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DISTANCES FROM BLOCH FUNCTIONS TO SOME MÖBIUS INVARIANT SPACES

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Abstract. Distance formulas from Bloch functions to some Möbius invariant function spaces are given. These results generalize the distance formula from Bloch functions to $BMOA$ by Peter Jones. As consequences, we have characterized the closures of these Möbius invariant function spaces in the Bloch space.

Let $H(D)$ be the space of all analytic functions on the unit disk D. For $a \in D$, let $g(z,a) = \log(1/|\varphi_a(z)|)$ be the Green's function for D with pole at a. where $\varphi_a(z) = (z-a)/(1-\overline{a}z)$. Let $0 < p < \infty$, $-2 < q < \infty$, $0 < s < \infty$, $-1 < q+s < \infty$, and let f be an analytic function on D. We say that $f \in F(p,q,s)$, if

$$\|f\|_{p,q,s}^p = \sup_{a\in D} \int_D |f'(z)|^p (1-|z|^2)^q g^s(z,a)\, dA(z) < \infty;$$

$f \in F_0(p,q,s)$, if

$$\lim_{|a|\to 1} \int_D |f'(z)|^p (1-|z|^2)^q g^s(z,a)\, dA(z) = 0$$

(see [Zha]). Here $dA(z) = dxdy/\pi$ is Lebesgue area measure normalized so that $A(D) = 1$.

For $p > 1$, the analytic Besov space B_p is the space of analytic functions f on D satisfying

$$\|f\|_{B_p}^p = \int_D |f'(z)|^p (1-|z|^2)^{p-2}\, dA(z) < \infty.$$

We note here that B_p can be viewed as $F(p, p-2, 0)$. When $p = 1$, the Besov space B_1 can be defined as the space of analytic functions f on D satisfying

$$\|f\|_{B_1} = \int_D |f''(z)|\, dA(z) < \infty.$$

We recall also that the Bloch space B is the space of analytic functions on D satisfying

$$\|f\|_B = \sup_{z\in D} |f'(z)|(1-|z|^2) < \infty,$$

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and the little Bloch space B_0 is the space of functions f analytic on D for which $|f'(z)|(1-|z|^2) \to 0$ as $|z| \to 1$. It is well known that B is a Banach space under the norm

$$\|f\|_B^* = |f(0)| + \|f\|_B$$

and B_0 is the closure of polynomials in B.

It is known that for $s > 1$, $F(p, p-2, s) = B$ and $F_0(p, p-2, s) = B_0$ (see, [Zha, p13]). It is also known that $F(2, 0, s) = Q_s$ and $F_0(2, 0, s) = Q_{s,0}$, which were introduced in [AL], [AXZ] and studied by many authors (see, for example, [AC], [ASX], [ASZ], [ALXZ], [EX] and [NX]). For the case $s = 1$, we have $F(2, 0, 1) = Q_1 = BMOA$ and $F_0(2, 0, 1) = Q_{1,0} = VMOA$ (see, for example, [B]). We note that, for $0 \le s < \infty$, $F(p, p-2, s)$ and $F_0(p, p-2, s)$ are Möbius invariant function spaces (see, [AFP]), for $0 \le s < 1$, $F(p, p-2, s)$ and $F_0(p, p-2, s)$ are subspaces of $BMOA$ and $VMOA$, respectively.

For $0 < s < \infty$, we say that a positive measure μ defined on D is an s-Carleson measure provided $\mu(S(I)) = O(|I|^s)$ for all subarcs I of ∂D, where $|I|$ denotes the arc length of I and $S(I)$ denotes the usual Carleson box based on I. If $\mu(S(I)) = o(|I|^s)$, as $|I| \to 0$, then we say that μ is a vanishing s-Carleson measure (cf. [ASX]). For f an analytic function on D, we define

$$d\mu_f = |f'(z)|^p (1-|z|^2)^{q+s}\, dA(z).$$

In [Zha, Theorem 2.4 and Theorem 2.5], it was proved that $f \in F(p, q, s)$ if and only if $d\mu_f$ is an s-Carleson measure, and $f \in F_0(p, q, s)$ if and only if $d\mu_f$ is a vanishing s-Carleson measure. Thus we can replace $g(z, a)$ by $(1 - |\varphi_a(z)|^2)$ in the definition of $F(p, q, s)$ and $F_0(p, q, s)$.

For a subspace X of B, we will denote the distance from a function $f \in B$ to the space X by $\operatorname{dist}_B(f, X)$. The following is the well-known distance formula by Jones (see [GZ, p. 503]).

Jones' theorem. *Let $f \in B$. Then the following quantities are equivalent:*

(A) $\operatorname{dist}_B(f, BMOA)$;

(B) $\inf\{\varepsilon\colon \chi_{\Omega_\varepsilon(f)} \frac{dA(z)}{1-|z|^2}$ *is a Carleson measure*$\}$,

where $\Omega_\varepsilon(f) = \{z \in D\colon |f'(z)|(1-|z|^2) \ge \varepsilon\}$, χ denotes the characteristic function of a set.

The purpose of the paper is to extend Jones' theorem from $BMOA$ to the space $F(p, p-2, s)$, for $1 \le p < \infty$ and $0 < s \le 1$. In Jones' proof presented in [GZ], the Fefferman duality theorem is used. However, the method cannot be used in our situation since in general we do not know what is the predual of the space $F(p, p-2, s)$. Here we give a relatively direct proof of our result. We need the following lemma.

The following inequality is from [OF2], Lemma 2.5. Since a proof was not given in [OF2], we give a proof here for the convenience of a reader. I would like to thank J. M. Ortega and J. Fàbrega for providing me the following proof. In what follows, C will be a positive constant which may vary from line to line.

Lemma 1. *Let $s > -1$, $r, t > 0$, and $r + t - s > 2$. If $t < s + 2 < r$ then we have*

$$\int_D \frac{(1-|\eta|^2)^s}{|1-\bar{\eta}z|^r|1-\bar{\eta}\zeta|^t}\,dA(\eta) \le \frac{C}{(1-|z|^2)^{r-s-2}|1-\bar{\zeta}z|^t}.$$

Proof. Let

$$I = \int_D \frac{(1-|\eta|^2)^s}{|1-\bar{\eta}z|^r|1-\bar{\eta}\zeta|^t}\,dA(\eta).$$

We will use the following well-known inequality (see Lemma 4.2.2 in [Zhu]): Let $-1 < s < r - 2$. Then

$$\int_D \frac{(1-|\eta|^2)^s}{|1-\bar{\eta}z|^r}\,dA(\eta) \le \frac{C}{(1-|z|^2)^{r-s-2}}.$$

Let $d(\zeta, z) = |\bar{z}(z-\zeta)| + |\bar{\zeta}(\zeta - z)|$ be the non isotropic pseudodistance and c_d a constant such that

$$d(\zeta, z) \le c_d(d(\zeta, w) + d(w, z)).$$

Given ζ, z in D, according Definition 3.2 in [OF1], we take the following partition of D:

$$\begin{aligned}
\Omega_1 &= \left\{\eta \in D\colon d(\eta, z) \le \frac{d(\zeta, z)}{2c_d}\right\},\\
\Omega_2 &= \left\{\eta \in D\colon d(\eta, \zeta) \le \frac{d(\zeta, z)}{2c_d}\right\},\\
\Omega_3 &= \left\{\eta \in D\colon \frac{d(\zeta, z)}{2c_d} < d(\eta, z) \le d(\eta, \zeta)\right\},\\
\Omega_4 &= \left\{\eta \in D\colon \frac{d(\zeta, z)}{2c_d} < d(\eta, \zeta) \le d(\eta, z)\right\}.
\end{aligned}$$

By Lemma 3.3 in [OF1],

$$\begin{aligned}
&|1-\bar{\eta}z| \le C|1-\bar{\zeta}z| \le C|1-\bar{\eta}\zeta|, \quad \eta \in \Omega_1,\\
&|1-\bar{\eta}\zeta| \le C|1-\bar{\zeta}z| \le C|1-\bar{\eta}z|, \quad \eta \in \Omega_2,\\
&|1-\bar{\zeta}z| \le C|1-\bar{\eta}z| \le C|1-\bar{\eta}\zeta|, \quad \eta \in \Omega_3,\\
&|1-\bar{\zeta}z| \le C|1-\bar{\eta}\zeta| \le C|1-\bar{\eta}z|, \quad \eta \in \Omega_4.
\end{aligned}$$

Divide the integral I into two integrals, I_1 and I_2, on $\Omega_1 \cup \Omega_3$ and $\Omega_2 \cup \Omega_4$, respectively. From above inequalities, it is clear that

$$I_1 \le \frac{1}{|1-\bar{\zeta}z|^t}\int_D \frac{(1-|\eta|^2)^s}{|1-\bar{\eta}z|^r}\,dA(\eta) \le \frac{C}{(1-|z|^2)^{r-s-2}|1-\bar{\zeta}z|^t}.$$

Now we estimate I_2. It is easy to see that I_2 is bounded by a multiple of

$$J_2 = \int_D \frac{(1-|\eta|^2)^s}{(|1-\bar{\zeta}z| + |1-\bar{\eta}\zeta|)^r|1-\bar{\eta}\zeta|^t}\,dA(\eta).$$

Let $\zeta = |\zeta|e^{i\theta}$. Using the change of variable $\lambda = e^{-i\theta}\eta$, we have $\zeta\bar{\eta} = |\zeta|\bar{\lambda}$. Thus

$$J_2 = \int_D \frac{(1-|\lambda|^2)^s}{(|1-\bar{\zeta}z| + |1-\bar{\lambda}|\zeta||)^r |1-\bar{\lambda}|\zeta||^t} \, dA(\lambda).$$

Since for any $\zeta \in D$ we have

$$|1-\lambda| \le 2|1-\bar{\lambda}|\zeta||,$$

we get J_2 is bounded by a constant times

$$M_2 = \int_D \frac{(1-|\lambda|^2)^s}{(|1-\bar{\zeta}z| + |1-\lambda|)^r |1-\lambda|^t} \, dA(\lambda).$$

If $s \ge 0$ then by integration in polar coordinates on a disk of center 1 and radius 2, we obtain

$$\text{(1)} \qquad I_2 \le C \int_0^2 \frac{R^{s+1-t}}{(|1-\bar{\zeta}z| + R)^r} dR \le \frac{C}{|1-\bar{\zeta}z|^{r+t-s-2}}.$$

In the last inequality, we used $r+t-s-2 > 0$ and $s > t-2$ (Note that, to get the above estimate for I_2 we have not used the condition $s+2 < r$ yet. This is important for the proof of the case $-1 < s < 0$).

Since $1-|z|^2 \le 2|1-\bar{\zeta}z|$ and $r-s-2 > 0$, we have

$$\frac{1}{|1-\bar{\zeta}z|^{r+t-s-2}} \le \frac{C}{(1-|z|^2)^{r-s-2}|1-\bar{\zeta}z|^t},$$

which concludes the proof.

Now, we consider the case $-1 < s < 0$. For simplicity let $K = |1-\bar{\zeta}z|$. Then

$$\begin{aligned} M_2 &= \int_D \frac{(1-|\lambda|^2)^s}{(K + |1-\lambda|)^r |1-\lambda|^t} \, dA(\lambda) \\ &= \int_0^1 (1-R^2)^s R \, dR \int_0^{2\pi} \frac{1}{(K + |1-Re^{i\theta}|)^r |1-Re^{i\theta}|^t} \, d\theta. \end{aligned}$$

Let

$$u(R) = -(1-R^2)^{s+1}/(2(s+1))$$

and

$$v(R) = \int_0^{2\pi} \frac{1}{(K + |1-Re^{i\theta}|)^r |1-Re^{i\theta}|^t} \, d\theta.$$

Using integration by parts we get

$$
\begin{aligned}
M_2 &= \int_0^1 u'(R)v(R)\,dR \\
&= u(R)v(R)\big|_0^1 - \int_0^1 u(R)v'(R)\,dR \\
&\le \frac{\pi}{s+1} + \frac{r}{2(s+1)} \int_0^1 (1-R^2)^{s+1}\,dR \int_0^{2\pi} \frac{d\theta}{(K+|1-Re^{i\theta}|)^{r+1}|1-Re^{i\theta}|^t} \\
&\quad + \frac{t}{2(s+1)} \int_0^1 (1-R^2)^{s+1}\,dR \int_0^{2\pi} \frac{d\theta}{(K+|1-Re^{i\theta}|)^{r}|1-Re^{i\theta}|^{t+1}} \\
&\le \frac{\pi}{s+1} + C_1 \int_\Delta \frac{(1-|\lambda|^2)^{s+1}}{(|1-\bar\zeta z|+|1-\lambda|)^{r+1}|1-\lambda|^t} dA(\lambda) \\
&\quad + C_2 \int_\Delta \frac{(1-|\lambda|^2)^{s+1}}{(|1-\bar\zeta z|+|1-\lambda|)^{r}|1-\lambda|^{t+1}} dA(\lambda).
\end{aligned}
$$

Thus by (1) both integrals are bounded by

$$\frac{C}{|1-\bar\zeta z|^{r+t-s-2}}.$$

Since $1-|z|^2 \le 2|1-\bar\zeta z|$ and $r-s-2>0$, we have

$$\frac{1}{|1-\bar\zeta z|^{r+t-s-2}} \le \frac{C}{(1-|z|^2)^{r-s-2}|1-\bar\zeta z|^t},$$

which completes the proof. □

The following is our main result.

Theorem 2. *Let $0 < s \le 1$, $1 \le p < \infty$, $0 \le t < \infty$, and let $f \in B$. Then the following quantities are equivalent:*

(A) $\operatorname{dist}_B\big(f, F(p, p-2, s)\big)$;

(B) $\inf\{\varepsilon\colon \chi_{\Omega_\varepsilon(f)} \frac{dA(z)}{(1-|z|^2)^{2-s}}$ *is an s-Carleson measure*$\}$;

(C) $\inf\{\varepsilon\colon \sup_{a\in D} \int_{\Omega_\varepsilon(f)} |f'(z)|^t (1-|z|^2)^{t-2}(1-|\varphi_a(z)|^2)^s\,dA(z) < \infty\}$;

(D) $\inf\{\varepsilon\colon \sup_{a\in D} \int_{\Omega_\varepsilon(f)} |f'(z)|^t (1-|z|^2)^{t-2} g^s(z,a)\,dA(z) < \infty\}$,

where $\Omega_\varepsilon(f) = \{z \in D\colon |f'(z)|(1-|z|^2) \ge \varepsilon\}$.

Remark. Notice that since $Q_s = F(2,0,s)$, Theorem 2 gives us the same estimates for $\operatorname{dist}_B(f, Q_s)$. Also since $Q_1 = BMOA$, Theorem 2 includes Jones' result mentioned above.

Proof. Let $f \in B$. By [Zhu, Lemma 4.2.8],

$$f(z) - f(0) = \int_D \frac{f'(w)(1-|w|^2)}{(1-z\bar w)^2 \bar w}\,dA(w). \tag{2}$$

Define

$$f_1(z) = f(0) + \int_{\Omega_\varepsilon(f)} \frac{f'(w)(1-|w|^2)}{(1-z\bar{w})^2\bar{w}}\,dA(w)$$

and

$$f_2(z) = \int_{D\setminus\Omega_\varepsilon(f)} \frac{f'(w)(1-|w|^2)}{(1-z\bar{w})^2\bar{w}}\,dA(w).$$

Then by (2),

$$f(z) = f_1(z) + f_2(z).$$

Let

$$f_3(z) = f_1(z) + f_2(0)$$

and

$$f_4(z) = f_2(z) - f_2(0).$$

Then

$$f = f_3 + f_4.$$

Now we are going to show that $f_3 \in F(p, p-2, s)$. Since

$$f_3''(z) = 6\int_{\Omega_\varepsilon(f)} \frac{\bar{w}f'(w)(1-|w|^2)}{(1-z\bar{w})^4}\,dA(w),$$

we get by Fubini's theorem,

$$\begin{aligned} I &= \sup_{a\in D}\int_D |f_3''(z)|(1-|\varphi_a(z)|^2)^s\,dA(z) \\ &\le 6\sup_{a\in D}\int_D\int_{\Omega_\varepsilon(f)} \frac{|f'(w)|(1-|w|^2)}{|1-z\bar{w}|^4}\,dA(w)\frac{(1-|a|^2)^s(1-|z|^2)^s}{|1-\bar{a}z|^{2s}}\,dA(z) \\ &\le 6\|f\|_B \sup_{a\in D}\int_{\Omega_\varepsilon(f)} (1-|a|^2)^s\,dA(w)\int_D \frac{(1-|z|^2)^s}{|1-w\bar{z}|^4|1-a\bar{z}|^{2s}}\,dA(z). \end{aligned}$$

By Lemma 1,

$$\int_D \frac{(1-|z|^2)^s}{|1-w\bar{z}|^4|1-a\bar{z}|^{2s}}\,dA(z) \le \frac{C}{(1-|w|^2)^{2-s}|1-\bar{a}w|^{2s}}.$$

Thus

$$I \le 6C\|f\|_B \sup_{a\in D}\int_{\Omega_\varepsilon(f)} \frac{(1-|a|^2)^s}{|1-\bar{a}w|^{2s}}\,\frac{dA(w)}{(1-|w|^2)^{2-s}}. \tag{3}$$

If $\chi_{\Omega_\varepsilon(f)}\frac{dA(z)}{(1-|z|^2)^{2-s}}$ is an s-Carleson measure, by [ASX] we know

$$\sup_{a\in D}\int_{\Omega_\varepsilon(f)} \left(\frac{(1-|a|^2)}{|1-\bar{a}w|^2}\right)^s \frac{dA(w)}{(1-|w|^2)^{2-s}} < \infty.$$

Combining with (3) we get that

$$\sup_{a\in D}\int_D |f_3''(z)|(1-|\varphi_a(z)|^2)^s\,dA(z) < \infty.$$

Thus $f_3' \in F(1,0,s)$. By Theorem 3.2 in [R], we get that $f_3 \in F(1,-1,s)$. By [Zha, Proposition 6.4], we see that $F(1,-1,s) \subset F(p,p-2,s)$. Thus $f_3 \in F(p,p-2,s)$.

Next we prove that

$$\|f_4\|_B^* \le C\varepsilon. \tag{4}$$

Since $f_4(0)=0$, we see that

$$\|f_4\|_B^* = \|f_2\|_B.$$

But

$$f_2'(z) = 2\int_{D\setminus\Omega_\varepsilon(f)} \frac{f'(w)(1-|w|^2)}{(1-z\bar{w})^3}\,dA(w).$$

So

$$|f_2'(z)| \le 2\varepsilon \int_D \frac{1}{|1-z\bar{w}|^3}\,dA(w) \le \frac{2C\varepsilon}{1-|z|^2}.$$

Thus

$$\|f_2\|_B = \sup_{z\in D} |f_2'(z)|(1-|z|^2) \le 2C\varepsilon.$$

Thus we get (4).

Therefore,

$$\operatorname{dist}_B\big(f, F(p,p-2,s)\big) \le \|f-f_3\|_B^* = \|f_4\|_B^* \le 2C\varepsilon,$$

which implies that $\operatorname{dist}_B\big(f, F(p,p-2,s)\big)$ is bounded by a multiple of quantity (B).

If quantity (B) > quantity (A), there are two positive constants ε and ε_1 and a function $f_{\varepsilon_1} \in F(p,p-2,s)$ so that $\chi_{\Omega_\varepsilon(f)} \frac{dA(z)}{(1-|z|^2)^{2-s}}$ is not an s-Carleson measure, $\varepsilon > \varepsilon_1$ and $\|f-f_{\varepsilon_1}\|_B^* \le \varepsilon_1$. Since

$$|f_{\varepsilon_1}'(z)|(1-|z|^2) > |f'(z)|(1-|z|^2) - \|f-f_{\varepsilon_1}\|_B^* > |f'(z)|(1-|z|^2) - \varepsilon_1,$$

we have $\Omega_\varepsilon(f) \subset \Omega_{\varepsilon-\varepsilon_1}(f_{\varepsilon_1})$, and so for every p, $1 \le p < \infty$,

$$\chi_{\Omega_\varepsilon(f)} \frac{dA(z)}{(1-|z|^2)^{2-s}} \le \frac{|f_{\varepsilon_1}'(z)|^p (1-|z|^2)^{s+p-2}}{(\varepsilon-\varepsilon_1)^p}\,dA(z).$$

Since $f_{\varepsilon_1} \in F(p,p-2,s)$, we get by [Zha, Theorem 2.4], $|f_{\varepsilon_1}'(z)|^p(1-|z|^2)^{s+p-2}\,dA(z)$ is an s-Carleson measure. Thus $\chi_{\Omega_\varepsilon(f)} \frac{dA(z)}{(1-|z|^2)^{2-s}}$ is an s-Carleson measure, which is a contradiction. Thus quantity (A) is equivalent to quantity (B).

Notice that $\chi_{\Omega_\varepsilon(f)} \frac{dA(z)}{(1-|z|^2)^{2-s}}$ is an s-Carleson measure means

$$\sup_{a\in D} \int_{\Omega_\varepsilon(f)} \frac{|\varphi_a'(z)|^s}{(1-|z|^2)^{2-s}}\,dA(z) < \infty,$$

which is equivalent to

$$\sup_{a\in D} \int_{\Omega_\varepsilon(f)} (1-|z|^2)^{-2}(1-|\varphi_a(z)|^2)^s\,dA(z) < \infty.$$

This is the case $t = 0$ in (C). For $t > 0$, the equivalence between (B) and (C) can be easily obtained from the above inequality by noticing that for any $z \in \Omega_\varepsilon(f)$,

$$\varepsilon \le |f'(z)|(1-|z|^2) \le \|f\|_B.$$

That quantity (C) is bounded by a multiple of quantity (D) is obvious from the fact that

$$1 - |\varphi_a(z)|^2 \le C \log \frac{1}{|\varphi_a(z)|} = Cg(z,a).$$

To show that quantity (D) is bounded by a multiple of quantity (C), we split the integral

$$I = \int_{\Omega_\varepsilon(f)} |f'(z)|^t (1-|z|^2)^{t-2} g^s(z,a)\, dA(z)$$

into the sum of two integrals

$$I_1 = \int_{\Omega_\varepsilon(f)\cap D_{1/4}} |f'(z)|^t (1-|z|^2)^{t-2} g^s(z,a)\, dA(z)$$

and

$$I_2 = \int_{\Omega_\varepsilon(f)\setminus D_{1/4}} |f'(z)|^t (1-|z|^2)^{t-2} g^s(z,a)\, dA(z),$$

where $D_{1/4} = \{z \in D\colon |z| < 1/4\}$. Using the following simple inequality:

$$g(z,a) = \log \frac{1}{|\varphi_a(z)|} \begin{cases} \ge \log 4 \ge 1, & |\varphi_a(z)| \le \frac{1}{4}, \\ \le 4(1-|\varphi_a(z)|^2), & |\varphi_a(z)| \ge \frac{1}{4}, \end{cases}$$

we get that

$$I_2 \le 4 \int_{\Omega_\varepsilon(f)} |f'(z)|^t (1-|z|^2)^{t-2} (1-|\varphi_a(z)|^2)^s\, dA(z),$$

and

$$\begin{aligned} I_1 &\le \int_{\Omega_\varepsilon(f)} |f'(z)|^t (1-|z|^2)^{t-2} g^2(z,a)\, dA(z) \\ &\le \|f\|_B^t \int_{\Omega_\varepsilon(f)} (1-|z|^2)^{-2} g^2(z,a)\, dA(z) \le C < \infty, \end{aligned}$$

where C is a constant independent of a. Therefore, quantity (D) is bounded by a multiple of quantity (C). The proof is complete. □

From Theorem 2 we immediately obtain the following corollaries.

Corollary 3. *Let $0 < s \le 1$, $1 \le p_1 < p_2 < \infty$. Then*

$$\operatorname{dist}_B(f, F(p_1, p_1 - 2, s)) = \operatorname{dist}_B(f, F(p_2, p_2 - 2, s)).$$

Corollary 4. *Let $0 < s \le 1$, $1 \le p < \infty$ and $0 \le t < \infty$. Let f be an analytic function on D. Then the following conditions are equivalent.*

(A) *f is in the closure of $F(p, p-2, s)$ in B;*

(B) $\chi_{\Omega_\varepsilon(f)} \frac{dA(z)}{(1-|z|^2)^{2-s}}$ *is an* s*-Carleson measure for every* $\varepsilon > 0$;
(C) $\sup_{a\in D} \int_{\Omega_\varepsilon(f)} |f'(z)|^t (1-|z|^2)^{t-2}(1-|\varphi_a(z)|^2)^s \, dA(z) < \infty$ *for every* $\varepsilon > 0$;
(D) $\sup_{a\in D} \int_{\Omega_\varepsilon(f)} |f'(z)|^t (1-|z|^2)^{t-2} g^s(z,a) \, dA(z) < \infty$ *for every* $\varepsilon > 0$.

Corollary 5. *Let* $0 < s \le 1$, $1 \le p_1 < p_2 < \infty$. *Then the closure of* $F(p_1, p_1 - 2, s)$ *and* $F(p_2, p_2 - 2, s)$ *in* B *are the same.*

For the "little-oh" case, we have

Theorem 6. *Let* $0 < s \le 1$, $1 \le p < \infty$, $0 \le t < \infty$, *and let* $f \in B$. *Then the following quantities are equivalent:*

(A) $\mathrm{dist}_B(f, B_0)$;
(B) $\mathrm{dist}_B\big(f, F_0(p, p-2, s)\big)$;
(C) $\inf\{\varepsilon\colon \chi_{\Omega_\varepsilon(f)} \frac{dA(z)}{(1-|z|^2)^{2-s}}$ *is a vanishing* s*-Carleson measure*$\}$;
(D) $\inf\{\varepsilon\colon \lim_{|a|\to 1} \int_{\Omega_\varepsilon(f)} |f'(z)|^t (1-|z|^2)^{t-2}(1-|\varphi_a(z)|^2)^s \, dA(z) = 0\}$;
(E) $\inf\{\varepsilon\colon \lim_{|a|\to 1} \int_{\Omega_\varepsilon(f)} |f'(z)|^t (1-|z|^2)^{t-2} g^s(z,a) \, dA(z) = 0\}$.

Proof. Let $0 < s \le 1$ and let $1 \le p < \infty$. Since $F_0(p, p-2, s)$ contains all polynomials, and it is well known that the closure of the set of all polynomials in B is just B_0 (see, for example, [Ax]), we see that the closure of $F_0(p, p-2, s)$ in B contains B_0.

On the other hand, by [Zha, Corollary 2.8], $F_0(p, p-2, s) \subset B_0$. It is obvious that the closure of $F_0(p, p-2, s)$ in B is included in B_0. Thus B_0 equals to the closure of $F_0(p, p-2, s)$ in B, and so quantity (A) is equivalent to quantity (B).

The proof of the equivalence of quantities (B), (C), (D) and (E) is similar to the proof of the equivalence of quantity (A), (B), (C) and (D) in Theorem 1, we leave the details to readers. □

Corollary 7. *Let* $0 < s \le 1$ *and let* f *be an analytic function in* D. *Then* $f \in B_0$ *if and only if* $\chi_{\Omega_\varepsilon(f)} \frac{dA(z)}{(1-|z|^2)^{2-s}}$ *is a vanishing* s*-Carleson measure for every* $\varepsilon > 0$.

Remark. For $s = 1$, the result of Corollary 7 is proved in [GZ, Theorem 3].

For the case $s = 0$, we give the following result:

Theorem 8. *Let* $1 \le p < \infty$, *and let* $f \in B$. *Then the following quantities are equivalent:*

(A) $\mathrm{dist}_B(f, B_0)$;
(B) $\mathrm{dist}_B(f, B_p)$;
(C) $\inf\{\varepsilon\colon \lambda(\Omega_\varepsilon(f)) < \infty\}$,

where $\lambda(\Omega_\varepsilon(f)) = \int_{\Omega_\varepsilon(f)} \frac{dA(z)}{(1-|z|^2)^2}$ *is the hyperbolic area of the set* $\Omega_\varepsilon(f)$.

Proof. Since $B_p \subset B_0$, for $1 \le p < \infty$ (see, for example, [AFP]), by the same reason as the proof of Theorem 6, we know that quantity (A) is equivalent to quantity (B).

To prove that the quantity (B) is bounded by a multiple of quantity (C), we proceed as in the proof of Theorem 2. Let f_1 and f_2 be the same as in the proof of Theorem 1. We need only prove that $f_1 \in B_p$, for $1 \le p < \infty$. We may assume that $f_1(0) = 0$. Since

$$f_1''(z) = 6 \int_{\Omega_\varepsilon(f)} \frac{f'(w)(1-|w|^2)\bar{w}}{(1-\bar{z}w)^4}\, dA(w),$$

we get by Fubini's theorem,

$$\begin{aligned}\int_D |f_1''(z)|\, dA(z) &\le 6\|f\|_B \int_{\Omega_\varepsilon(f)} \int_D \frac{dA(z)}{|1-z\bar{w}|^4}\, dA(w) = 6\|f\|_B \int_{\Omega_\varepsilon(f)} \frac{dA(w)}{(1-|w|^2)^2} \\ &= 6\|f\|_B \lambda(\Omega_\varepsilon(f)).\end{aligned}$$

Thus $f_1 \in B_1$ if $\lambda(\Omega_\varepsilon(f)) < \infty$. Since $B_1 \subset B_p$, for $1 < p < \infty$, we get that for $1 \le p < \infty$, $f_1 \in B_p$ if $\lambda(\Omega_\varepsilon(f)) < \infty$. Thus $\operatorname{dist}_B(f, B_p)$ is bounded by a multiple of quantity (C).

To prove the converse, suppose that the quantity (C) > quantity (B). Without loss of generality, since $B_1 \subset B_p$, we may assume that $1 < p < \infty$. Then there are two constants $\varepsilon > \varepsilon_1 > 0$ and a function $f_{\varepsilon_1} \in B_p$ such that $\lambda(\Omega_\varepsilon(f)) = \infty$ and $\|f - f_{\varepsilon_1}\|_B \le \varepsilon_1$. As before, we have, for $1 < p < \infty$,

$$\chi_{\Omega_\varepsilon(f)} \frac{dA(z)}{(1-|z|^2)^2} \le \frac{|f_{\varepsilon_1}'(z)|^p (1-|z|^2)^{p-2}}{(\varepsilon-\varepsilon_1)^p}\, dA(z).$$

Since $f_{\varepsilon_1} \in B_p$, we have

$$\int_D |f_{\varepsilon_1}'(z)|^p (1-|z|^2)^{p-2}\, dA(z) < \infty.$$

Thus

$$\lambda(\Omega_\varepsilon(f)) = \int_D \chi_{\Omega_\varepsilon(f)} \frac{dA(z)}{(1-|z|^2)^2} \le \frac{1}{(\varepsilon-\varepsilon_1)^p} \int_D |f_{\varepsilon_1}'(z)|^p (1-|z|^2)^{p-2}\, dA(z) < \infty,$$

which is a contradiction. □

The following result is an immediate consequence of Theorem 8.

Corollary 9. *Let f be analytic in D. Then $f \in B_0$ if and only if $\lambda(\Omega_\varepsilon(f)) < \infty$ for every $\varepsilon > 0$.*

The author learnt after submitting this paper that Lindström and Palmberg found the predual of the space $F(p,q,s)$, in their paper [LP]. Although, it is not clear whether their result could be used to provide a proof of Theorem 2 here.

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