Jordan disk* in $\mathbf{R}^2$ is a simply connected plane domain whose boundary is a single Jordan curve in $\mathbf{R}^2$. The celebrated Jordan Curve Theorem tells us that each Jordan curve C in $\mathbf{R}^2$ divides the extended plane $\hat{\mathbf{R}}^2$ into two disjoint Jordan disks, say G and G^*, each having common boundary $\partial G = C = \partial G^*$. (Of course when C is a Jordan loop in $\mathbf{R}^2$, the êxterior of C is a Jordan disk in $\hat{\mathbf{R}}^2$.) Whenever G is a Jordan disk in $\hat{\mathbf{R}}^2$ we write $G^* := \hat{\mathbf{R}}^2 \setminus \hat{G}$ to denote the complementary Jordan disk. Also, every boundary point of a Jordan disk is accessible.

By a *crosscut* of a domain D we mean the trajectory of an arc in $\bar{D}$ with endpoints in ∂D and all other points in D. An *endcut* of D is the trajectory of an arc having one endpoint in ∂D but all other points in D.

As is well known, for a simply connected plane domain D there is a close connection between local connectivity of ∂D, D being locally connected along its boundary, D being a Jordan disk, and the homeomorphic extendability of any Riemann map for D (i.e., any conformal map from the unit disk $\mathbf{D}$ to D). For example, see [New51, Theorems 14.1 & 16.2, pp. 161 & 167], [Näk70, 4.2], [Pal91, Theorems 4.8 & 4.9, pp. 443 & 445] or [Pom92, Theorem 2.6, p. 24]. In addition, these notions also come into play in describing when Riemann maps have continuous extensions, which is the case precisely when ∂D is locally connected or when D is finitely connected at every boundary point; see [Näk70, 4.2], [Pal91, Theorem 4.7, p. 441] or [Pom92, Theorem 2.1, p. 20]. (We caution the reader that here we must use the extended plane topology!)



A simply connected Jordan curve domain

Figure 3. Infinitely many unbounded boundary components.

The following surely belongs to folklore; lacking a precise reference, we present a proof. A result for domains in $\hat{\mathbf{R}}^2$, similar to Corollary 3.3, is [New51, Theorem VI.16.3, p. 168].

3.2. Proposition. *Let $G \subsetneq \mathbf{R}^2$ be a domain with $G^c = \mathbf{R}^2 \setminus G$ connected and non-degenerate (i.e., not an isolated point). Suppose G is locally connected along its boundary. Then ∂G is a Jordan curve in $\mathbf{R}^2$.*

Proof. First, suppose ∂G is bounded. We show that it is a Jordan loop in $\mathbf{R}^2$. If G is also bounded, then G is simply connected and so this follows from [New51, Theorems VI.13.1 & VI.16.2, pp. 160 & 167]. If G is unbounded, then $G' = G \cup \{\infty\}$ is a simply connected domain in $\hat{\mathbf{R}}^2$ and the same argument confirms that $\partial G' = \partial G$ is a Jordan loop in $\mathbf{R}^2$.

Next, suppose ∂G is unbounded. We show that it is a Jordan line in $\mathbf{R}^2$. According to [HK95, 2.1], when viewed as a domain in $\hat{\mathbf{R}}^2$, G is finitely connected at $\infty \in \hat{\partial} G$. It then follows, e.g., from [Pal91, Theorem 4.7, p. 441], that any Riemann map $f\colon \mathbf{D} \to G$ has a continuous extension to a map $f\colon \bar{\mathbf{D}} \to \hat{G} = \bar{G} \cup \{\infty\}$, that f^{-1} has a continuous extension to a map $g\colon \bar{G} \to \bar{\mathbf{D}}$, and thus that $g\colon \bar{G} \to \bar{\mathbf{D}} \setminus I$ is a homeomorphism where $I = f^{-1}\{\infty\}$.

Since G^c is connected, [New51, Theorem V.14.5, p. 124] tells us that ∂G is connected. Thus ∂G is homeomorphic to an open arc $g(\partial G) \subset \partial \mathbf{D}$ and has 'endpoints at ∞', so it is a Jordan line. $\square$

3.3. Corollary. *Suppose D is a plane domain which is locally connected along its boundary. Then D is a Jordan curve domain.*

Proof. Let B be a non-degenerate component of ∂D, let C be the component of $\mathbf{R}^2 \setminus D$ containing B and let $G = \mathbf{R}^2 \setminus C$. Then by Fact 3.1, G is a domain with $\partial G = B$. According to Lemma 2.5, G is also locally connected along its boundary. Since $\mathbf{R}^2 \setminus G$ is connected, we can appeal to Proposition 3.2 and conclude that $B = \partial G$ is a Jordan curve. $\square$

We record the following 'folklore fact' as it may have independent interest. Roughly speaking, it says that the interior of each non-degenerate complementary component of a c-quasiconvex (or c-BT) plane domain is a b-John disk with $b = b(c)$. Here we employ the terminology of [NV91, 2.26]. The converse of this is false: there are even Jordan John disks whose complements fail to be quasiconvex.

By a *hole* of $A \subset \mathbf{R}^2$ we mean a ĉomponent of $\hat{\mathbf{R}}^2 \setminus \hat{A}$. For example, the unit disk $\mathbf{D}$ has one hole, $\mathbf{D}^*$, whereas $[0,1] \times \mathbf{R}$ has two holes each being an open half-plane. Notice that when A is unbounded, $\hat{\mathbf{R}}^2 \setminus \hat{A} = \mathbf{R}^2 \setminus \bar{A}$ and these two spaces have the same components; whereas when A is bounded, $\hat{\mathbf{R}}^2 \setminus \hat{A} = (\mathbf{R}^2 \setminus \bar{A}) \cup \{\infty\}$, so each bounded component of $\mathbf{R}^2 \setminus \bar{A}$ is a ĉomponent of $\hat{\mathbf{R}}^2 \setminus \hat{A}$ and the unique unbounded component C of $\mathbf{R}^2 \setminus \bar{A}$ has the property that $C \cup \{\infty\}$ is the ĉomponent of $\hat{\mathbf{R}}^2 \setminus \hat{A}$ containing ∞.

3.4. Corollary. *Each hole of a c-quasiconvex (or a c-BT) plane domain is a b-John disk with $b = b(c)$.*

Proof. It suffices to consider a hole, say G, of a c-BT domain $D \subsetneq \mathbf{R}^2$. According to [New51, Theorem VI.4.4, p. 144], G is simply connected. Appealing to [NV91, 4.2, 4.5(6)], it suffices to show that $\mathbf{R}^2 \setminus G = (\hat{\mathbf{R}}^2 \setminus G) \cap \mathbf{R}^2$ has the bounded turning property.

Let C be the component of $\mathbf{R}^2 \setminus \bar{D}$ which corresponds to G; that is, when G contains the point at infinity, $G = C \cup \{\infty\}$, and otherwise $G = C$. Evidently, $\mathbf{R}^2 \setminus G = \mathbf{R}^2 \setminus C$. Thanks to Lemmas 2.2 and 2.4, respectively, we deduce that for any $b > c$, $\mathbf{R}^2 \setminus C$ is $(b+1)$-BT. □

Recall that points are *separated* by a closed set if they lie in different components of its complement. When the topology is simple, it is easy to understand separation. For example, every crosscut of a Jordan disk G divides the disk into two simply connected regions each of which is separated in G from the other by the crosscut. On the other hand, an endcut does not separate any points.

We require the following information; this follows from [New51, Theorem V.14.3, p. 123], because a simply connected plane domain is homeomorphic to $\mathbf{R}^2$.

3.5. Fact. Let $G \subset \mathbf{R}^2$ be a simply connected domain and $F \subset G$ be a relatively closed set. If F separates points $x, y \in G$ in G, then some component of F separates x, y in G.

3.B. Technical details. Here we establish several geometric facts required for the proofs of our main theorems. Our first result provides a quantitative estimate describing the size of the complement of a quasiconvex or BT Jordan disk.

3.6. Proposition. *Let G be Jordan disk in $\mathbf{R}^2$ with ∂G unbounded. Suppose each pair of points $\xi, \eta \in \partial G$ can be joined in $G \cup \{\xi, \eta\}$ either by a b-quasiconvex path or by a continuum whose diameter is at most $a\,|\xi - \eta|$. Then for all $0 < \tau < 1$, there exists an $R > 1$, depending only on τ and $\operatorname{dist}(0, \partial G)$, such that for all $r > R$ there is a subarc A of $G^* \cap S(0; r)$ with*

$$\ell(A) \geq r\vartheta \quad \text{where} \quad \vartheta = \begin{cases} 2\arcsin(\tau/b) & \text{in the quasiconvex case,} \\ 2\arcsin(\tau/2a) & \text{in the bounded turning case.} \end{cases}$$

Proof. Choose $\zeta \in \partial G$ with $|\zeta| = \operatorname{dist}(0, \partial G)$, set $R = |\zeta|/(1-\tau)$ and fix $r > R$. Note that $(r - |\zeta|) \geq \tau r$. Put $S = S(0; r)$. Then S separates ζ and the point at infinity, so there are endcuts of G^*—which do not meet S—joining ζ to some point w_0 and joining ∞ to some point w_1 with $w_0, w_1 \in G^*$. Since $S \cap G^*$ separates w_0, w_1 in G^*, Fact 3.5 says that there is a component A of $S \cap G^*$ which also separates w_0, w_1 in G^*.

Now $\bar{A}$ is a crosscut of G^* and $\bar{A}$ also separates ζ and ∞ in G^*. Let $\xi, \eta \in S \cap \partial G$ be the endpoints of A; note that ξ and η belong to different components of $\partial G \setminus \{\zeta\}$. Let ϑ be the angular measure of A, so $\ell(A) = r\vartheta$. If $\vartheta \geq \pi$, then we are done.

Suppose $0 < \vartheta < \pi$. By rotating, and relabeling if necessary, we can assume that $\xi = re^{i\vartheta/2}$ and $\eta = re^{-i\vartheta/2}$. (Note that this rotation does not change $|\zeta|$.)

By hypothesis, ξ and η can be joined in $G \cup \{\xi, \eta\}$ via a b-quasiconvex path γ, or by a continuum K with $\operatorname{diam} K \le a|\xi - \eta|$. In either case we obtain a closed set C, $C := \bar{A} \cup |\gamma|$ or $C := \bar{A} \cup K$, which separates ζ and the point at infinity. Since $L = (-\infty, 0] \cup [0, \zeta]$ joins ζ to ∞, it must intersect C. (Here $(-\infty, 0]$ is the closed negative real axis.) Let $z \in L \cap C$.

When $C = \bar{A} \cup K$ we obtain

$$\operatorname{diam} K \ge |z - \xi| \ge \operatorname{dist}(\xi, L) \ge r - |\zeta| \,,$$

and when $C = \bar{A} \cup |\gamma|$ we obtain

$$\ell(\gamma) \ge |z - \xi| + |z - \eta| \ge \operatorname{dist}(\xi, L) + \operatorname{dist}(\eta, L) \ge 2(r - |\zeta|) \,.$$

Finally, since $|\xi - \eta| = 2r \sin(\vartheta/2)$, we either have

$$\tau\, r \le (r - |\zeta|) \le \operatorname{diam} K \le a|\xi - \eta| = 2a\, r \sin(\vartheta/2) \implies \vartheta \ge 2 \arcsin(\tau/2a)$$

or

$$2\tau\, r \le 2(r - |\zeta|) \le \ell(\gamma) \le b|\xi - \eta| = 2b\, r \sin(\vartheta/2) \implies \vartheta \ge 2 \arcsin(\tau/b)$$

as asserted. □

3.7. Corollary. *Let D be a Jordan curve domain with the property that each pair of points $\xi, \eta \in \partial D$ can be joined in $D \cup \{\xi, \eta\}$ by a b-quasiconvex path (or by a continuum whose diameter is at most $a\,|\xi - \eta|$). Then ∂D has at most $\pi/\arcsin(1/b)$ (or $\pi/\arcsin(1/2a)$, respectively) unbounded components.*

Proof. Let B be an unbounded component of ∂D. Then B is a Jordan line in $\mathbf{R}^2$. Let G and G^* be the components of $\mathbf{R}^2 \setminus B$ with $G \supset D$. Notice that if A is another different unbounded component of ∂D and H, H^* are the components of $\mathbf{R}^2 \setminus A$ with $H \supset D$, then $G^* \cap H^* = \emptyset$ (e.g., by the Jordan Curve Theorem). Assume the quasiconvexity hypothesis holds; the argument for the BT version is identical.

Fix $0 < \tau < 1$. According to Proposition 3.6, once r is large enough, $G^* \cap S(0; r)$ contains an arc with angular measure at least $2 \arcsin(\tau/b)$. Clearly there can be at most $\pi/\arcsin(\tau/b)$ such disjoint arcs. Since different unbounded components B of ∂D correspond to disjoint components G^* of $\mathbf{R}^2 \setminus \bar{D}$ (see Fact 3.1), it follows—by letting $\tau \nearrow 1$—that there are at most $\pi/\arcsin(1/b)$ unbounded components of ∂D. □

Next we examine paths which join interior points to a boundary point.

3.8. Lemma. *Let $D \subsetneq \mathbf{R}^2$ be a Jordan curve domain having finitely many boundary components. Fix points $x, y \in D$ and $\zeta \in \partial D$. Suppose E and F are continuua in $D \cup \{\zeta\}$ which join x and y to ζ respectively. Then for all sufficiently small $r > 0$, there is a component A of $D \cap S(\zeta; r)$ with*

$$A \cap E \ne \emptyset \ne A \cap F \,.$$

Proof. First, let γ be any path joining x and y in D. Then $\operatorname{dist}(\zeta,|\gamma|) \geq \operatorname{dist}(|\gamma|,\partial D) > 0$. Next, let C be the ζ-component of ∂D. Since ∂D has finitely many components, there is little to do if $C = \{\zeta\}$: in this case, for all $0 < r < \operatorname{dist}(\zeta,\partial D \setminus \{\zeta\})$ we have $S(\zeta;r) \subset D$. We assume C is non-degenerate. Thus C is a Jordan curve and $\operatorname{diam}(C) > 0$.

Let G be the component of $\hat{\mathbf{R}}^2 \setminus \hat{C}$ containing D. Then G is a Jordan disk with $\partial G = C$. Since ∂D has finitely many components, so does D^c by Fact 3.1; therefore $(G \cap \mathbf{R}^2) \setminus D$ is a closed set, whence $d := \operatorname{dist}(\zeta, |\gamma| \cup (G \setminus D)) > 0$. We establish the claim for $r \in (0,d)$.

Let $r \in (0,d)$. Since G is locally connected at ζ, we can find points $u \in E \cap B(\zeta;r)$ and $v \in F \cap B(\zeta;r)$ which can be joined by an arc α in $G \cap B(\zeta;r)$. Since $D \cap S(\zeta;r) = G \cap S(\zeta;r)$ separates u and x in G, an appeal to Fact 3.5 produces a component A of $D \cap S(\zeta;r)$ which also separates these points in G. Then A separates $|\gamma|$ and $|\alpha|$ in G, and hence A must meet both E and F. □

3.C. Proofs of Theorems D & E. Here we establish these results as well as Corollary F.

3.9. *Proof of Theorem D.* Let D be a c-quasiconvex proper subdomain of $\mathbf{R}^2$. Since quasiconvex sets are locally connected along their boundaries (see §2.C), Corollary 3.3 validates (1). As indicated in Lemma 2.2, (3) holds in the general metric space context. To corroborate (2) we appeal to Corollary 3.7—which is permissable because (1) and (3) hold—and then let $b \searrow c$. □

3.10 *Proof of Theorem E.* We assume that $D \subsetneq \mathbf{R}^2$ is a Jordan curve domain with finitely many boundary components and that there is a constant $c \geq 1$ such that all rectifiably accessible points $\xi, \eta \in \partial D$ can be joined by a c-quasiconvex path in $D \cup \{\xi,\eta\}$.

First, suppose $c = 1$. Let F be the set of all points $\zeta \in \partial D$ with $\{\zeta\}$ being a component of ∂D. Then F is a finite set and $G := D \cup F$ is a domain. We claim that $\bar{G}$ is convex. This is an easy consequence of Motzkin's Theorem (see [Val76, Theorem 7.8, p. 94]): we must check that each point $z \in \mathbf{R}^2$ has a unique nearest point in $\bar{G}$, and this is clear from our hypotheses. It now follows that G is convex (cf. [Val76, Theorem 1.11, p. 10]), and hence that G is strictly convex.

Now suppose $c > 1$. We demonstrate that D is c-quasiconvex. Let $x, y \in D$. There is nothing to prove if $[x,y] \subset D$, so we assume $[x,y] \cap \partial D \neq \emptyset$. Select $\xi, \eta \in [x,y] \cap \partial D$ so that $[x,\xi) \cup [y,\eta) \subset D$. Let γ be a c-quasiconvex path joining ξ and η in $D \cup \{\xi,\eta\}$. Then the concatenation $[x,\xi] \star \gamma \star [\eta,y]$ is a c-quasiconvex path joining x and y, but it does not lie in D.

Let $r > 0$ be small; precisely how small to be explained below. According to Lemma 3.8, there are components A and B (respectively) of $D \cap S(\xi;r)$ and $D \cap S(\eta;r)$ with

$$[x,\xi] \cap A \neq \emptyset \neq |\gamma| \cap A \quad \text{and} \quad [y,\eta] \cap B \neq \emptyset \neq |\gamma| \cap B\,.$$

Let u, v be the unique points of $[x,\xi]\cap A$, $[y,\eta]\cap B$ (respectively) and select points $w\in|\gamma|\cap A$, $z\in|\gamma|\cap B$. Next let α and β denote the subarcs of A and B (respectively) joining the points u to w and v to z. Put

$$\delta=[x,u]\star\alpha\star\gamma[w,z]\star\beta\star[v,y]\,.$$

Clearly δ is a path joining x and y in D. It remains to confirm that δ is c-quasiconvex. Note that x,u,ξ,η,v,y are successive points along the Euclidean line segment $[x,y]$. Thus

$$|x-y|=|x-u|+|u-\xi|+|\xi-\eta|+|\eta-v|+|v-y|=|x-u|+|\xi-\eta|+|v-y|+2r\,.$$

Note too that

$$\ell(\gamma)=\ell(\gamma[\xi,w])+\ell(\gamma[w,z])+\ell(\gamma[z,\eta])\geq\ell(\gamma[w,z])+2r\,.$$

Using the c-quasiconvexity of γ, and recalling that α and β are subarcs of circles of radius r, we obtain

$$\begin{aligned}\ell(\delta)&\leq|x-\xi|+\ell(\gamma)+|\eta-y|+4(\pi-1)r\\&\leq|x-\xi|+c\,|\xi-\eta|+|\eta-y|+4(\pi-1)r\\&=c\,(|x-\xi|+|\xi-\eta|+|\eta-y|)-(c-1)\,(|x-\xi|+|y-\eta|)+4(\pi-1)r\,.\end{aligned}$$

Thus $\ell(\delta)\leq c\,|x-y|$ if and only if $4(\pi-1)r\leq(c-1)\,(|x-\xi|+|y-\eta|)$. Since $c>1$, we certainly can choose $r>0$ small enough so that this latter inequality holds. ☐

3.11. *Proof of Corollary F.* The necessity follows immediately from parts (1) and (3) of Theorem D. For the sufficiency, we first appeal to Corollary 3.7 to see that ∂D has finitely many unbounded components. This together with D being finitely connected now permits us to apply Theorem E. ☐

3.D. Bounded turning analogs. For the sake of completeness, here we state the bounded turning versions of Theorems D, E and Corollary F. Their proofs are similar to those for the quasiconvex versions.

3.12. Theorem. *Suppose $D\subsetneq\mathbf{R}^2$ is an a-BT domain. Then:*

(1) *D is a Jordan curve domain*[‡],
(2) *∂D has at most $\pi/\arcsin(1/2a)$ unbounded components, and*
(3) *for any $b>a$, each pair of points $\xi,\eta\in\bar D$ can be joined by a continuum K in $D\cup\{\xi,\eta\}$ with $\operatorname{diam}K\leq|\xi-\eta|$.*

3.13. Theorem. *Let $D\subsetneq\mathbf{R}^2$ be a Jordan curve domain with ∂D having finitely many components. Suppose $b\geq1$ and each pair of rectifiably accessible points $\xi,\eta\in\partial D$ can be joined by a continuum K in $D\cup\{\xi,\eta\}$ with $\operatorname{diam}K\leq b\,|\xi-\eta|$. Then D is b-BT.*

[‡]In fact, each non-degenerate component of D^c is a closed b-John disk with $b=b(a)$; see Corollary 3.4.

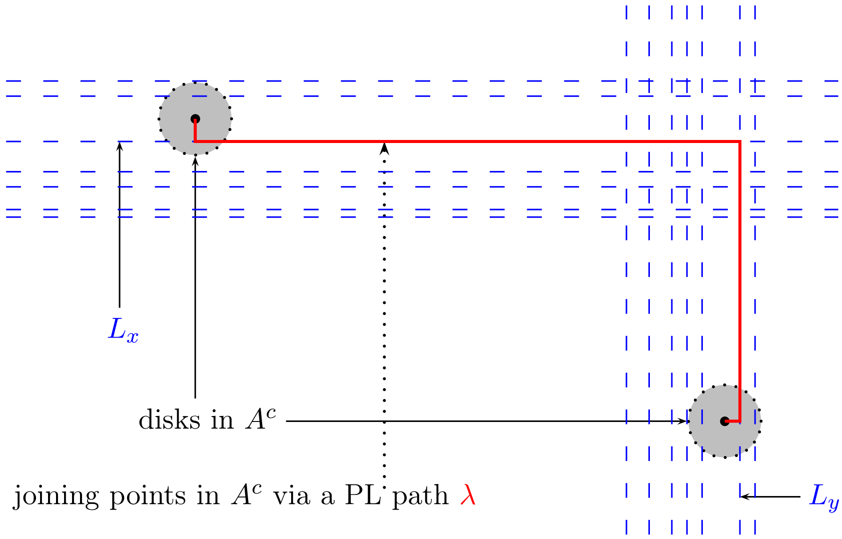


Figure 4. $A \subset \mathbf{R}^2$ closed with nowhere dense horizontal and vertical projections.

3.14. Corollary. *Let $D \subsetneq \mathbf{R}^2$ be a finitely connected domain. Then D is a-BT if and only if*

(1) *D is a Jordan curve domain, and*
(2) *each pair of points $\xi, \eta \in \partial D$ can be joined by a continuum K in $D \cup \{\xi, \eta\}$ with $\operatorname{diam} K \le b\,|\xi - \eta|$.*

For the necessity, we can take any $b > a$; for the sufficiency, $a = b$ works.

4. Complements of closed sets

Proof of Theorem A. We assume that each projection of a closed set $A \subset \mathbf{R}^n$ onto a coordinate $(n-1)$-plane is nowhere dense. We show that for any $c > \sqrt{n}$, A^c is c-quasiconvex.

Since the plane case is easy, we start by explaining the argument in this special setting. See Figure 4. The hypotheses ensure that there are plenty of horizontal and vertical lines in A^c. Given points $a, b \in A^c$, we select open disks $D(a;r), D(b;r) \subset A^c$. Next we pick a horizontal line $L_x \subset A^c$ which meets $D(a;r)$ and a vertical line $L_y \subset A^c$ which meets $D(b;r)$. Now the pictured PL (i.e., 'broken-line-segment') path λ—from a to L_x, along L_x to L_y, along L_y into $D(b;r)$, and then to b—lies in A^c and has length

$$\ell(\lambda) \le 2r + |a-b|_1 \le 2r + \sqrt{2}|a-b|\,.^*$$

Thus for any $c > \sqrt{2}$, we can choose $r > 0$ and small enough so that such a path λ will be c-quasiconvex.

*Here $|\cdot|_1$ denotes the ℓ^1 or box distance: $|(x_1,\dots,x_n)-(y_1,\dots,y_n)|_1 = |x_1-y_1|+\dots+|x_n-y_n|$.

The proof in higher dimensions follows the above idea, but—in addition to cumbersome notation—requires a little care. Fix $c > \sqrt{n}$. Let a, b be points in A^c. Choose $\varepsilon > 0$ so that $B(a;\varepsilon), B(b;\varepsilon) \subset A^c$ and so that

$$\varepsilon + \sqrt{n}|a-b| \le c\,|a-b|\,.$$

We construct a piecewise linear path in A^c which joins a to b and has length at most $\varepsilon + |a-b|_1$ (and thus is c-quasiconvex). In fact, we find points z^k and w^k $(1 \le k \le n)$ such that the 'broken-line-segment'

$$[a,z^1]\star[z^1,w^1]\star[w^1,z^2]\star[z^2,w^2]\star\cdots\star[w^{n-1},z^n]\star[z^n,w^n]\star[w^n,b]$$

has these properties.

Let $\mathbf{R}^n \xrightarrow{P_i} \mathbf{R}^n_i := \{(x_1,\dots,x_n) \in \mathbf{R}^n : x_i = 0\}$ denote orthogonal projection onto the i^{th} coordinate $(n-1)$-plane. Then $A_i := P_i(A)$ is a nowhere dense subspace of $\mathbf{R}^n_i$. The points w^k will be chosen to lie on $\mathbf{R}^n_k$; that is, we will have

$$w^k = (w^k_1,\dots,w^k_n) \quad \text{where} \quad w^k_k = 0\,.$$

The points z^k will be chosen to lie on the 'coordinate line' L_k which goes through w^k and is normal to $\mathbf{R}^n_k$; thus we will have

$$z^k = (z^k_1,\dots,z^k_n) \quad \text{with} \quad z^k_i = w^k_i \quad \text{for all } 1 \le i \le n, i \ne k\,.$$

Put $r := \varepsilon/n^2$. We may assume that $b = o = (0,\dots,0)$. Write $a = (a_1,\dots,a_n)$. Since A_1 is nowhere dense in $\mathbf{R}^n_1$, we can select a point $w^1 \in P_1[B(a;r)] \setminus A_1$. Then the normal line L_1 to $\mathbf{R}^n_1$ at w^1 lies in A^c and meets $B(a;r)$, so there exist points $z^1 = (z^1_1,\dots,z^1_n) \in L_1 \cap B(a;r)$. We choose the point z^1 with $z^1_1 = a_1$ (and $z^1_i = w^1_i$ for $i = 2,\dots,n$). Note that

$$|a - z^1| < r \quad \text{and} \quad |z^1 - w^1| = |a_1|\,.$$

Since A_2 is nowhere dense in $\mathbf{R}^n_2$, we can select a point $w^2 \in P_2[B(w^1;r)] \setminus A_2$. Then the normal line L_2 to $\mathbf{R}^n_2$ at w^2 lies in A^c and meets $B(w^1;r)$, so there exist points $z^2 = (z^2_1,\dots,z^2_n) \in L_2 \cap B(w^1;r)$. We choose the point z^2 with $z^2_2 = w^1_2$ (and $z^2_i = w^2_i$ for $i = 1,3,\dots,n$). Note that

$$|w^1 - z^2| < r \quad \text{and} \quad |z^2 - w^2| = |w^1_2| = |z^1_2| < |a_2| + r\,,$$

where the last inequality holds because $z^1 \in B(a;r)$.

Since A_3 is nowhere dense in $\mathbf{R}^n_3$, we can select a point $w^3 \in P_3[B(w^2;r)] \setminus A_3$. Then the normal line L_3 to $\mathbf{R}^n_3$ at w^3 lies in A^c and meets $B(w^2;r)$, so there exist points $z^3 = (z^3_1,\dots,z^3_n) \in L_3 \cap B(w^2;r)$. We choose the point z^3 with $z^3_3 = w^2_3$ (and $z^3_i = w^3_i$ for $i = 1,2,4,\dots,n$). Note that

$$|w^2 - z^3| < r \quad \text{and} \quad |z^3 - w^3| = |w^2_3| = |z^2_3| < |w^1_3| + r < |a_3| + 2r\,,$$

where the last two inequalities hold because $z^2 \in B(w^1;r)$ and $w^1_3 = z^1_3$ with $z^1 \in B(a;r)$.

Continuing in this manner, we select points $w^k \in P_k[B(w^{k-1};r)] \setminus A_k$ and $z^k = (z^k_1,\dots,z^k_n) \in L_k \cap B(w^{k-1};r)$ where L_k is the normal line to $\mathbf{R}^n_k$ at w^k (which lies in A^c and meets $B(w^{k-1};r)$). We choose the point z^k with $z^k_k = w^{k-1}_k$ (and $z^k_i = w^k_i$ for $i = 1, \dots, k-1, k+1, \dots, n$). Note that

$$|w^{k-1} - z^k| < r \quad \text{and} \quad |z^k - w^k| = |w^{k-1}_k| = |z^{k-1}_k| < |a_k| + (k-1)r\,.$$

Now the coordinates of w^n satisfy $|w^n_i| < (n-i)r$ for $1 \le i \le n$. Thus

$$|w^n| \le |w^n|_1 = \sum_{i=1}^{n} |w^n_i| < \sum_{i=1}^{n} (n-i)r = [(n-1)n/2]r < \varepsilon\,.$$

In particular, $w^n \in B(o;\varepsilon) \subset A^c$. Therefore the PL (i.e., 'broken-line-segment') path

$$\Lambda := [a, z^1] \star [z^1, w^1] \star [w^1, z^2] \star [z^2, w^2] \star \cdots \star [w^{n-1}, z^n] \star [z^n, w^n] \star [w^n, o]$$

joins a to $b = o$ in A^c. Finally,

$$\begin{aligned} \ell(\Lambda) &= |a - z^1| + \sum_{k=1}^{n} |z^k - w^k| + \sum_{k=1}^{n-1} |w^k - z^{k+1}| + |w^n| \\ &\le n^2 r + \sum_{k=1}^{n} |a_k| = \varepsilon + |a-b|_1\,. \end{aligned}$$

□

4.B. Linear local connectivity. A subspace A of a metric space X is *b-linearly locally connected*, or *b-LLC*, if $b \ge 1$ and the following two conditions hold for all points $x \in X$ and all $r > 0$:

(LLC$_1$) points in $A \cap \bar{B}(x;r)$ can be joined in $A \cap \bar{B}(x;b\,r)$

and

(LLC$_2$) points in $A \setminus B(x;r)$ can be joined in $A \setminus B(x;r/b)$.

Here the phrase 'can be joined' means 'can be joined by a continuum'. We also employ the terminology *LLC with respect to paths* in which case 'can be joined' means 'can be joined by a path'. Note that quasiconvexity implies LLC$_1$ with respect to rectifiable paths, but the converse is false. The LLC$_1$ and LLC$_2$ conditions were first introduced by Gehring to characterize quasidisks and are well known in the literature.

The bounded turning property is quantitatively equivalent to the LLC$_1$ property: If A is a-BT, it is $(2a+1)$-LLC$_1$. If A is b-LLC$_1$, it is $2b$-BT. For subspaces of Euclidean space we find that b-LLC$_1$ $\implies$ b-BT. All these implications hold both for 'joining by continua' as well as 'joining by paths'.

The following result is probably folklore, but it does not seem to appear in the literature. We thank the referee for drawing this to our attention.

4.1. Proposition. *Let $A \subset \mathbf{R}^n$ be closed and totally disconnected. Then A^c is 1-LLC with respect to paths.*

Proof. First we note that A has topological dimension zero, and hence does not disconnect any open subset of $\mathbf{R}^n$; see [HW41, Corollary 1, p. 48]. Fix a point $z \in \mathbf{R}^n$, let $r > 0$, and set $B := B(z; r)$.

Suppose $x, y \in A^c \cap \bar{B}$. Since A^c is open, we can select points $u, v \in B$ with $u \in [z, x]$, $v \in [z, y]$ and so that $A \cap [x, u] = \emptyset = A \cap [y, v]$. Since A does not disconnect B, $B \setminus A$ is a domain, so there is an arc α joining u and v in $B \setminus A$. Then the concatenation $\gamma := [x, u] \star \alpha \star [v, y]$ joins x, y in $A^c \cap \bar{B}$ as desired.

Suppose $x, y \in A^c \setminus B$. Now choose $u, v \in \mathbf{R}^n \setminus B$ with $x \in [z, u]$, $y \in [z, v]$ and so that $A \cap [x, u] = \emptyset = A \cap [y, v]$. Since A does not disconnect $\mathbf{R}^n \setminus \bar{B}$, there is an arc α joining u and v in $A^c \setminus \bar{B}$. Now $\gamma := [x, u] \star \alpha \star [v, y]$ joins x, y in $A^c \setminus B$ as desired. □

4.2. Corollary. *Let $A \subset \mathbf{R}^n$ be closed and totally disconnected. Then points $x, y \in A^c$ can be joined by a path γ in A^c with* $\operatorname{diam}|\gamma| = |x - y|$. *In particular, A^c is 1-BT.*

4.C. The main example. Here we prove Theorem B. Our construction is based on the following result.

4.3. Proposition. *Given any $M > 0$, there exists a compact totally disconnected set $A \subset [-M, M]^{n-1} \times [-1/2, 1/2] \subset \mathbf{R}^n$ with Hausdorff dimension* $\dim_{\mathscr{H}} A \leq n - 1$ *and such that each rectifiable path γ joining $\pm e := (0, \ldots, \pm 1)$ in A^c has length $\ell(\gamma) \geq M$.*

Assuming the above, we proceed as follows.

4.4. *Proof of Theorem B.* For each $m \in \mathbf{N}$, select a set A_m as given by Proposition 4.3 for $M = m$. Let B_m be the scaled and translated copy of A_m defined via

$$B_m := t_m A_m + b_m , \quad \text{where } t_m := (2^{m+2} \operatorname{diam} A_m)^{-1} \text{ and } b_m := (1/2^m, 0, \ldots, 0) .$$

Thus $\operatorname{diam} B_m = 1/2^{m+2}$ and $\operatorname{dist}(B_m, B_{m+1}) \geq 1/2^{m+3}$ (so the sets $B_1, B_2, \ldots$ are 'far apart'). In addition, there are points $x_m, y_m = b_m \pm t_m e$ (corresponding to the points $\pm e$ scaled and translated) in B_m^c with the property that each rectifiable path γ joining x_m, y_m in B_m^c has $\ell(\gamma) \geq (m/2)|x_m - y_m|$.

We now see that $A := \{0\} \cup \bigcup_1^\infty B_m$ is compact (because it is closed and bounded) and totally disconnected (because the component of A containing the origin is $\{0\}$) with $\dim_{\mathscr{H}}(A) \leq n - 1$ and A^c non-quasiconvex. The latter assertion follows from the fact that all of the points x_m, y_m lie in A^c.

Finally, A must have non-zero $(n-1)$-dimensional Hausdorff measure, for otherwise the comments preceding Theorem A would tell us that A^c is quasiconvex. □

It remains to establish Proposition 4.3, a task which we complete in §4.C.7.

4.C.1. Main idea. The set A, whose existence is asserted by Proposition 4.3, is constructed as a Cantor type set, $A := \cap_i E_i$ where $E_1 \supset E_2 \supset \ldots$ are decreasing compact sets given as $E_i = \cup_j B_{ij}$ with B_{ij} closed rectangular boxes which are appropriately nested and satisfy

$$\lim_{i\to\infty} \max_j \operatorname{diam} B_{ij} = 0\,.$$

This easily gives A closed and totally disconnected. Below we provide an explicit description of the sets B_{ij}. The idea is as follows. We start with a thin flat closed rectangular box. Inside this box we create a 'maze' by placing 'barriers' which are even thinner closed rectangular boxes parallel to the 'top' and 'bottom' faces of B. We do this so that each path joining these faces in B is long. Then we repeat this process for each of the 'barriers'.

Here, briefly, is the idea for our construction in the $\mathbf{R}^2$ case. See Figure 5. We start with a thin flat rectangle, say $[0,s]\times[0,t]$ with $t << s$. We divide this into four horizontal corridors ($[0,s]\times[0,t/4]$, etc.) and place thin barriers of size $(2s/3)\times(\varepsilon\, t)$ in the vertical middles of each of these corridors. We alternate the horizontal placement of the barriers putting them first at the left, then at the right, etc. Any path in the original rectangle which joins the two horizontal edges and avoids all the barriers must have 'horizontal length' at least s. Such a 'penetrating path' can be replaced—without increasing 'horizontal length'—by an 'avoiding path' which stays on the original rectangle's boundary. Now we repeat this process replacing each barrier with four more even thinner barriers.

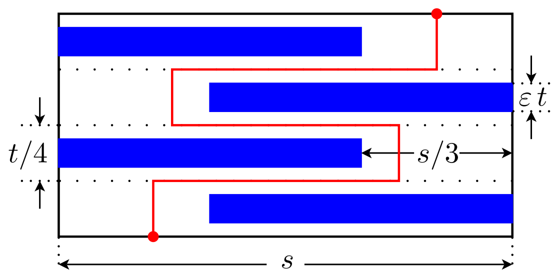


Figure 5. A plane maze.

We give explicit construction details for the case $n = 3$ and leave the general case for the industrious reader. We consider points (x, y, v) in $\mathbf{R}^3$. By the *vertical*, or V, direction we mean parallel to the v-axis whereas the *horizontal*, or H, directions are parallel to the xy-plane. The H_x and H_y directions are parallel to the x-axis and y-axis respectively.

We call B a *rectangular* $s\times s\times t$ *box* if B is congruent to $[0,s]^2\times[0,t]$ via some translation of $\mathbf{R}^3$ (so no rotations are allowed). In general we will consider thin flat boxes meaning that $t << s$. The *top, bottom, front, back, left, right* faces of $[0,s]^2\times[0,t]$ are, respectively,

$$\begin{gathered}[0,s]^2\times\{t\}\,,\ [0,s]^2\times\{0\}\,,\\ \{s\}\times[0,s]\times[0,t]\,,\ \{0\}\times[0,s]\times[0,t]\,,\\ [0,s]\times\{0\}\times[0,t]\,,\ [0,s]\times\{s\}\times[0,t]\,.\end{gathered}$$

Of course, the top and bottom faces are horizontal whereas the other four faces each have a vertical component.

4.C.2. Piecewise horizontal-vertical paths. We call λ a *piecewise horizontal-vertical*, or PHV, path if it is PL and each line segment is either vertical (parallel to the v-axis) or parallel to the x-axis or parallel to the y-axis. Thus a PHV path is a 'broken-line-segment' path whose segments all have direction either V or H_x or H_y. Given a PHV path λ, we write $\ell_H(\lambda)$ to denote the sum of the lengths of all the horizontal segments of λ. We note that if π is the orthogonal projection of a PHV path λ onto some plane which is parallel to a coordinate plane, then

$$\ell_H(\pi) \leq \ell_H(\lambda)\,.$$

It is straightforward to approximate an arbitrary path by PL and PHV paths.

4.5. Lemma. *Let $F \subset \mathbf{R}^3$ be closed. Suppose γ is any path in F^c. There exist a PL path λ and a PHV path κ, both in F^c and having the same endpoints as γ, and satisfying*

$$\ell(\gamma) \geq \ell(\lambda) \geq \frac{1}{\sqrt{2}}\,\ell_H(\kappa)\,.$$

Proof. Suppose $[0,1] \xrightarrow{\gamma} F^c$. Choose $0 = t_0 < t_1 < \cdots < t_m = 1$ so that

$$\forall i: \quad t, s \in [t_{i-1}, t_{i+1}] \implies |\gamma(t) - \gamma(s)| \leq \delta := \operatorname{dist}(|\gamma|, F)\,.$$

Setting $z_i := \gamma(t_i)$ we obtain a PL path $\lambda := [z_0, z_1] \star \cdots \star [z_{m-1}, z_m] \subset F^c$ and evidently, $\ell(\gamma) \geq \ell(\lambda)$. Since each ball $B(z_i;\delta)$ lies in F^c, we can replace each segment $\lambda_i := [z_{i-1}, z_i]$ by a PHV path, say $\kappa_i := \xi \star \eta \star \nu$, with H_x, H_y, V segments respectively. Then

$$\ell_H(\kappa_i) = \ell(\xi) + \ell(\eta) \leq \sqrt{2}[\ell(\xi)^2 + \ell(\eta)^2]^{1/2} \leq \sqrt{2}\,\ell(\lambda_i)\,. \qquad \square$$

4.C.3. Box mazes. We start with a parameter $\varepsilon \in (0, 1/24)$ and a thin flat rectangular $s \times s \times t$ box B with $0 < t << s$. We divide B into six congruent rectangular horizontal corridors with dimensions $s \times s \times (t/6)$. In the vertical middle of each of these corridors, we place a submaze—described below and pictured in Figure 8—consisting of four $s \times s \times \varepsilon t$ walls. We call the region so constructed a *box maze* based on B.

Consider one of the rectangular $s \times s \times (t/6)$ horizontal corridors, say C, in B. Divide C into four congruent (real thin) rectangular horizontal subcorridors with dimensions $s \times s \times (t/24)$. In the vertical middle of each of these subcorridors we construct (really thin) $s \times s \times \varepsilon t$ *walls* which we now describe.

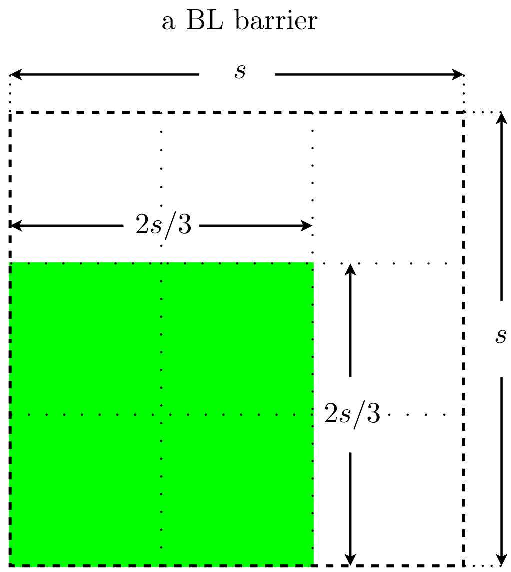


Figure 6. A bird's-eye view of a BL barrier.

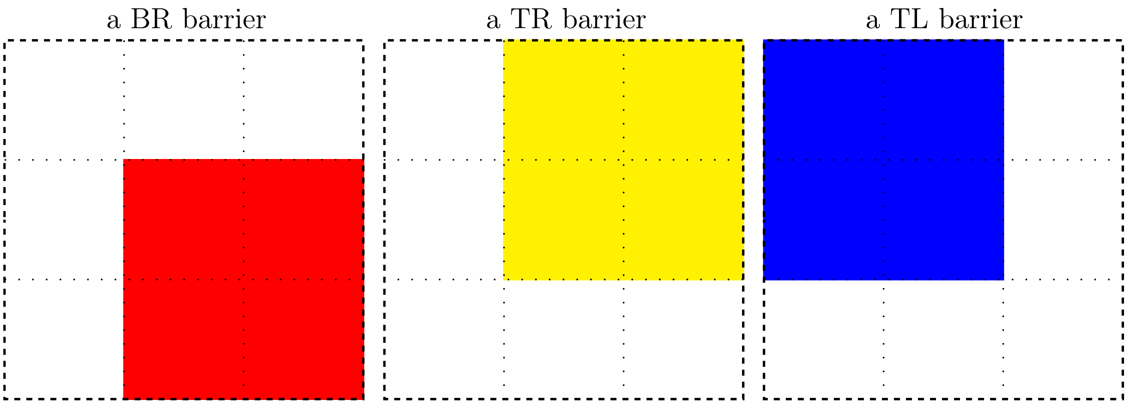


Figure 7. The other three types of barriers.

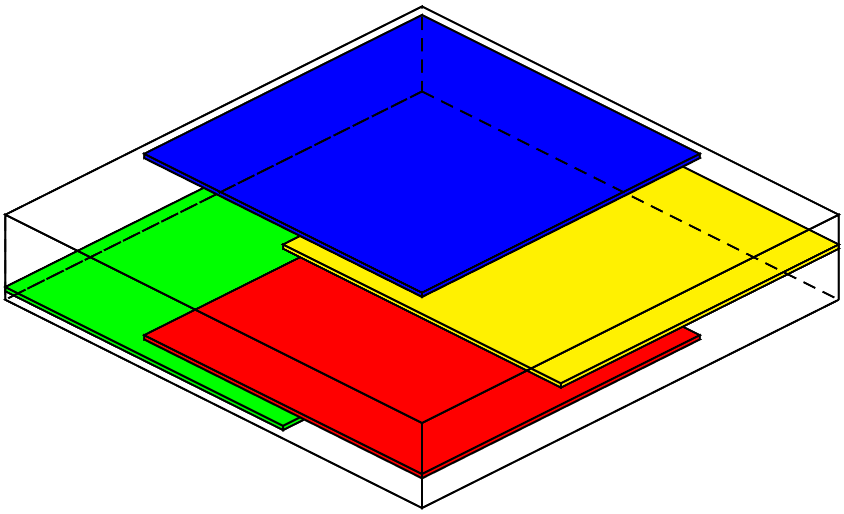


Figure 8. A submaze in C with four $(2s/3) \times (2s/3) \times (\varepsilon\, t)$ barriers.

Each *wall* consists of a $(2s/3) \times (2s/3) \times (\varepsilon\, t)$ *barrier* which is partially surrounded on two sides by open space. There are four types of walls which we label as BL, BR, TR, TL for bottom or top left or right. (See Figures 6 and 7.) In each of these the associated barrier is attached at a different corner of the wall as indicated in the accompanying pictures. We place these four walls, one per subcorridor and in the described order, into the vertical middle of each of the four horizontal subcorridors of C. See Figure 8. We call the region just constructed in C a *submaze*.

Note that any PHV path λ in C which joins the top and bottom faces of C and avoids all the barriers must have 'horizontal length' $\ell_H(\lambda) \geq s/3$.

Now the *box maze* based on B is constructed by stacking six such submazes on top of each other, one into each of the six (real thin) horizontal corridors C of B. Thus this box maze consists of 24 rectangular $(2s/3) \times (2s/3) \times (\varepsilon\, t)$ subboxes of B (i.e., all the different, but congruent, barriers). We call these subboxes the *children* of B and write $\mathscr{C}_\varepsilon(B)$ to denote the collection of these 24 rectangular box barriers. Notice that each child $C \in \mathscr{C}_\varepsilon(B)$ satisfies

$$t_C = \varepsilon\, t_B\,,\ s_C = (2/3)s_B\,,\ \text{and}\quad \operatorname{diam}(C) \le (2/3)\operatorname{diam}(B)$$

where B and C are $t_B \times s_B \times s_B$ and $t_C \times s_C \times s_C$ boxes respectively.

4.C.4. Key lemmas.

4.6. Lemma. *Let B be a $s \times s \times t$ rectangular box. Each pair of points on ∂B can be joined by a PHV path λ in ∂B with $\ell_H(\lambda) \le 2s$.*

Proof. We consider the cases where the points lie on the same face, adjacent faces, or opposite faces. The first two cases are left to the reader (but see the proof of Lemma 4.7).

Suppose our points are on opposite faces. If these two faces have a vertical component, then we can use 'purely vertical' paths to join our points to, say, the top face and then appeal to an earlier case. Thus we are left with, say, a point $p = (x, y, 0)$ on the bottom face and a point $q = (a, b, t)$ on the top face.

Consider the two PHV paths from p to q given by

$$\lambda := [p, (s, y, 0)] \star [(s, y, 0), (s, y, t)] \star [(s, y, t), (s, b, t)] \star [(s, b, t), q]$$

and

$$\kappa := [p, (0, y, 0)] \star [(0, y, 0), (0, y, t)] \star [(0, y, t), (0, b, t)] \star [(0, b, t), q]\,.$$

Writing P for orthogonal projection onto the xy-plane, we see that $P(|\lambda| \cup |\kappa|)$ is a rectangle inside an $s \times s$ square. Thus $\ell_H(\lambda)+\ell_H(\kappa) \le 4s$, and so $\min\{\ell_H(\lambda), \ell_H(\kappa)\} \le 2s$. $\square$

Given a collection $\mathscr{C}$ of sets (e.g., a finite collection of closed sets) we write

$$\bigcup \mathscr{C} := \bigcup_{C \in \mathscr{C}} C\,.$$

The following provides the crucial step in verifying that our construction has the property that every path joining $\pm e$ must be long. It says that we can replace 'penetrating' PHV paths with associated 'avoiding' PHV paths without increasing 'horizontal length'. That is, 'going around is no longer than going through'.

4.7. Lemma. *Let B be a $s \times s \times t$ rectangular box and fix $0 < \varepsilon < 1/24$. Put $F := \bigcup \mathscr{C}_\varepsilon(B)$. Suppose λ is a PHV path in $F^c \cup \partial F$. Then there exists a PHV path κ in $B^c \cup \partial B$ with the same endpoints as λ and satisfying $\ell_H(\kappa) \le \ell_H(\lambda)$.*

Proof. By looking at the components of $|\lambda| \cap B$ we see that it suffices to assume that λ has endpoints $p, q \in \partial B$ and that $|\lambda| \setminus \{p, q\}$ lies in the interior of B. As in the proof of Lemma 4.6 we consider the cases where p, q lie on the same face, adjacent faces, or opposite faces. Again, except for brief comments, we leave the first two cases for the reader.

If p, q lie on some face F, we let κ be the concatenation of two adjacent edges of the rectangle on F with opposite vertices p, q; then, writing P for the orthogonal projection onto the plane determined by F, we see that

$$\ell_H(\lambda) \geq \ell_H(P \circ \lambda) \geq \ell_H(\kappa) .$$

If p, q lie on adjacent faces with common edge E, then we join them by going 'straight' to E and then along E.

Suppose p, q are on opposite faces. If these two faces have a vertical component, then we can use 'purely vertical' paths to join p, q to, say, points p', q' on the top face and then join p', q' by a PHV path on the top face. Here we get a PHV path κ joining p, q with

$$\ell_H(\kappa) = \ell_H(P \circ \kappa) \leq \ell_H(P \circ \lambda) = \ell_H(\lambda)$$

where now P denotes projection onto the horizontal plane determined by the top face.

Finally, we are left with, say, p on the bottom face and q on the top face. In this situation, λ must pass through six submazes. As we noted above, each submaze forces λ to travel at least $s/3$ in some horizontal direction. Thus $\ell_H(\lambda) \geq 6(s/3) = 2s$. On the other hand, an appeal to Lemma 4.6 produces a path κ on ∂B which joins p, q and has $\ell_H(\kappa) \leq 2s$. $\square$

4.C.5. The construction. Let $M > 0$ be given. We start with the rectangular box

$$B_0 := [-M, M]^2 \times [-1/2, 1/2] \subset \mathbf{R}^3 \quad \text{and} \quad \mathscr{G}_0 := \{B_0\} .$$

In §4.C.6 below we indicate exactly how we choose the sequence $(\varepsilon_n)_{n=1}^{\infty}$. Our first generation of subboxes is $\mathscr{G}_1 := \mathscr{C}_{\varepsilon_1}(B_0)$, and $E_1 := \bigcup \mathscr{G}_1$. Our second generation of subboxes is

$$\mathscr{G}_2 := \bigcup_{B \in \mathscr{G}_1} \mathscr{C}_{\varepsilon_2}(B) , \quad \text{and then} \quad E_2 := \bigcup \mathscr{G}_2 .$$

In general, our n^{th} generation of subboxes is

$$\mathscr{G}_n := \bigcup_{B \in \mathscr{G}_{n-1}} \mathscr{C}_{\varepsilon_n}(B) , \quad \text{and then} \quad E_n := \bigcup \mathscr{G}_n = \bigcup_{B \in \mathscr{G}_n} B .$$

Then $E_1 \supset E_2 \supset \dots$ are decreasing compact sets and we put $A := \cap_{n=1}^{\infty} E_n$.

Moreover, if C is a component of E_n (i.e., $C \in \mathscr{G}_n$), then there is some $B \in \mathscr{G}_{n-1}$ with $C \in \mathscr{C}_{\varepsilon_n}(B)$. So the components of the E_n are appropriately nested. We also see that

$$\operatorname{diam}(C) \leq (2/3) \operatorname{diam}(B) , \quad \text{and so} , \quad \operatorname{diam}(C) \leq (2/3)^n \operatorname{diam}(B_0) .$$

Thus A is a Cantor type set and in particular A is compact and totally disconnected.

4.C.6. Estimating dimension. Here we describe how to choose $(\varepsilon_n)_{n=1}^{\infty}$ to ensure that A has Hausdorff dimension $\dim_{\mathscr{H}}(A) \leq 2$. In fact, we shall see that it suffices to have

$$\lim_{n \to \infty} [n / \log(1/\textstyle\prod_{i=1}^{n} \varepsilon_i)] = 0 .$$

Write $N(E;r)$ to denote the smallest number of closed $r\times r\times r$ cubes needed to cover the set E. Note that if B is a closed $s\times s\times t$ rectangular box with $t<<s$, then $N(B;t)\le 2(s/t)^2$.

We examine the set E_n which has $(24)^n$ components, each an $s_n\times s_n\times t_n$ rectangular box with $s_n=(2/3)^n 2M$ and $t_n=\varepsilon_1\times\cdots\times\varepsilon_n$. Every $C\in\mathscr{G}_n$ satisfies $N(C;t_n)\le 2(s_n/t_n)^2$, so

$$N(E_n;t_n)=(24)^n N(C;t_n)\le 8M^2(32/3)^n/t_n^2\,.$$

Thus $\log N(E_n;t_n)\le c\,n+2\log(1/t_n)$ where $c=c(M)$ depends only on M.

We estimate the Hausdorff dimension of A by

$$\dim_{\mathscr{H}}(A)\le\liminf_{r\to 0}\frac{\log N(A;r)}{\log(1/r)}\le\lim_{n\to\infty}\frac{\log N(E_n;t_n)}{\log(1/t_n)}=2+\lim_{n\to\infty}\frac{c\,n}{\log(1/t_n)}\,.$$

Taking, e.g., $\varepsilon_n=1/n!$, we get $n/\log(1/t_n)\to 0$, and then $\dim_{\mathscr{H}}(A)\le 2$ as desired.

4.C.7. *Proof of Proposition 4.3.* We have constructed, above, a totally disconnected compact set $A\subset[-M,M]^2\times[-1/2,1/2]$ with $\dim_{\mathscr{H}}(A)\le 2$. It remains to check that any path in A^c which joins $\pm e=(0,0,\pm 1)$ has length at least M.

We begin by corroborating a stronger statement for H-lengths of PHV paths. We claim that any PHV path joining $\pm e$ in some $E_n^c\cup\partial E_n$ has H-length at least $2M$. Clearly any PHV path λ joining $\pm e$ in $B_0^c\cup\partial B_0$ has $\ell_H(\lambda)\ge 2M$ (because any such path must meet one of the planes $x=\pm M$ or $y=\pm M$). Suppose λ is a PHV path joining $\pm e$ in $E_1^c\cup\partial E_1$. We appeal to Lemma 4.7, with $B=B_0$, to find a PHV path κ in $B_0^c\cup\partial B_0$ which joins $\pm e$ and has $\ell_H(\lambda)\ge\ell_H(\kappa)\ge 2M$ (the latter inequality holding by our first case).

Suppose our claim holds for PHV paths joining $\pm e$ in $E_n^c\cup\partial E_n$ and let λ be a PHV path joining $\pm e$ in $E_{n+1}^c\cup\partial E_{n+1}$. We may assume $|\lambda|$ meets the interior of some box $B\in\mathscr{G}_n$. Since $|\lambda|$ lies in $E_{n+1}^c\cup\partial E_{n+1}$, it is also in $F^c\cup\partial F$ with $F=\bigcup\mathscr{C}_{\varepsilon_{n+1}}(B)$. According to Lemma 4.7, we can replace λ—without increasing horizontal length—by a PHV path in $B^c\cup\partial B$. Doing this for each such B we find a PHV path κ joining $\pm e$ in $E_n^c\cup\partial E_n$ and with $\ell_H(\lambda)\ge\ell_H(\kappa)\ge 2M$ (the latter inequality holding by our induction hypothesis).

Finally, suppose γ is a path joining $\pm e$ in A^c. The open sets E_n^c, which increase to A^c, form an open cover of $|\gamma|$. Since $|\gamma|$ is compact, there is an $n\in\mathbf{N}$ with $|\gamma|\subset E_n^c$. Employing Lemma 4.5, we find a PHV path λ in E_n^c with the same endpoints as γ and $\ell(\gamma)\ge(1/\sqrt{2})\ell_H(\lambda)\ge 2M/\sqrt{2}\ge M$ (the penultimate inequality holding by our claim). □

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