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FINE TOPOLOGY OF VARIABLE EXPONENT ENERGY SUPERMINIMIZERS

Petteri Harjulehto and Visa Latvala

University of Helsinki, Department of Mathematics and Statistics
P.O. Box 68, FI-00014 University of Helsinki, Finland; petteri.harjulehto@helsinki.fi

University of Joensuu, Department of Physics and Mathematics
P.O. Box 111, FI-80101 Joensuu, Finland; visa.latvala@joensuu.fi

Abstract. We study the $p(\cdot)$-fine continuity in the variable exponent Sobolev spaces under the standard assumptions that $p\colon \Omega \to \mathbf{R}$ is log-Hölder continuous and $1 < p^- \leq p^+ < \infty$. As a by-product we obtain improvements in the variational exponent capacity theory and in the non-linear potential theory based on $p(\cdot)$-Laplacian.

1. Introduction

Quasicontinuity is a central notion in the Sobolev function theory and in potential theory. This is so since many crucial ideas in the theory of pde's require the use of quasicontinuous representatives of Sobolev functions. For the pointwise study of quasicontinuous functions the Euclidean topology is not relevant in general, instead one can use the fine topology, which dates back to Cartan [6] in the linear case. Nowadays it is well-known that, for any constant $1 < p < \infty$, the function u is p-quasicontinuous if and only if u is p-finely continuous outside a set of p-capacity zero. This result is deep, the implication from left to right requires sharp energy estimates for supersolutions of p-Laplace equation. The converse implication which is related to Choquet's property is even deeper, it was first established for general p in [24], Theorem 3; see also [12] and [2]. Another proof based on the pointwise estimates of p-supersolutions of [30] can be found in [32], p. 145.

The goal of this paper is to introduce the fine topology in the variable exponent case and show that quasicontinuity implies fine continuity outside a set of capacity zero even in this setting (Theorem 6.5). In particular we show that each $p(\cdot)$-superharmonic function is $p(\cdot)$-finely continuous and $p(\cdot)$-quasicontinuous (Theorems 5.3 and 6.7). As a by-product we obtain several improvements related to the variational exponent capacity theory or the non-linear potential theory based on $p(\cdot)$-Laplacian. For instance we establish the existence of the capacitary extremal for arbitrary subsets compactly contained in a given bounded open set (Theorem 6.3).

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This result extends a similar result in [5] proved for compact subsets by a different method.

To obtain the $p(\cdot)$-fine continuity for $p(\cdot)$-superharmonic functions we modify the fixed exponent argumentation from [32]. We use energy estimates and pointwise estimates for supersolutions of the $p(\cdot)$-Laplace equation proved in [5]. We make the standard assumptions that the variable exponent p satisfies the condition $1 < p^- \le p^+ < \infty$ and is log-Hölder continuous.

Roughly speaking the effect of the variable exponent is that many crucial estimates include an additional term which however appears to be irrelevant from the fine topological point of view. Typically the additional terms and difficulties are related to the use of Hölder's inequality, Poincaré's inequality, or the standard mollification procedure. It is also crucial that in the case of variable exponent the product αu is not necessarily $p(\cdot)$-superharmonic if u is $p(\cdot)$-superharmonic and $\alpha > 0$ is constant. As a consequence of this fact certain potential theoretic properties require new ideas. For instance the strict minimum principle for $p(\cdot)$-superharmonic functions seems to be an open problem. Also it is not known in general, see [20], Corollary 4.7, whether the infinity set of a $p(\cdot)$-superharmonic function is of zero $p(\cdot)$-capacity.

2. Variable exponent spaces

A measurable function $p\colon \mathbf{R}^n \to [1,\infty)$ is called a *variable exponent*. We assume that p is bounded and denote

$$p^+ = \sup_{x\in\mathbf{R}^n} p(x), \quad p^- = \inf_{x\in\mathbf{R}^n} p(x).$$

For each $A \subset \mathbf{R}^n$ we write

$$p^+_A = \sup_{x\in A} p(x), \quad p^-_A = \inf_{x\in A} p(x).$$

Let Ω be an open subset of $\mathbf{R}^n, n \ge 2$. The *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ consists of all measurable functions u defined on Ω for which the modular

$$\varrho_{L^{p(\cdot)}(\Omega)}(u) := \int_\Omega |u(x)|^{p(x)}\,dx$$

is finite. The Luxemburg norm on this space is defined as

$$\|u\|_{p(\cdot)} = \inf\Big\{\lambda > 0 : \varrho_{L^{p(\cdot)}(\Omega)}\big(\tfrac{u}{\lambda}\big) \le 1\Big\}.$$

Equipped with this norm $L^{p(\cdot)}(\Omega)$ is a Banach space. The variable exponent Lebesgue space is a special case of an Orlicz–Musielak space studied in [33]. For a constant function p the variable exponent Lebesgue space coincides with the standard Lebesgue space.

The *variable exponent Sobolev space* $W^{1,p(\cdot)}(\Omega)$ consists of functions $u \in L^{p(\cdot)}(\Omega)$ whose distributional gradient ∇u exists almost everywhere and belongs to $L^{p(\cdot)}(\Omega)$.

The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is a Banach space with the norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

The local Sobolev space $W^{1,p(\cdot)}_{\mathrm{loc}}(\Omega)$ is defined in the usual way. For basic results on variable exponent spaces we refer to [31].

An interesting feature here is that smooth functions need not to be dense in variable exponent Sobolev spaces. This was observed by Zhikov in connection with Lavrentiev phenomenon, see [35]. However, if the exponent p satisfies a logarithmic Hölder continuity property, or briefly "p is log-*Hölder continuous*", then the maximal operator is locally bounded and consequently smooth functions are dense, see [7, 26, 34]. Recall that the log-Hölder condition means that there is a constant $C > 0$ such that

$$|p(x) - p(y)| \le \frac{C}{-\log(|x-y|)}$$

for all $x, y \in \Omega$ with $|x - y| \le 1/2$. The exponent p is log-Hölder continuous in an open set Ω if and only if there exists a constant $C > 0$ such that

$$|B|^{p^-_{B\cap\Omega} - p^+_{B\cap\Omega}} \le C \tag{2.1}$$

for every ball $B \cap \Omega \neq \emptyset$, see [7].

When smooth functions are dense in variable exponent Sobolev spaces, there is no confusion to define the Sobolev space with zero boundary values, $W^{1,p(\cdot)}_0(\Omega)$, as the completion of $C^\infty_0(\Omega)$ with respect to the norm $\|u\|_{1,p(\cdot)}$. For more about this see [14].

Assumptions and conventions. Throughout we assume that $p\colon \Omega \to \mathbf{R}$ is log-Hölder continuous and $1 < p^- \le p^+ < \infty$. For brevity we write $C = C(p)$ and say that C depends only on p, if C is a constant which depends only on p^+, p^- and the constant C in (2.1). Moreover, we write $a \approx b$ for two non-negative quantities if there is a constant $C = C(n, p)$ such that $\frac{1}{C} a \le b \le Ca$.

3. Capacities

We begin by recalling some capacities appearing in the existing literature. Throughout this section $\Omega \subset \mathbf{R}^n$ is an open set.

The Sobolev capacity was extended to variable exponent case in [18], Section 3. For $E \subset \mathbf{R}^n$ we denote

$$S_{p(\cdot)}(E) = \big\{u \in W^{1,p(\cdot)}(\mathbf{R}^n) : u \ge 1 \text{ in an open set } U \Subset \mathbf{R}^n \text{ containing } E\big\}$$

and define

$$C_{p(\cdot)}(E) = \inf_{u \in S_{p(\cdot)}(E)} \int_{\mathbb{R}^n} \big(|u|^{p(x)} + |\nabla u|^{p(x)}\big)\, dx.$$

Here we make the convention that $C_{p(\cdot)}(E) = \infty$ if $S_{p(\cdot)}(E) = \emptyset$. Recall that under our assumption $1 < p^- \le p^+ < \infty$ the *Sobolev* $p(\cdot)$*-capacity* $C_{p(\cdot)}(\cdot)$ is an outer measure and a Choquet capacity, see [18], Corollaries 3.3 and 3.4.

A variable exponent version of the relative $p(\cdot)$-capacity of the condenser has been used in [5], [16]. This is defined for any compact $K \subset \Omega$ by setting

$$\operatorname{cap}_{p(\cdot)}(K,\Omega) = \inf_u \int_\Omega |\nabla u|^{p(x)}\,dx,$$

where the infimum is taken over all $u \in C_0^\infty(\Omega)$ such that $u \geq 1$ in K. Further, if $U \subset \Omega$ is open, define

$$\operatorname{cap}_{p(\cdot)}(U,\Omega) = \sup_{K\subset U \text{ compact}} \operatorname{cap}_{p(\cdot)}(K,\Omega),$$

and for an arbitrary $E \subset \Omega$, define

$$\operatorname{cap}_{p(\cdot)}(E,\Omega) = \sup_{E\subset U\subset\Omega \text{ open}} \operatorname{cap}_{p(\cdot)}(K,\Omega),$$

If p is bounded, then the relative $p(\cdot)$-capacity is a Choquet capacity. If $1 < p^- \leq p^+$ and if smooth functions are dense in the Sobolev space, then for $E \subset \Omega$ holds that $C_{p(\cdot)}(E) = 0$ if and only if $\operatorname{cap}_{p(\cdot)}(E,\Omega) = 0$. For the proof see [16].

3.1. Remark. In [5], the definition of relative $p(\cdot)$-capacity slightly differs from the one above. However, the resulting capacities are equivalent, and hence for our purposes the difference is irrelevant.

Next we present another version of the relative $p(\cdot)$-capacity. For every $E \subset \Omega$ we define

$$C_{p(\cdot)}(E,\Omega) = \inf \int_\Omega |\nabla u(x)|^{p(x)}\,dx,$$

where the infimum is taken over all $u \in W_0^{1,p(\cdot)}(\Omega)$ which are at least one in a neighborhood of E.

For the sake of clarity we first prove a lemma which connects the relative capacities $C_{p(\cdot)}$ and $\operatorname{cap}_{p(\cdot)}$. In what follows, we don't need this but we feel that the result has some independent interest.

3.2. Lemma. *Let $\Omega \subset \mathbf{R}^n$ be bounded. Then for every compact $K \subset \Omega$, we have*

$$C_{p(\cdot)}(K,\Omega) = \operatorname{cap}_{p(\cdot)}(K,\Omega).$$

Proof. For the proof we modify the argumentation in [32], p. 65. Note that the convolution approximation requires somewhat involved estimates in the variable exponent case.

Let $K \subset \Omega$ be compact. Then the inequality

$$C_{p(\cdot)}(K,\Omega) \leq \operatorname{cap}_{p(\cdot)}(K,\Omega)$$

is easy, since for all $u \in C_0^\infty(\Omega)$ satisfying $u \geq 1$ in K, and for all $\alpha > 1$, the function αu satisfies $u \geq 1$ in an open neighborhood of K.

To prove the converse inequality, let $u \in W_0^{1,p(\cdot)}(\Omega)$ be non-negative such that $u \geq 1$ in an open neighborhood $U \subset \Omega$ of K. Choose $\eta \in C_0^\infty(\Omega)$ so that $0 \leq \eta \leq 1$ and $\eta = 1$ on K. By definition smooth functions are dense in $W_0^{1,p(\cdot)}(\Omega)$ and hence we may choose a sequence (v_j) of functions in $C_0^\infty(\Omega)$ so that $v_j \to u$ in $W^{1,p(\cdot)}(\Omega)$. Set

$$u_j := (\phi_{\varepsilon_j} * u)\eta + v_j(1-\eta),$$

where $\varepsilon_j = j^{-1}$ and ϕ_{ε_j} is the standard mollifier. We choose j large enough so that $0 < \varepsilon_j < \min\{\operatorname{dist}(K, \partial U), \operatorname{dist}(\operatorname{spt}\eta, \partial\Omega)\}$. Then $u_j \in C_0^\infty(\Omega)$ and $u_j \geq 1$ in K. The inequality

$$\operatorname{cap}_{p(\cdot)}(K, \Omega) \leq C_{p(\cdot)}(K, \Omega)$$

follows, once we show that

$$\lim_{j\to\infty} \int_\Omega |\nabla u_j|^{p(x)}\,dx = \int_\Omega |\nabla u|^{p(x)}\,dx. \tag{3.3}$$

To do this, denote $u_{\varepsilon_j} := \phi_{\varepsilon_j} * u$ and notice that $D_i u_{\varepsilon_j} = (D_i u)_{\varepsilon_j}$ for $i = 1, \ldots, n$. Clearly,

$$|D_i u_j(x)|^{p(x)} \leq C(p) g_{ij}(x)$$

for $x \in \Omega$, where

$$g_{ij} := \eta^{p(x)} |D_i u_{\varepsilon_j}|^{p(x)} + |D_i\eta|^{p(x)} |u_{\varepsilon_j}|^{p(x)} + |D_i v_j|^{p(x)} |\eta|^{p(x)} + |v_j||D_i\eta|^{p(x)}.$$

By [22], Lemma 4.6, we have the estimate

$$|(D_i u)_{\varepsilon_j}(x)|^{p(x)} \leq C\big(\varrho_{L^{p(\cdot)}(\Omega)}(\nabla u) + 1 + |\Omega|\big)^{p^+/p^-} \big((\phi_{\varepsilon_j} * |D_i u|^{p(\cdot)})(x) + 1\big)$$

for all $x \in \Omega$. Integrating this gives

$$\begin{aligned}\int_\Omega |(D_i u)_{\varepsilon_j}|^{p(x)}\,dx &\leq M\left(\int_\Omega \phi_{\varepsilon_j} * |D_i u|^{p(\cdot)}\,dx + |\Omega|\right)\\ &\leq M\left(\int_\Omega |D_i u|^{p(x)}\,dx + |\Omega|\right)\end{aligned}$$

for $M := C(\varrho_{L^{p(\cdot)}(\Omega)}(\nabla u) + 1 + |\Omega|)^{p^+/p^-}$ since the mollification does not increase the L^1-norm. The same reasoning works for u_{ε_j}, and hence we obtain that g_{ij} is integrable over Ω. Since $v_j \to u$ in $W^{1,p}(\Omega)$, we may pick a subsequence still denoted by (v_j) such that $v_j \to u$ a.e. in Ω and $|\nabla v_j| \to |\nabla u|$ a.e. in Ω. Since $|D_i u_{\varepsilon_j}| \to |D_i u|$ a.e. in Ω and $|u_{\varepsilon_j}| \to |u|$ a.e. in Ω and $u_{\varepsilon_j} \to u$ in $W^{1,p(\cdot)}(\Omega)$, see for example the proof of Theorem 2.6 in [10], as $j \to \infty$, we infer by a variant of dominated convergence theorem (see [8], p. 21) that (3.3) holds for a subsequence. □

The relative capacity $C_{p(\cdot)}(E, \Omega)$ has the advantage that the extremal function can be directly studied for all subsets of Ω, not only for compact subsets. In Section 6 below, we characterize $C_{p(\cdot)}(E, \Omega)$ by means of $p(\cdot)$-quasicontinuous representatives. This gives the most natural version of capacity in the study of $p(\cdot)$-fine topology.

Quasicontinuity. Recall that a property holds $p(\cdot)$-*quasieverywhere* if it holds outside a set of zero Sobolev $p(\cdot)$-capacity. Recall also that $u\colon \Omega \to [-\infty, \infty]$ is $p(\cdot)$-*quasicontinuous* if for every $\varepsilon > 0$ there exists a set E, with $C_{p(\cdot)}(E) \leq \varepsilon$, so that u is continuous when restricted to $\Omega \setminus E$. Since $C_{p(\cdot)}$ is an outer capacity, we can assume that E is open.

3.4. Remark. Under our assumptions u belongs to $W_0^{1,p(\cdot)}(\Omega)$ if and only if there is a $p(\cdot)$-quasicontinuous function $\tilde{u} \in W^{1,p(\cdot)}(\mathbf{R}^n)$ such that $u = \tilde{u}$ a.e. in Ω and $\tilde{u} = 0$ $p(\cdot)$-q.e. in $\mathbf{R}^n \setminus \Omega$, see [19], Theorem 3.3 for the proof.

We also recall a uniqueness property [19], Lemma 2.1 for $p(\cdot)$-quasicontinuous functions. The proof of this property is based on an abstract result of [29].

3.5. Lemma. *Let u and v be $p(\cdot)$-quasicontinuous functions in Ω such that $u = v$ a.e. in Ω. Then $u = v$ $p(\cdot)$-q.e. in Ω.*

We are now prepared to establish the fundamental relationships between the two types of capacities. To do this, we need a modular version of the Poincaré inequality.

3.6. Lemma. *Let $B = B(x_0, r) \subset \Omega$ be a ball. Then for all $u \in W_0^{1,p(\cdot)}(B)$ with $\int_B |\nabla u|^{p(x)}\,dx \le 1$, there is a constant $C = C(n,p)$ so that*

$$\int_B \left(\frac{|u|}{r}\right)^{p(x)} dx \le C \int_B |\nabla u|^{p(x)} + C|B|.$$

Proof. We have by [13], Lemma 7.14, for every $u \in W_0^{1,1}(B)$ and for almost all $x \in B$

$$|u(x)| \le C \int_B \frac{|\nabla u|}{|x-y|^{n-1}}\,dy.$$

By the estimate [36], Lemma 2.8.3

$$\int_B \frac{|\nabla u|}{|x-y|^{n-1}}\,dy \le CrM|\nabla u|(x),$$

where M is the Hardy–Littlewood maximal operator. Hence we arrive at

$$\frac{|u(x)|}{r} \le CM|\nabla u|(x).$$

We raise both sides of this inequality to the power $p(x)$ and integrate over B to obtain

$$\int_B \left(\frac{|u|}{r}\right)^{p(x)} dx \le C \int_B \big(M|\nabla u|\big)^{p(x)}\,dx.$$

By [7], Lemma 3.3 (here $\varrho_{L^{p(\cdot)}(B)}(\nabla u) \le 1$ is needed) and by the fact that $M\colon L^{p_B^-}(B) \to L^{p_B^-}(B)$ is bounded, we have

$$\int_B \big(M|\nabla u|\big)^{p(x)}\,dx \le \int_B C\left(M\big(|\nabla u|^{p(\cdot)/p_B^-}\big) + 1\right)^{p_B^-} dx \le C\int_B |\nabla u|^{p(x)}\,dx + C|B|.$$

Combining this with the previous inequality gives the claim. □

3.7. Lemma. *Let $B = B(x_0, r) \subset \mathbf{R}^n$ be a ball with $r \le 1$ and let $E \subset B$. Then there is a constant $C = C(n,p)$ so that*

$$C_{p(\cdot)}(E) \le (Cr^{p(x_0)} + 1)C_{p(\cdot)}(E, 2B) + Cr^{n+p(x_0)} \tag{3.8}$$

and

$$C_{p(\cdot)}(E,2B) \le \left(\frac{C2^{p^+-1}}{r^{p(x_0)}} + 2^{p^+-1}\right) C_{p(\cdot)}(E). \tag{3.9}$$

Moreover, there is a constant $C = C(n,p)$ such that for any $x \in \mathbf{R}^n$ and $r > 0$

$$\frac{1}{C} r^{n-p(x)} \le C_{p(\cdot)}(B(x,r),B(x,2r)) \le C r^{n-p(x)} \tag{3.10}$$

if $p(x) < n$ and

$$\frac{1}{C} \le C_{p(\cdot)}(B(x,r),B(x,2r)) \le C \tag{3.11}$$

if $p(x) = n$.

Proof. Let u be an admissible test function for $C_{p(\cdot)}(E,2B)$. Then it is also a test function for $C_{p(\cdot)}(E)$. By (2.1) we have $r^{-p(x)} \approx r^{-p(x_0)}$ for all $x \in 2B$. Hence by Lemma 3.6

$$|E| \le \int_{2B} |u|^{p(x)}\,dx \le C r^{p(x_0)} \int_{2B} |\nabla u|^{p(x)}\,dx + C r^{n+p(x_0)}. \tag{3.12}$$

Therefore

$$\begin{aligned} C_{p(\cdot)}(E) &\le \int_{2B} |u|^{p(x)}\,dx + \int_{2B} |\nabla u|^{p(x)}\,dx \\ &\le \big(C r^{p(x_0)} + 1\big) \int_{2B} |\nabla u|^{p(x)}\,dx + C r^{n+p(x_0)}. \end{aligned}$$

Taking infimum over all admissible functions for $C_p(E,2B)$ yields (3.8).

Next, let u be an admissible test function for $C_{p(\cdot)}(E)$ and let $\eta \in C_0^\infty(2B)$ be such that $0 \le \eta \le 1$, $\eta = 1$ on B, and $|\nabla \eta| \le \frac{C}{r}$. Then $u\eta$ is an admissible test function for $C_{p(\cdot)}(E,2B)$, and hence

$$\begin{aligned} C_{p(\cdot)}(E,2B) &\le \int_{2B} |\nabla(u\eta)|^{p(x)}\,dx \le \frac{C2^{p^+-1}}{r^{p(x_0)}} \int_{2B} |u|^{p(x)}\,dx + 2^{p^+-1} \int_{2B} |\nabla u|^{p(x)}\,dx \\ &\le \left(\frac{C2^{p^+-1}}{r^{p(x_0)}} + 2^{p^+-1}\right) \int_{2B} |u|^{p(x)} + |\nabla u|^{p(x)}\,dx. \end{aligned}$$

The claim (3.9) follows by taking infimum over all u.

The inequalities (3.10) and (3.11) follow from [5], Proposition 5.1 and 5.2 together with basic properties of relative $p(\cdot)$-capacity. □

3.13. Remark. (a) Let $\Omega \subset \mathbf{R}^n$ be a bounded open set and let $E \Subset \Omega$. Then we have $C_{p(\cdot)}(E) = 0$ if and only if $C_{p(\cdot)}(E,\Omega) = 0$.

Assume first that $C_{p(\cdot)}(E) = 0$. Then essentially the same argument which proves (3.9) gives $C_{p(\cdot)}(E,\Omega) = 0$.

Assume then that $C_{p(\cdot)}(E,\Omega)=0$. Choose a minimizing sequence (u_i) of test functions of $C_{p(\cdot)}(E,\Omega)$. Then each u_i is a test function for $C_{p(\cdot)}(E)$. Thus we have

$$C_{p(\cdot)}(E) \le \int_\Omega |u_i|^{p(x)} + |\nabla u_i|^{p(x)}\,dx.$$

We may assume that $\|\nabla u_i\|_{L^{p(\cdot)}(\Omega)} \le 1$ for every i. Hence a norm version of the Poincaré inequality, see [19], Theorem 4.1, implies

$$\|u_i\|_{L^{p(\cdot)}(\Omega)} \le C\|\nabla u_i\|_{L^{p(\cdot)}(\Omega)} \le C\left(\int_\Omega |\nabla u_i|^{p(x)}\,dx\right)^{1/p_\Omega^+}.$$

Here the last estimate is based on [10], Theorem 1.3, which also gives

$$\int_\Omega |u_i|^{p(x)}\,dx \le C\left(\int_\Omega |\nabla u_i|^{p(x)}\,dx\right)^{p_\Omega^-/p_\Omega^+}.$$

Thus the claim $C_{p(\cdot)}(E)=0$ follows.

(b) Lemma 3.6 gives an easy proof for the estimates (3.10) and (3.11) in the local sense. By choosing $E=B$ the inequality (3.12) implies that

$$|B| - Cr^{n+p(x_0)} \le Cr^{p(x_0)} C_{p(\cdot)}(B,2B).$$

For small values r, the left hand side is comparable to Cr^n, and therefore

$$r^{n-p(x_0)} \le CC_{p(\cdot)}(B,2B) \tag{3.14}$$

for all $0 < r \le r_0$. Here r_0 depends only on n, $p(x_0)$, and the constant in the Poincaré inequality.

Conversely, let $\eta \in C_0^\infty(2B)$ be such that $\eta = 1$ on $\frac{3}{2}B$ and $|\nabla \eta| \le \frac{C}{r}$. Then η is admissible for $C_{p(\cdot)}(B,2B)$ and we obtain by (2.1) that

$$C_{p(\cdot)}(B,2B) \le \int_{2B} |\nabla \eta|^{p(x)} \le Cr^{n-p(x_0)}. \tag{3.15}$$

The claims (3.10) and (3.11) follow for $0<r\le r_0$ by combining (3.14) and (3.15).

4. Energy estimates for supersolutions

To obtain the main results, we need certain sharp energy estimates for supersolutions. These are essentially included in the paper [5]. Therefore we do not give details here but instead refer to [5]. Throughout $\Omega \subset \mathbf{R}^n$, $n \ge 2$, is an open set.

We say that a function $u \in W^{1,p(\cdot)}_{\mathrm{loc}}(\Omega)$ is a (weak) $p(\cdot)$-*supersolution* in Ω, if

$$\int_\Omega p(x)|\nabla u|^{p(x)-2}\nabla u \cdot \nabla \varphi\,dx \ge 0 \tag{4.1}$$

for every non-negative test function $\varphi \in C_0^\infty(\Omega)$. A function u is a $p(\cdot)$-*subsolution* in Ω if $-u$ is a $p(\cdot)$-supersolution in Ω, and a $p(\cdot)$-*solution* in Ω if it is both a $p(\cdot)$-super- and a $p(\cdot)$-subsolution in Ω.

The dual of $L^{p(\cdot)}(\Omega)$ is the space $L^{p'(\cdot)}(\Omega)$ obtained by conjugating the exponent pointwise. This together with our definition of $W_0^{-1,p(\cdot)}(\Omega)$ as the completion of $C_0^\infty(\Omega)$ implies that we can also test with functions $\varphi \in W_0^{1,p(\cdot)}(\Omega)$.

Existence of solutions has been discussed in [11, 19, 25]. Under our conditions on p, every $p(\cdot)$-solution has a locally Hölder continuous representative, see [1, 3, 4, 9].

We say that a function $u\colon \Omega \to (-\infty, \infty]$ is $p(\cdot)$-*superharmonic* in Ω if

(1) u is lower semicontinuous,
(2) u is finite almost everywhere and
(3) The comparison principle holds: Let $D \Subset \Omega$ be an open set. If h is a $p(\cdot)$-solution in D, which is continuous in $\overline{D}$, and satisfies $u \geq h$ on ∂D, then $u \geq h$ in D.

4.2. Remark. It turns out that every $p(\cdot)$-supersolution in Ω, which satisfies

$$u(x) = \operatorname*{ess\,lim\,inf}_{y \to x} u(y)$$

for every $x \in \Omega$, is $p(\cdot)$-superharmonic in Ω. On the other hand every locally bounded $p(\cdot)$-superharmonic function is a $p(\cdot)$-supersolution. Moreover, $\min(u, \lambda)$ is $p(\cdot)$-superharmonic in Ω whenever u is $p(\cdot)$-superharmonic in Ω and $\lambda \in \mathbf{R}$. For the proofs of these claims, see [17], Section 6.

We recall a Caccioppoli inequality for $p(\cdot)$-supersolutions. This is obtained as in the proof of [5], Proposition 6.1. A version of Caccioppoli inequality for unbounded $p(\cdot)$-supersolutions can be found in [20], Theorem 3.15.

4.3. Lemma. *Assume that u is a bounded non-negative $p(\cdot)$-supersolution in $\Omega \subset \mathbf{R}^n$, $x_0 \in \Omega$, and $B = B(x_0, R)$ is a ball with radius so small that $4B \subset \Omega$. Let $\gamma < \gamma_0 < 0$ and $\eta \in C_0^\infty(4B)$ with $0 \leq \eta \leq 1$. Then*

$$\int_B (u+R)^{\gamma-1} |\nabla u|^{p(x)} \eta^{p_{4B}^+} \, dx \leq C \int_{4B} (u+R)^{\gamma+p(x)-1} |\nabla \eta|^{p(x)} \, dx. \tag{4.4}$$

Here the constant C depends only on p^+ and γ_0.

The Caccioppoli inequality implies the weak Harnack inequality, see [5], Lemma 6.4.

4.5. Lemma. *Assume that u is a bounded non-negative $p(\cdot)$-supersolution in $\Omega \subset \mathbf{R}^n$, $x_0 \in \Omega$, and $B = B(x_0, R)$ is a ball with radius so small that $4B \subset \Omega$. Then, for every $0 < q < n(p(x_0) - 1)/(n-1)$, we have*

$$\left(\fint_{2B} (u+R)^q \, dx \right)^{1/q} \leq C \left(\inf_B u + R \right).$$

Here the constant C depends only on n, p, q, and $M := \sup_{x \in \Omega} u(x)$.

By combining the Caccioppoli inequality and the weak Harnack estimate we obtain the following inequality, see [5], Lemma 7.1.

4.6. Lemma. *Assume that u is a bounded non-negative $p(\cdot)$-supersolution in $\Omega \subset \mathbf{R}^n$, $x_0 \in \Omega$, and $B = B(x_0, R)$ is a ball with radius so small that $4B \subset \Omega$. Then, for every $0 < q < p(x_0) - 1$, we have*

$$\fint_{2B} (u+R)^{q-p(x_0)} |\nabla u|^{p(x)}\, dx \le C R^{-p(x_0)} \Big(\inf_B u + R\Big)^q .$$

Here the constant C depends only on n, p, q, and $M := \sup_{x\in\Omega} u(x)$.

Now we are prepared to formulate our key lemma:

4.7. Lemma. *Assume that u is a bounded non-negative $p(\cdot)$-supersolution in $\Omega \subset \mathbf{R}^n$, $x_0 \in \Omega$, and $B = B(x_0, R)$ is a ball with radius so small that $4B \subset \Omega$. Then, for every $\eta \in C_0^\infty(2B)$ with $0 \le \eta \le 1$ and $|\nabla \eta| \le \frac{C}{R}$, we have*

$$\int_{2B} |\nabla u|^{p(x)} \eta\, dx \le C R^{n-p(x_0)} \Big(\inf_B u + R\Big)^{p(x_0)-1} . \tag{4.8}$$

Here the constant C depends only on n, p, and $M := \sup_{x\in\Omega} u(x)$.

Proof. By imitating the proof of [5], Lemma 7.2, we obtain

$$\fint_{2B} |\nabla u|^{p(x)-1}\, dx \le C R^{1-p(x_0)} \Big(\inf_B u + R\Big)^{p(x_0)-1} . \tag{4.9}$$

Let $\eta \in C_0^\infty(2B)$ with $\eta = 1$ in B and $|\nabla \eta| \le C(n)/R$. We test u by $(M-u)\eta$ and obtain by standard argumentation that

$$\int_{2B} |\nabla u|^{p(x)} \eta \le C(p) \int_{2B} |\nabla u|^{p(x)-1} |\nabla \eta|\, dx \le C(n,p,M) R^{-1} \int_{2B} |\nabla u|^{p(x)-1}\, dx.$$

The claim follows by combining this last estimate with (4.9). □

Notice here that the left hand side of (4.8) is usually written in terms of $\eta^{p(x)}$ instead of η. This slight modification is needed in the application of Lemma 4.7 in the proof of Theorem 5.3.

5. Fine continuity

The fine topology is defined by means of thinness just the same way as in the fixed exponent case.

5.1. Definition. The set $E \subset \mathbf{R}^n$ is called *$p(\cdot)$-thin at $x_0 \in \mathbf{R}^n$* if

$$\int_0^1 \left(\frac{C_{p(\cdot)}(E \cap B(x_0,r), B(x_0,2r))}{C_{p(\cdot)}(B(x_0,r), B(x_0,2r))} \right)^{1/(p(x_0)-1)} \frac{dr}{r} < \infty.$$

We say that $U \subset \mathbf{R}^n$ is *$p(\cdot)$-finely open* if $\mathbf{R}^n \setminus U$ is $p(\cdot)$-thin at x for all $x \in U$.

Hence the $p(\cdot)$-thinness at x_0 depends on the point x_0. However, it is clear that $p(\cdot)$-finely open sets give a rise to a topology which we call *$p(\cdot)$-fine topology.* It is also clear that $p(\cdot)$-fine topology is finer than the Euclidean topology.

We say that a function $u\colon U \to \mathbf{R}$ defined in a $p(\cdot)$-finely open set U is $p(\cdot)$-*finely continuous* at $x_0 \in U$ if $\{ x \in U : |u(x) - u(x_0)| \geq \varepsilon\}$ is $p(\cdot)$-thin at x_0 for each $\varepsilon > 0$.

5.2. Remark. The notion of $p(\cdot)$-fine continuity implies the continuity with respect to the $p(\cdot)$-fine topology on U. In fact, if $\{ x \in U : |u(x) - u(x_0)| \geq \varepsilon\}$ is $p(\cdot)$-thin at $x_0 \in U$ for $\varepsilon > 0$, then $\{ x \in U : |u(x) - u(x_0)| < \varepsilon\}$ is $p(\cdot)$-finely open by definition. Hence u is continuous at x_0 if U is equipped with the $p(\cdot)$-fine topology. The converse implication, which is based on deep results even for constant exponent, remains open here, see [32], Theorem 2.136.

5.3. Theorem. *Let u be $p(\cdot)$-superharmonic in Ω and let $x_0 \in \Omega$ such that $p(x_0) \leq n$. Then u is $p(\cdot)$-finely continuous at x_0.*

Proof. To prove the claim we modify the argumentation in [32], Theorem 2.121. Observe first that we are free to assume $u(x_0) < +\infty$ since u is lower semicontinuous. Since $C_{p(\cdot)}(B(x_0, r), B(x_0, 2r))$ is comparable to $r^{n-p(x_0)}$ (Lemma 3.7), it is enough to show that

$$\int_0^1 \left(\frac{C_{p(\cdot)}(\{ u \geq l \} \cap B(x_0, r), B(x_0, 2r))}{r^{n-p(x_0)}} \right)^{1/(p(x_0)-1)} \frac{dr}{r} < \infty$$

for all $l \in \mathbf{R}$ so that $u(x_0) < l$.

We denote $E_l = \{ u \geq l \}$ and fix $R > 0$ with $B(4R) := B(x_0, 4R) \subset \Omega$. Choose l so that $l > u(x_0)$ and denote $u_l = \min(u, l)$. Since u is lower semicontinuous, we have

$$u(x_0) = \lim_{r \to 0^+} m(r)$$

for $m(r) = \inf_{B(r)} u_l$ and $B(r) = B(x_0, r)$. Let $r \in (0, R)$ and denote

$$v := u_l - m(4r).$$

Note that v is a bounded non-negative $p(\cdot)$-supersolution on $B(4r)$. Let $\eta \in C_0^\infty(B(2r))$ such that $0 \leq \eta \leq 1$, $\eta = 1$ in $B(r)$, and $|\nabla \eta| \leq C/r$. By lower semicontinuity the function $2(l - u(x_0))^{-1} v\eta$ is an admissible test function for the capacity $C_{p(\cdot)}(E_l \cap B(r), B(2r))$. We are free to choose l so close to $u(x_0)$ and $r \leq 1$ so small that $0 \leq v \leq 1$ and $p^+_{B(4r)} - p^-_{B(4r)} \leq 1$. We conclude from Lemma 4.7 that

$$\begin{aligned}\int_{B(2r)} |\nabla v|^{p(x)} \eta^{p(x)}\, dx &\leq \int_{B(2r)} |\nabla v|^{p(x)} \eta\, dx \leq C r^{n-p(x_0)} (\inf_{B(r)} v + r)^{p(x_0)-1} \\ &\leq C r^{n-p(x_0)} \left((m(r) - m(4r))^{p(x_0)-1} + r^{p(x_0)-1} \right).\end{aligned}$$

On the other hand we obtain

$$\begin{aligned}\int_{B(2r)} v^{p(x)} |\nabla \eta|^{p(x)} &\leq C r^{-p(x_0)} \int_{B(2r)} (v + r)^{p(x)}\, dx \\ &= C r^{-p(x_0)} \int_{B(2r)} (v + r)^{p(x)-p(x_0)+1} (v + r)^{p(x_0)-1}\, dx\end{aligned}$$

$$\leq C r^{n-p(x_0)} \fint_{B(2r)} (v+r)^{p(x_0)-1}\, dx$$
$$\leq C r^{n-p(x_0)} \left((m(r)-m(4r))^{p(x_0)-1} + r^{p(x_0)-1} \right).$$

Here the log-Hölder continuity has been used in the first inequality, the facts that $0 \leq v \leq 1$, $0 < r \leq 1$, and $p^+_{B(4r)} - p^-_{B(4r)} \leq 1$ in the second inequality, and Lemma 4.5 in the third inequality.

By combining the two estimates we arrive at

$$\int_{B(2r)} |\nabla(v\eta)|^{p(x)}\, dx \leq C r^{n-p(x_0)} \left((m(r)-m(4r))^{p(x_0)-1} + r^{p(x_0)-1} \right).$$

Hence

$$\begin{aligned} \varphi(r) := \frac{C_{p(\cdot)}(E_l \cap B(r), B(2r))}{r^{n-p(x_0)}} &\leq C \frac{\int_{B(2r)} |\nabla(v\eta)|^{p(x)}\, dx}{(l-u(x_0)) r^{n-p(x_0)}} \\ &\leq \frac{C}{l-u(x_0)} \left((m(r)-m(4r))^{p(x_0)-1} + r^{p(x_0)-1} \right). \end{aligned}$$

By choosing $R > 0$ small enough and by assuming that $4\rho < R$ we obtain by a simple change of variable on the first line that

$$\begin{aligned} \int_\rho^R \frac{m(r)-m(4r)}{r}\, dr &= \left(\int_\rho^R \frac{m(t)}{t}\, dt - \int_{4\rho}^{4R} \frac{m(t)}{t}\, dt \right) \\ &= \left(\int_\rho^{4\rho} \frac{m(t)}{t}\, dt - \int_R^{4R} \frac{m(t)}{t}\, dt \right) \\ &\leq \left(\int_\rho^{4\rho} \frac{m(\rho)}{t}\, dt - \int_R^{4R} \frac{m(4R)}{t}\, dt \right) \\ &\leq (u(x_0) - m(4R)) \log 4. \end{aligned}$$

Since the upper bound is independent of ρ, we easily infer that

$$\int_0^R \varphi(r)^{1/(p(x_0)-1)} \frac{dr}{r} < \infty. \qquad \square$$

5.4. Remark. Theorem 5.3 clearly holds if $p(x_0) > n$. Indeed, if $p(x_0) > n$, then x_0 has a neighborhood $U \Subset \Omega$ satisfying $p^-_U > n$. Since u is $p(\cdot)$-superharmonic, $u_l = \min\{u, l\}$ is a $p(\cdot)$-supersolution and hence in $W^{1,p(\cdot)}(U)$. Since $W^{1,p(\cdot)}(U) \subset W^{1,p^-_U}(U)$, the function u_l has a continuous representative in U. By choosing $l > u(x_0)$ we infer that u is continuous at x_0.

The following comparison theorem allow us to compare the variable exponent thinness with the constant exponent thinness.

5.5. Theorem. *Let* $1 < p^- \le p^+ < \infty$ *and* $1 < q^- \le q^+ < \infty$ *be* log-*Hölder continuous exponents so that* $p \le q$ *and* $p(x_0) < q(x_0) < n$. *If* $E \subset \mathbf{R}^n$ *is* $q(\cdot)$-*thin at* x_0, *then* E *is* $p(\cdot)$-*thin at* x_0.

Proof. We write $B = B(x_0, r)$, $2B = B(x_0, 2r)$, $p_0 = p(x_0)$, and $q_0 = q(x_0)$. The claim follows if we can show that

$$\left(\frac{C_{p(\cdot)}(E \cap B, 2B)}{C_{p(\cdot)}(B, 2B)}\right)^{1/(p_0-1)} \le C\left(\frac{C_{q(\cdot)}(E \cap B, 2B)}{C_{q(\cdot)}(B, 2B)}\right)^{1/(q_0-1)} \tag{5.6}$$

for every $0 < r < R$, where $0 < R \le 1$ is chosen later.

The basic idea of the proof is the same as in [27], Lemma 3.16; we estimate $C_{p(\cdot)}(E \cap B, 2B)$ with the aid of variable exponent Hölder's inequality. Since

$$C_{q(\cdot)}(E \cap B, 2B) \le C_{q(\cdot)}(B, 2B) \approx r^{n-q_0} \tag{5.7}$$

by Lemma 3.7, we may choose R so small that $C_{q(\cdot)}(E \cap B, 2B) < 1$ (by the assumption $q_0 < n$). Let $u \in W_0^{1,q(\cdot)}(2B)$ be such that $u \ge 1$ in an open neighborhood of $E \cap B$ and $\int_{2B} |\nabla u|^{q(x)} dx \le 1$. By Hölder's inequality we obtain

$$\begin{aligned}\int_{2B} |\nabla u|^{p(x)} dx &\le 3\|1\|_{L^{(q(\cdot)/p(\cdot))'}(2B)} \| |\nabla u|^{p(\cdot)} \|_{L^{q(\cdot)/p(\cdot)}(2B)} \\ &\le C r^{\frac{n(q_0-p_0)}{q_0}} \left(\int_{2B} |\nabla u|^{q(x)} dx\right)^{\frac{1}{(q/p)^+_{2B}}}.\end{aligned}$$

Here in the second inequality we estimate the norm of 1 by [23], Lemma 2.4, see [10], Theorem 1.3 for the estimate concerning the modular. By taking infimum over all admissible test functions u for the capacity $C_{q(\cdot)}(E \cap B, 2B)$ we obtain

$$C_{p(\cdot)}(E \cap B, 2B) \le C r^{n(q_0-p_0)/q_0} \left(C_{q(\cdot)}(E \cap B, 2B)\right)^{1/(q/p)^+_{2B}}.$$

This yields

$$\begin{aligned}\left(\frac{C_{p(\cdot)}(E \cap B, 2B)}{C_{p(\cdot)}(B, 2B)}\right)^{1/(p_0-1)} &\le \left(\frac{C r^{n(q_0-p_0)/q_0} C_{q(\cdot)}(E \cap B, 2B)^{1/(q/p)^+_{2B}}}{r^{n-p_0}}\right)^{1/(p_0-1)} \\ &\le \left(C r^{-\frac{p_0}{q_0}(n-q_0)} C_{q(\cdot)}(E \cap B, 2B)^{1/(q/p)^+_{2B}}\right)^{1/(p_0-1)}.\end{aligned}$$

Let $z \in \overline{2B}$ be such that $(q/p)^+_{2B} = q(z)/p(z)$. Since p and q are log-Hölder continuous we obtain

$$\begin{aligned}r^{-p_0/q_0} &= r^{-p_0/q_0 + p(z)/q(z) - p(z)/q(z)} \\ &\le r^{(-p_0 q(z) + q_0 p_0 - q_0 p_0 + p(z) q_0)/q_0 q(z)} r^{-p(z)/q(z)} \\ &\le r^{p_0(q_0 - q(z))/q_0 q(z)} r^{q_0(p(z) - p(x_0))/q_0 q(z)} r^{-p(z)/q(z)} \\ &\le C^{p^+} C^{q^+} r^{-p(z)/q(z)}.\end{aligned}$$

This yields

$$\left(\frac{C_{p(\cdot)}(E\cap B,2B)}{C_{p(\cdot)}(B,2B)}\right)^{1/(p_0-1)} \le C\left(r^{-p(z)(n-q_0)/q(z)}C_{q(\cdot)}(E\cap B,2B)^{1/(q/p)^+_{2B}}\right)^{1/(p_0-1)}$$
$$\le C\left(\frac{C_{q(\cdot)}(E\cap B,2B)}{C_{q(\cdot)}(B,2B)}\right)^{\frac{p(z)}{q(z)(p_0-1)}}.$$

Since $C_{q(\cdot)}(E\cap B,2B)/C_{q(\cdot)}(B,2B)\le 1$ we are done if only

$$\frac{p(z)}{q(z)(p_0-1)}\ge\frac{1}{q_0-1}.$$

This condition holds whenever

$$\left(\frac{q}{p}\right)^+_{2B}=\frac{q(z)}{p(z)}\le\frac{q_0-1}{p_0-1}. \tag{5.8}$$

Since $(q/p)^+_{2B}\to q_0/p_0$ as $r\to 0$ and $q_0/p_0<(q_0-1)/(p_0-1)$ (because $q_0>p_0$), there exists R so that (5.8) holds for $0<r<R$. Thus (5.6) holds for $0<r<R$. □

5.9. Remark. The assumption $q(x_0)<n$ in Theorem 5.5 is made just for convenience. We could allow the assumption $q(x_0)=n$ and prove instead of using (5.7) that

$$C_{q(\cdot)}(E\cap B,2B)\to 0 \tag{5.10}$$

as $r\to 0^+$. The proof of this is based on the definition of thinness. Notice here that in the proof of Theorem 5.5 the assumption $q(x_0)<n$ is used only for the estimate $C_{q(\cdot)}(E\cap B,2B)<1$. Since we do not need the claim in the borderline case $q(x_0)=n$, we skip the somewhat technical proof of the fact (5.10). Notice also that the claim is trivial if we assume $q(x_0)>n$.

Recall that a measurable function u in Ω is called *approximately continuous* at $x_0\in\Omega$ if there is a measurable set E with measure density 1 at x_0 such that u is continuous at x_0 relative to E.

5.11. Corollary. *Every $p(\cdot)$-superharmonic function in Ω is approximately continuous in Ω.*

Proof. Let u be $p(\cdot)$-superharmonic and fix a point x_0. Since u is lower semicontinuous we have

$$u(x_0)=\liminf_{x\to x_0}u(x).$$

Therefore we may assume that $u(x_0)<\infty$. By Remark 5.4, we are free to assume that $p(x_0)\le n$.

It is enough to show that for any $\varepsilon>0$ the set $E_\varepsilon:=\{\,u\ge u(x_0)+\varepsilon\,\}$ has the measure density 0 at x_0, see [36], p. 170. This however holds by Theorems 5.3 and 5.5 together with known results for constant exponent, see [32], p. 86. □

6. Fine continuity and quasicontinuity

We finish this paper by showing that $p(\cdot)$-quasicontinuous functions are $p(\cdot)$-finely continuous $p(\cdot)$-quasieverywhere. As a consequence of this we also prove that $p(\cdot)$-superharmonic functions are $p(\cdot)$-quasicontinuous. To obtain these results we first show that for any $E \Subset \Omega$ there is a unique capacitary extremal for the capacity $C_{p(\cdot)}(E,\Omega)$. This has been shown in [5], Theorem 5.2 for compact sets by using a different method. We use the Banach–Saks theorem since the standard application of weak lower semicontinuity does not yield the result in the variable exponent setting.

Throughout this section, let $\Omega \subset \mathbf{R}^n$ be a bounded open set and let $E \Subset \Omega$ be arbitrary. For convenience, we denote

$$S(E,\Omega) = \{\, u \in W_0^{1,p(\cdot)}(\Omega) : u \geq 1 \text{ in a neighborhood of } E \,\}$$

and

$$\tilde{S}(E,\Omega) = \{\, u \in W_0^{1,p(\cdot)}(\Omega) : u \text{ is } p(\cdot)\text{-qc. with } u \geq 1 \ p(\cdot)\text{-q.e. in } E \,\}.$$

Here qc. is an abbreviation for the word quasicontinuous. We define

$$\tilde{C}_{p(\cdot)}(E,\Omega) = \inf_{u \in \tilde{S}(E,\Omega)} \int_\Omega |\nabla u(x)|^{p(x)}\,dx.$$

6.1. Lemma. *Let $E \Subset \Omega$ and let $u \in \tilde{S}(E,\Omega)$ be non-negative. Then for any $\varepsilon > 0$ there is $v \in S(E,\Omega)$ such that $\|u - v\|_{1,p(\cdot)} < \varepsilon$.*

Proof. We choose $0 < \varepsilon < 1$ and fix an open set U with $E \subset U \Subset \Omega$. Since u is $p(\cdot)$-quasicontinuous there is an open set $V \subset U$ such that $C_{p(\cdot)}(V,\Omega) < \varepsilon$, u restricted to $U \setminus V$ is continuous, and $u \geq 1$ on $E \setminus V$. Here a priori the Sobolev $p(\cdot)$-capacity of V can be assumed to be small: however, by Remark 3.13 we obtain that the relative Sobolev capacity is small as well. Since $C_{p(\cdot)}(V,\Omega) < \varepsilon$, we find $w \in W_0^{1,p(\cdot)}(\Omega)$ such that $w \geq 1$ in an open set containing V and $\|\nabla w\|_{p(\cdot)} < \varepsilon$. Now the Poincaré inequality, [19], Theorem 4.1, implies that $\|w\|_{p(\cdot)} < C\varepsilon$. By setting $v := (1+\varepsilon)u + w$ we have $v \geq 1$ on an open set containing E. Since

$$\|v - u\|_{1,p(\cdot)} \leq \varepsilon(\|u\|_{1,p(\cdot)} + C)$$

we easily infer the claim. □

6.2. Lemma. *For any $E \Subset \Omega$ we have*

$$C_{p(\cdot)}(E,\Omega) = \tilde{C}_{p(\cdot)}(E,\Omega).$$

Proof. The inequality

$$C_{p(\cdot)}(E,\Omega) \leq \tilde{C}_{p(\cdot)}(E,\Omega)$$

follows from Lemma 6.1. The converse inequality follows since any $p(\cdot)$-quasicontinuous representative $\tilde{u}$ of $u \in S(E,\Omega)$ satisfies $\tilde{u} \geq 1$ $p(\cdot)$-q.e. in an open neighborhood of E by Lemma 3.5. □

6.3. Theorem. *For every $E \Subset \Omega$ there exists a unique capacitary extremal $u \in \tilde{S}(E,\Omega)$, $0 \le u \le 1$, for $C_{p(\cdot)}(E,\Omega) = \tilde{C}_{p(\cdot)}(E,\Omega)$.*

Proof. Let (u_i) be a minimizing sequence in $\tilde{S}(E,\Omega)$ for the capacity $\tilde{C}_{p(\cdot)}(E,\Omega)$, that is $u_i \in \tilde{S}(E,\Omega)$ and

$$\lim_{i\to\infty} \int_\Omega |\nabla u_i|^{p(x)}\,dx = \tilde{C}_{p(\cdot)}(E,\Omega).$$

Without loss of generality we may assume that $(\varrho_{p(\cdot)}(\nabla u_i))$ is decreasing and $0 \le u_i \le 1$. Thus (u_i) is bounded in $W_0^{1,p(\cdot)}(\Omega)$ and has a subsequence that converges weakly to $u \in W_0^{1,p(\cdot)}(\Omega)$. Since $t \mapsto t^{p(x)}$ is uniformly convex, we obtain that $L^{p(\cdot)}(\Omega)$ is uniformly convex, see [33], Theorem 11.6. By the Banach–Saks property of [28] we have $i^{-1}(u_1 + \ldots + u_i) \to u$ in $L^{p(\cdot)}(\Omega)$ and $i^{-1}(\nabla u_1 + \ldots + \nabla u_i) \to \nabla u$ in $L^{p(\cdot)}(\Omega)$. We denote

$$v_i = i^{-2}(u_{i+1} + \ldots + u_{i^2}) \quad \text{and} \quad \nabla v_i = i^{-2}(\nabla u_{i+1} + \ldots + \nabla u_{i^2}).$$

Since (∇u_i) is bounded in $L^{p(\cdot)}(\Omega)$, we have $i^{-2}(u_1 + \ldots + u_i) \to 0$ in $L^{p(\cdot)}(\Omega)$, and therefore $v_i \to u$ in $L^{p(\cdot)}(\Omega)$. Similarly we have $\nabla v_i \to \nabla u$ in $L^{p(\cdot)}(\Omega)$. By passing to a subsequence we obtain by Fatou's lemma

$$\begin{aligned}\int_\Omega |\nabla u|^{p(x)}\,dx &\le \liminf_{i\to\infty} \int_\Omega |\nabla v_i|^{p(x)}\,dx \le \liminf_{i\to\infty} \int_\Omega i^{-2} \sum_{j=i+1}^{i^2} |\nabla u_j|^{p(x)}\,dx \\ &\le \liminf_{i\to\infty} \frac{i^2 - i}{i^2} \int_\Omega |\nabla u_{i+1}|^{p(x)}\,dx \le \liminf_{i\to\infty} \int_\Omega |\nabla u_i|^{p(x)}\,dx.\end{aligned}$$

Here the inequality

$$\Big(\sum_{j=1}^{i^2} t_j\Big)^{p(x)} \le (i^2)^{p(x)-1} \sum_{j=1}^{i^2} t_j^{p(x)}$$

with $t_j = 0$ for $1 \le j \le i$ and $t_j = i^{-2}|\nabla u_j|$ for $i+1 \le j \le i^2$ has been used in the second inequality. We have also used the assumption that $(\varrho_{p(\cdot)}(\nabla u_i))$ is decreasing.

Hence the existence of the minimizer is established once we show that $u \in \tilde{S}(E,\Omega)$. Since $i^{-1}(u_1 + \ldots u_i) \to u$ in $W_0^{1,p(\cdot)}(\Omega)$ and each u_i is $p(\cdot)$-quasicontinuous, we infer from [19], Lemma 2.3 that u is $p(\cdot)$-quasicontinuous. Moreover, there is a subsequence so that $i_j^{-1}(u_1 + \ldots u_{i_j}) \to u$ $p(\cdot)$-q.e. in Ω. Since $i_j^{-1}(u_1 + \ldots u_{i_j}) = 1$ $p(\cdot)$-q.e. in E for each j, we conclude that $u = 1$ $p(\cdot)$-q.e. in E. Clearly by truncation we have $0 \le u \le 1$.

The uniqueness of the minimizer follows by standard reasoning. Assume that v is another capacitary extremal for the capacity $\tilde{C}_{p(\cdot)}(E,\Omega)$. By the strict convexity of $t \mapsto t^{p(x)}$ we obtain

$$\left|\frac{1}{2}\nabla u + \frac{1}{2}\nabla v\right|^{p(x)} \le \left(\frac{1}{2}|\nabla u| + \frac{1}{2}|\nabla v|\right)^{p(x)} < \frac{1}{2}|\nabla u|^{p(x)} + \frac{1}{2}|\nabla v|^{p(x)}.$$

This contradicts the minimality assumption if $u \neq v$ because $\frac{1}{2}(u+v)$ is an admissible test function for the capacity $\tilde{C}_{p(\cdot)}(E, \Omega)$. □

We complete Theorem 6.3 by showing that the capacity extremal is a $p(\cdot)$-supersolution.

6.4. Lemma. *Let $E \Subset \Omega$. If u is the capacitary extremal of $C_{p(\cdot)}(E, \Omega)$, then u is a $p(\cdot)$-supersolution on Ω and a $p(\cdot)$-solution on $\Omega \setminus \overline{E}$.*

Proof. Let $\phi \in C_0^\infty(\Omega)$ be non-negative and $\varepsilon > 0$. Then $u+\varepsilon\phi$ is a test function for $\tilde{C}_{p(\cdot)}(E, \Omega)$ and hence

$$\int_\Omega \frac{|\nabla u + \varepsilon \nabla \phi|^{p(x)} - |\nabla u|^{p(x)}}{\varepsilon}\, dx \geq 0.$$

Letting $\varepsilon \to 0$ we obtain

$$\int_\Omega p(x)|\nabla u|^{p(x)-2}\nabla u \cdot \nabla \phi \, dx \geq 0$$

in the same way as in [19], Theorem 5.4.

To prove the second claim, let $\phi \in C_0^\infty(\Omega \setminus \overline{E})$ and $\varepsilon > 0$. Since u is the capacitary extremal and $u + \varepsilon\phi$ is a test function for $\tilde{C}_{p(\cdot)}(E, \Omega)$, we obtain that

$$J(\varepsilon) = \int_\Omega |\nabla(u + \varepsilon\phi)|^{p(x)} dx$$

attains its infimum for $\varepsilon = 0$. So we must have $J'(0) = 0$. □

6.5. Theorem. *Let $u\colon \Omega \to [-\infty, \infty]$ be $p(\cdot)$-quasicontinuous. Then u is $p(\cdot)$-finely continuous $p(\cdot)$-quasieverywhere in Ω.*

Proof. In the proof we denote by E^* the $p(\cdot)$-fine closure of $E \subset \mathbf{R}^n$. By subadditivity it is enough to prove the claim for $u_{|B}$, where B is an open ball such that $2B \Subset \Omega$.

Let (E_i) be a sequence of open subsets of B with the positive $p(\cdot)$-capacity such that $\lim_{i\to\infty} C_{p(\cdot)}(E_i) = 0$ and u is continuous when restricted to $B \setminus E_i$. It is enough to show that

$$C_{p(\cdot)}\Big(\bigcap_i E_i^*\Big) = 0, \tag{6.6}$$

since then for any $x \in B \setminus \bigcap_i E_i^*$ there is an index i such that u is continuous when restricted to a $p(\cdot)$-finely open neighborhood $B \setminus E_i^*$ of x. By Theorem 6.3 and Lemma 6.4 there is a bounded $p(\cdot)$-quasicontinuous $p(\cdot)$-supersolution u_i in $2B$ so that $u_i \in W_0^{1,p(\cdot)}(2B)$,

$$C_{p(\cdot)}(E_i, 2B) = \int_{2B} |\nabla u_i|^{p(x)}\, dx,$$

and $u_i = 1$ $p(\cdot)$-q.e. in E_i. We define the $p(\cdot)$-superharmonic representative $\tilde{u}_i$ of u_i by setting

$$\tilde{u}_i(x) = \operatorname{ess}\liminf_{y\to x} u_i(y)$$

for every $x \in 2B$, see Remark 4.2. By the proof of [20], Theorem 4.1 we have

$$\tilde{u}_i(x) = \lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} \tilde{u}_i(y)\, dy$$

for all $x \in 2B$. We know that the function $\tilde{u}_i$ is $p(\cdot)$-quasicontinuous; notice here that [21], Theorem 2, improves [15], Theorem 4.6 by showing that our standard assumptions are sufficient for the capacitary version of Lebesgue point theorem for variable exponent Sobolev functions. Hence we obtain $\tilde{u}_i = 1$ $p(\cdot)$-q.e. in E_i by using Lemma 3.5.

Since $\tilde{u}_i$ is $p(\cdot)$-finely continuous in $2B$ (Theorem 5.3), we have $\tilde{u}_i \geq 1$ in E_i^*. In fact, if $x \in E_i^*$, V_x is a $p(\cdot)$-fine neighborhood of x, and $F_i \subset E_i$ is a set of $p(\cdot)$-capacity zero so that $\tilde{u}_i \geq 1$ in $E_i \setminus F_i$, then $V_x \setminus (F_i \setminus \{x\})$ is a $p(\cdot)$-fine neighborhood of x, and therefore V_x intersects with $E_i \setminus F_i$. By the $p(\cdot)$-quasicontinuity of $\tilde{u}_i$ we conclude from Theorem 6.3 that

$$C_{p(\cdot)}(E_i^*, 2B) \leq C_{p(\cdot)}(E_i, 2B)$$

for all $i = 1, 2, \ldots$. Since $\lim_{i\to\infty} C_{p(\cdot)}(E_i, 2B) = 0$ (by Lemma 3.7), we infer that

$$\lim_{i\to\infty} C_{p(\cdot)}(E_i^*, 2B) = 0.$$

As in Remark 3.13 we obtain

$$\lim_{i\to\infty} C_{p(\cdot)}(E_i^*) = 0,$$

and hence $C_{p(\cdot)}(\bigcap_i E_i^*) = 0$. □

6.7. Theorem. *Let u be $p(\cdot)$-superharmonic in Ω. Then u is $p(\cdot)$-quasicontinuous in Ω.*

Proof. By [17, Corollary 6.7], for each $k \in \mathbf{N}$, the function $u_k = \min\{u, k\}$ is a $p(\cdot)$-supersolution and hence belongs to $W^{1,p(\cdot)}_{\mathrm{loc}}(\Omega)$. This implies that u_k has a $p(\cdot)$-quasicontinuous representative $\tilde{u}_k$ [18, Theorem 5.2]. By Corollary 5.11 the function u_k is approximately continuous in Ω and by Theorem 6.5 the function $\tilde{u}_k$ is approximately continuous $p(\cdot)$-q.e. in Ω. Since $u_k = \tilde{u}_k$ a.e. in Ω, we obtain that $u_k = \tilde{u}_k$ $p(\cdot)$-q.e. in Ω. This implies that u_k is $p(\cdot)$-quasicontinuous in Ω for each $k \in \mathbf{N}$.

Next we show that u is $p(\cdot)$-quasicontinuous as well. For each $\varepsilon > 0$ we choose $U_k \subset \Omega$ so that $C_{p(\cdot)}(U_k) \leq \varepsilon/2^k$ and the restriction of u_k to $\Omega \setminus U_k$ is continuous. Write $U = \bigcup_k U_k$. Then for all $k \in \mathbf{N}$ the restriction of u_k to $\Omega \setminus U$ is continuous and moreover $C_{p(\cdot)}(U_k) \leq \varepsilon$. Let $x \in \Omega \setminus U$ and assume first that $u(x) \in \mathbf{R}$. By choosing k large enough we have $u_k = u$ in a neighborhood of x in $\Omega \setminus U$ (since u_k is continuous at x in $\Omega \setminus U$). Hence u is continuous at x in $\Omega \setminus U$. On the other hand, if $u(x) = +\infty$ and $M > 0$, we may choose $k \in \mathbf{N}$ large enough so that

$u_k(x) = k \geq M + 1$. Since u_k is continuous at x in $\Omega \setminus U$, we have $u \geq u_k > M$ in a neighborhood of x in $\Omega \setminus U$. Hence u is continuous at x when restricted to $\Omega \setminus U$. $\square$

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