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VARIABLE BESOV AND TRIEBEL–LIZORKIN SPACES

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Dedicated to Professor Shanzhen Lu on the occasion of his 70th birthday.

Abstract. In this paper, variable Besov and Triebel–Lizorkin spaces are introduced. Then equivalent norms of these new spaces are given.

1. Introduction

Let p be a measurable function on $\mathbf{R}^n$ with range in $[1,\infty)$. $L^{p(\cdot)}(\mathbf{R}^n)$ denotes the set of measurable functions f on $\mathbf{R}^n$ such that for some $\lambda > 0$,

$$\int_{\mathbf{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx < \infty.$$

The set becomes a Banach function space when equipped with the norm

$$\|f\|_{L^{p(\cdot)}} = \inf\left\{\lambda > 0\colon \int_{\mathbf{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \le 1\right\}.$$

These spaces are referred to as variable Lebesgue spaces, since they generalize the standard Lebesgue spaces. It is remarked that one can define variable Lebesgue spaces on any measurable subset of $\mathbf{R}^n$, see [13]. However, in this paper we only work on the whole space $\mathbf{R}^n$.

Denote by $\mathscr{P}(\mathbf{R}^n)$ the set of measurable functions p on $\mathbf{R}^n$ with range in $[1,\infty)$ such that

$$1 < p_- = \operatorname*{ess\,inf}_{x\in\mathbf{R}^n} p(x), \quad \operatorname*{ess\,sup}_{x\in\mathbf{R}^n} p(x) = p_+ < \infty.$$

In the classical Lebesgue spaces we can work with L^p where $0 < p < 1$. In this paper, we need to consider analogous spaces with variable exponents. Define $\mathscr{P}^0(\mathbf{R}^n)$ to be the set of measurable functions p on $\mathbf{R}^n$ with range in $(0,\infty)$ such that

$$p_- = \operatorname*{ess\,inf}_{x\in\mathbf{R}^n} p(x) > 0, \quad \operatorname*{ess\,sup}_{x\in\mathbf{R}^n} p(x) = p_+ < \infty.$$

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Given $p(\cdot) \in \mathscr{P}^0(\mathbf{R}^n)$, one can define the space $L^{p(\cdot)}(\mathbf{R}^n)$ as above. This is equivalent to defining it to be the set of all functions f such that $|f|^{p_0} \in L^{q(\cdot)}(\mathbf{R}^n)$, where $0 < p_0 < p_-$, and $q(x) = \frac{p(x)}{p_0} \in \mathscr{P}(\mathbf{R}^n)$. One can define a quasi-norm on this space by

$$\|f\|_{L^{p(\cdot)}} = \||f|^{p_0}\|^{1/p_0}_{L^{q(\cdot)}}.$$

In recent decades, these spaces and the corresponding variable Sobolev spaces $W^{k,p(\cdot)}$ have attracted more attention and have been applied to partial differential equations and the calculus of variations, see [1]–[16], [18], [19], [26], [27].

It is well known that Besov and Triebel–Lizorkin spaces have played important roles in both classical analysis and modern analysis. In particular, these spaces contain many classical spaces as special cases, for example, the Hölder spaces, the Sobolev spaces, the Bessel-potential spaces, the Zygmund spaces, the local Hardy spaces and the space $\mathrm{bmo}(\mathbf{R}^n)$. All the above-mentioned classical spaces have been proved to be useful tools in the study of ordinary and partial differential equations; for details one can see Triebel's books [21], [22], [23] and [24] and other literature.

Inspired by the mentioned references, the purpose of this paper is to introduce variable Besov and Triebel–Lizorkin spaces. Before going on, we recall some notation.

Let $\mathscr{S}(\mathbf{R}^n)$ be the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbf{R}^n$. Let $\mathscr{S}'(\mathbf{R}^n)$ be the set of all tempered distributions on $\mathbf{R}^n$. If $\varphi \in \mathscr{S}(\mathbf{R}^n)$, then $\widehat{\varphi}$ denotes the Fourier transform of φ, and $\varphi^\vee$ denotes the inverse Fourier transform of φ. For $j \in \mathbf{N}$ we also set $\varphi_j(x) = 2^{nj}\varphi(2^j x)$, $x \in \mathbf{R}^n$. Let functions $A, \theta \in \mathscr{S}(\mathbf{R}^n)$ satisfy the following conditions:

$$|\widehat{A}(\xi)| > 0 \quad \text{on } \{|\xi| < 2\}, \quad \operatorname{supp} \widehat{A} \subset \{|\xi| < 4\},$$

$$|\widehat{\theta}(\xi)| > 0 \quad \text{on } \{1/2 < |\xi| < 2\}, \quad \operatorname{supp} \widehat{\theta} \subset \{1/4 < |\xi| < 4\}.$$

It is well known that Besov and Triebel–Lizorkin spaces (see, e.g., Triebel [21]) can be defined as follows.

Definition 1. (i) Let $-\infty < s < \infty$, $0 < q, p \le \infty$. Then the Besov space is

$$B^s_{p,q}(\mathbf{R}^n) = \Big\{ f \in \mathscr{S}'(\mathbf{R}^n) \colon \|f\|_{B^s_{p,q}} = \|A * f\|_{L_p} + \big\|\{2^{sj}\theta_j * f\}_1^\infty\big\|_{\ell_q(L_p)} < \infty \Big\}.$$

(ii) Let $-\infty < s < \infty$, $0 < q \le \infty$, $0 < p < \infty$. Then the Triebel–Lizorkin space is

$$F^s_{p,q}(\mathbf{R}^n) = \Big\{ f \in \mathscr{S}'(\mathbf{R}^n) \colon \|f\|_{F^s_{p,q}} = \|A * f\|_{L_p} + \big\|\{2^{sj}\theta_j * f\}_1^\infty\big\|_{L_p(\ell_q)} < \infty \Big\}.$$

Here $\ell_q(L_p)$ and $L_p(\ell_q)$ are the spaces of all sequences $\{g_j\}$ of measurable functions on $\mathbf{R}^n$ with finite quasi-norms

$$\|\{g_j\}\|_{\ell_q(L_p)} = \|\{\|g_j\|_{L_p}\}\|_{\ell_q} = \left(\sum_{j=1}^\infty \left(\int_{\mathbf{R}^n} |g_j(x)|^p \, dx \right)^{\frac{q}{p}} \right)^{\frac{1}{q}}$$

and

$$\|\{g_j\}\|_{L_p(\ell_q)} = \|\|\{g_j\}\|_{\ell_q}\|_{L_p} = \left(\int_{\mathbf{R}^n}\left(\sum_{j=1}^{\infty}|g_j(x)|^q\right)^{\frac{p}{q}}dx\right)^{\frac{1}{p}}.$$

Naturally, one can replace the Lebesgue norm by variable Lebesgue norms, then one can introduce the variable Besov space and the Triebel–Lizorkin space as follows.

Definition 2. Let $s \in \mathbf{R}$, $0 < q \le \infty$, $p(\cdot) \in \mathscr{P}^0(\mathbf{R}^n)$.

(i) The set

$$\left\{f \in \mathscr{S}'(\mathbf{R}^n)\colon \|A * f\|_{L^{p(\cdot)}} + \left\|\left\{2^{sj}\theta_j * f\right\}_1^{\infty}\right\|_{\ell_q(L^{p(\cdot)})} < \infty\right\}$$

is called the variable Besov space, denoted by $B^s_{p(\cdot),q}(\mathbf{R}^n)$. The norm of f in this space is

$$\|f\|_{B^s_{p(\cdot),q}} = \|A * f\|_{L^{p(\cdot)}} + \left\|\left\{2^{sj}\theta_j * f\right\}_1^{\infty}\right\|_{\ell_q(L^{p(\cdot)})};$$

(ii) The set

$$\left\{f \in \mathscr{S}'(\mathbf{R}^n)\colon \|A * f\|_{L^{p(\cdot)}} + \left\|\left\{2^{sj}\theta_j * f\right\}_1^{\infty}\right\|_{L^{p(\cdot)}(\ell_q)} < \infty\right\}$$

is called the variable Triebel–Lizorkin space, denoted by $F^s_{p(\cdot),q}(\mathbf{R}^n)$. The norm of f in this space is

$$\|f\|_{F^s_{p(\cdot),q}} = \|A * f\|_{L^{p(\cdot)}} + \left\|\left\{2^{sj}\theta_j * f\right\}_1^{\infty}\right\|_{L^{p(\cdot)}(\ell_q)},$$

where $L^{p(\cdot)}(\ell_q)$, $\ell_q(L^{p(\cdot)})$ are similar to $\ell_q(L_p)$ and $L_p(\ell)$.

To make these spaces definite, the primary point is to show them independent of the choice of functions A and θ. To this aim we need more notation.

Let $\Psi, \psi \in \mathscr{S}(\mathbf{R}^n)$, $\varepsilon > 0$, integer $S \ge -1$ be such that

$$\begin{aligned} &|\widehat{\Psi}(\xi)| > 0 \quad \text{on } \{|\xi| < 2\varepsilon\}, \\ &|\widehat{\psi}(\xi)| > 0 \quad \text{on } \{\varepsilon/2 < |\xi| < 2\varepsilon\}, \end{aligned} \tag{1}$$

and

$$D^\tau \widehat{\psi}(0) = 0 \quad \text{for all } |\tau| \le S. \tag{2}$$

Here (1) are Tauberian conditions, while (2) expresses moment conditions on ψ. For any $a > 0$, $f \in \mathscr{S}'(\mathbf{R}^n)$, and $x \in \mathbf{R}^n$, define maximal functions

$$\begin{aligned} \Psi^*_a f(x) &= \sup_{y\in\mathbf{R}^n}\frac{|\Psi * f(y)|}{(1+|x-y|)^a}, \\ \psi^*_{j,a} f(x) &= \sup_{y\in\mathbf{R}^n}\frac{|\psi_j * f(y)|}{(1+2^j|x-y|)^a}. \end{aligned} \tag{3}$$

It is well known that the boundedness of Hardy–Littlewood maximal operator on Lebesgue spaces plays a key role in analysis. So does it on variable exponent Lebesgue spaces. There are some sufficient conditions on $p(\cdot)$ for maximal operator $\mathscr{M}$ to be bounded on $L^{p(\cdot)}(\mathbf{R}^n)$. Since we do not need to use them in this paper, we omit the details here, one can see [3], [4], [5], [7], [14], [15]. Let $\mathscr{B}(\mathbf{R}^n)$ be the set of $p(\cdot) \in \mathscr{P}(\mathbf{R}^n)$ such that the Hardy–Littlewood maximal operator $\mathscr{M}$ is bounded on $L^{p(\cdot)}(\mathbf{R}^n)$.

Now we state our result.

Theorem 1. *Let $s < S+1$, $0 < q \le \infty$ and $p(\cdot) \in \mathscr{P}^0(\mathbf{R}^n)$ with $p_0 < p_-$ such that $p(\cdot)/p_0 \in \mathscr{B}(\mathbf{R}^n)$.*

(i) *If $n/a < p_0$, then for all $f \in \mathscr{S}'(\mathbf{R}^n)$*

$$\begin{aligned} \|\Psi_a^* f\|_{L^{p(\cdot)}} + \|\{2^{sj}\psi_{j,a}^* f\}_1^\infty\|_{\ell_q(L^{p(\cdot)})} &\lesssim \|f\|_{B^s_{p(\cdot),q}} \\ &\lesssim \|\Psi * f\|_{L^{p(\cdot)}} + \|\{2^{js}\psi_j * f\}_1^\infty\|_{\ell_q(L^{p(\cdot)})}. \end{aligned} \tag{4}$$

(ii) *If $n/a < \min(q, p_0)$, then for all $f \in \mathscr{S}'$*

$$\begin{aligned} \|\Psi_a^* f\|_{L^{p(\cdot)}} + \|\{2^{sj}\psi_{j,a}^* f\}_1^\infty\|_{L^{p(\cdot)}(\ell_q)} &\lesssim \|f\|_{F^s_{p(\cdot),q}} \\ &\lesssim \|\Psi * f\|_{L^{p(\cdot)}} + \|\{2^{js}\psi_j * f\}_1^\infty\|_{L^{p(\cdot)}(\ell_q)}. \end{aligned} \tag{5}$$

Remark 1. By writing $A_1 \lesssim A_2$ we mean that there exists a positive constant C such that $A_1 \le CA_2$. In (4) and (5) these constants are independent of $f \in \mathscr{S}'(\mathbf{R}^n)$. Letter C will denote various positive constants. Constants may in general depend on all fixed parameters, and sometimes we show this dependence explicitly by writing, e.g., C_N.

To prove Theorem 1, some lemmas are needed, which will be given in Section 2. Then the complete proof of Theorem 1 will be given in Section 3.

2. Preliminaries

Lemma 1. ([17]) *Let $\mu, \nu \in \mathscr{S}(\mathbf{R}^n)$, $M \ge -1$ be integer,*

$$D^\tau \widehat{\mu}(0) = 0 \quad \text{for all } |\tau| \le M.$$

Then for any $N > 0$ there is a constant C_N so that

$$\sup_{z \in \mathbf{R}^n} |\mu_t * \nu(z)|(1+|z|)^N \le C_N t^{M+1},$$

where $\mu_t(x) = t^{-n}\mu(\frac{x}{t})$ for all $t > 0$.

The following Lemma 2 is easy to obtain. For its proof one can also see [17].

Lemma 2. *Let $0 < q \le \infty$, $\delta > 0$. For any sequence $\{g_j\}_0^\infty$ of nonnegative measurable functions on $\mathbf{R}^n$ denote*

$$G_j = \sum_{k=0}^\infty 2^{-|k-j|\delta} g_k.$$

Then

$$\|\{G_j\}_0^\infty\|_{\ell_q} \le C\|\{g_j\}_0^\infty\|_{\ell_q} \tag{6}$$

holds, where C is a constant depending only on q, δ.

Lemma 3. *Let $0 < q \le \infty$, $\delta > 0$ and $p(\cdot) \in \mathscr{P}^0(\mathbf{R}^n)$. For any sequence $\{g_j\}_0^\infty$ of nonnegative measurable functions on $\mathbf{R}^n$ denote*

$$G_j(x) = \sum_{k=0}^{\infty} 2^{-|k-j|\delta} g_k(x), \quad x \in \mathbf{R}^n.$$

Then

$$\|\{G_j\}_0^\infty\|_{L^{p(\cdot)}(\ell_q)} \le C_1 \|\{g_j\}_0^\infty\|_{L^{p(\cdot)}(\ell_q)}, \tag{7}$$

and

$$\|\{G_j\}_0^\infty\|_{\ell_q(L^{p(\cdot)})} \le C_2 \|\{g_j\}_0^\infty\|_{\ell_q(L^{p(\cdot)})} \tag{8}$$

hold with some constants $C_1 = C_1(q, \delta)$ and $C_2 = C_2(p(\cdot), q, \delta)$.

Proof. By Lemma 2, (7) follows immediately from (6). Now we prove (8).

Firstly, let $p(\cdot) \in \mathscr{P}(\mathbf{R}^n)$. Since $\|\cdot\|_{L^{p(\cdot)}}$ is a norm, by Minkowski's inequality we have

$$\|G_j\|_{L^{p(\cdot)}} \le \sum_{k=0}^{\infty} 2^{-|k-j|\delta} \|g_k\|_{L^{p(\cdot)}}.$$

Hence (8) follows from Lemma 2.

Then, for general $p(\cdot) \in \mathscr{P}^0(\mathbf{R}^n)$, choose $0 < p_0 < p_-$ such that $\bar{p}(\cdot) = p(\cdot)/p_0 \in \mathscr{P}(\mathbf{R}^n)$. We have

$$\|\{G_j\}\|_{\ell_q(L^{p(\cdot)})}^{p_0} = \|\{G_j^{p_0}\}\|_{\ell_{q/p_0}(L^{\bar{p}(\cdot)})} \le C \|\{g_j^{p_0}\}\|_{\ell_{q/p_0}(L^{\bar{p}(\cdot)})} = C \|\{g_j\}\|_{\ell_q(L^{p(\cdot)})}^{p_0}.$$

Raising to the power $1/p_0$, we obtain (8). In the last inequality, we used (8) that have been proved for space $L^{\bar{p}(\cdot)}(\mathbf{R}^n)$. This ends the proof. □

The following lemma is the estimate for the vector-valued setting in variable Lebesgue spaces, one can see Corollary 2.1 in [4].

Lemma 4. *If $p(\cdot) \in \mathscr{B}(\mathbf{R}^n)$, then for all $1 < q \le \infty$,*

$$\|\{\mathscr{M} f_j\}\|_{L^{p(\cdot)}(\ell_q)} \le C \|\{f_j\}\|_{L^{p(\cdot)}(\ell_q)},$$

where $\mathscr{M}$ is the Hardy–Littlewood maximal operator.

Lemma 5. ([17]) *Let $0 < r \le 1$, and let $\{b_j\}_0^\infty$, $\{d_j\}_0^\infty$ be two sequences taking values in $(0, +\infty]$ and $(0, +\infty)$, respectively. Assume that for some $N_0 > 0$*

$$d_j = O(2^{jN_0}), \quad j \to \infty,$$

and that for any $N > 0$, there exists a constant C_N such that

$$d_j \le C_N \sum_{k=j}^{\infty} 2^{(j-k)N} b_k d_k^{1-r}, \quad j \in \mathbf{N}_0 = \mathbf{N} \cup \{0\}.$$

Then for any $N > 0$,

$$d_j^r \le C_N \sum_{k=j}^{\infty} 2^{(j-k)Nr} b_k, \quad j \in \mathbf{N}_0,$$

holds with the same constant C_N.

3. Proof of Theorem 1

The idea of the proof is from Rychkov in [17]. Since Theorem 1 is novel, we give the details here. In fact, we combine the method in [17] and Lemma 3 with Lemma 4 in the last section. The whole proof is divided to three steps.

Step 1. Take any pair of functions $\Phi, \varphi \in \mathscr{S}(\mathbf{R}^n)$ so that for an $\varepsilon' > 0$

$$
\begin{aligned}
&|\widehat{\Phi}(\xi)| > 0 \quad \text{on } \{|\xi| < 2\varepsilon'\},\\
&|\widehat{\varphi}(\xi)| > 0 \quad \text{on } \{\varepsilon'/2 < |\xi| < 2\varepsilon'\},
\end{aligned} \tag{9}
$$

and denote $\Phi_a^* f$, $\varphi_{j,a}^* f$ as in (3).

For any $a > 0$, $s < S + 1$, $0 < q \le \infty$ and $p(\cdot) \in \mathscr{P}^0(\mathbf{R}^n)$, we will prove that for all $f \in \mathscr{S}'(\mathbf{R}^n)$ the following estimates are true:

$$
\|\Psi_a^* f\|_{L^{p(\cdot)}} + \|\{2^{sj}\psi_{j,a}^* f\}_1^\infty\|_{\ell_q(L^{p(\cdot)})} \lesssim \|\Phi_a^* f\|_{L^{p(\cdot)}} + \|\{2^{js}\varphi_{j,a}^* f\}_1^\infty\|_{\ell_q(L^{p(\cdot)})}, \tag{10}
$$

and

$$
\|\Psi_a^* f\|_{L^{p(\cdot)}} + \|\{2^{sj}\psi_{j,a}^* f\}_1^\infty\|_{L^{p(\cdot)}(\ell_q)} \lesssim \|\Phi_a^* f\|_{L^{p(\cdot)}} + \|\{2^{js}\varphi_{j,a}^* f\}_1^\infty\|_{L^{p(\cdot)}(\ell_q)}. \tag{11}
$$

Let us start. It follows from (9) that there exist two functions $\Lambda, \lambda \in \mathscr{S}(\mathbf{R}^n)$ so that

$$
\begin{aligned}
&\operatorname{supp} \widehat{\Lambda} \subset \{|\xi| < 2\varepsilon'\},\\
&\operatorname{supp} \widehat{\lambda} \subset \{\varepsilon'/2 < |\xi| < 2\varepsilon'\},
\end{aligned}
$$

and

$$
\widehat{\Lambda}(\xi)\widehat{\Phi}(\xi) + \sum_{j=1}^{\infty} \widehat{\lambda}(2^{-j}\xi)\widehat{\varphi}(2^{-j}\xi) \equiv 1, \quad \xi \in \mathbf{R}^n.
$$

Then for all $f \in \mathscr{S}'(\mathbf{R}^n)$ the identity

$$
f = \Lambda * \Phi * f + \sum_{k=1}^{\infty} \lambda_k * \psi_k * f
$$

is true. Thus we can write

$$
\psi_j * f = \psi_j * \Lambda * \Phi * f + \sum_{k=1}^{\infty} \psi_j * \lambda_k * \psi_k * f.
$$

We have

$$
\begin{aligned}
|\psi_j * \lambda_k * \varphi_k * f(y)| &\le \int_{\mathbf{R}^n} |\psi_j * \lambda_k(z)||\varphi_k * f(y-z)|\, dz\\
&\le \varphi_{k,a}^* f(y) \int_{\mathbf{R}^n} |\psi_j * \lambda_k(z)|(1 + 2^k|z|)^a\, dz\\
&\equiv \varphi_{k,a}^* f(y) I_{j,k},
\end{aligned}
$$

where

$$I_{j,k} \le C(\lambda, \psi) \begin{cases} 2^{(k-j)(S+1)} & \text{if } k \le j, \\ 2^{(j-k)(S+a+1)} & \text{if } k \ge j, \end{cases}$$

which can be obtained from Lemma 1. In fact, if $j \ge k$, then $\psi_j * \lambda_k(z) = 2^{nk}\psi_{j-k} * \lambda(2^k z)$,

$$\begin{aligned} I_{j,k} &= \int_{\mathbf{R}^n} 2^{nk} |\psi_{j-k} * \lambda(2^k z)|(1 + 2^k|z|)^a \, dz \\ &= \int_{\mathbf{R}^n} |\psi_{j-k} * \lambda(z)|(1 + |z|)^a \, dz \\ &= \int_{\mathbf{R}^n} \frac{|\psi_{j-k} * \lambda(z)|(1 + |z|)^{a+n+1}}{(1+|z|)^{n+1}} \, dz \\ &\le C2^{-(j-k)(S+1)} \int_{\mathbf{R}^n} \frac{1}{(1+|z|)^{n+1}} \, dz \\ &= C2^{(k-j)(S+1)}, \end{aligned}$$

since by Lemma 1,

$$|\psi_{j-k} * \lambda(z)|(1 + |z|)^{a+n+1} \le C2^{-(j-k)(S+1)}.$$

If $k \ge j$, then $\psi_j * \lambda_k(z) = 2^{nj}\psi * \lambda_{k-j}(2^j z)$, and

$$\begin{aligned} I_{j,k} &= \int_{\mathbf{R}^n} 2^{nj} |\psi * \lambda_{k-j}(2^j z)|(1 + 2^k|z|)^a \, dz \\ &= \int_{\mathbf{R}^n} |\psi * \lambda_{k-j}(z)|(1 + 2^{k-j}|z|)^a \, dz \\ &\le 2^{(k-j)a} \int_{\mathbf{R}^n} \frac{|\psi * \lambda_{k-j}(z)|(1 + |z|)^{a+n+1}}{(1+|z|)^{n+1}} \, dz \\ &\le C2^{(j-k)(S+a+1)} \int_{\mathbf{R}^n} \frac{1}{(1+|z|)^{n+1}} \, dz \\ &= C2^{(j-k)(S+a+1)}. \end{aligned}$$

Since λ has arbitrary order vanishing moments, by Lemma 1,

$$|\psi * \lambda_{k-j}(z)|(1 + |z|)^{a+n+1} \le C2^{-(j-k)(S+2a+1)}.$$

Noting that for all $x, y \in \mathbf{R}^n$,

$$\varphi^*_{k,a} f(y) \le \varphi^*_{k,a} f(x)(1 + 2^k|x-y|)^a \le \varphi^*_{k,a} f(x) \max(1, 2^{(k-j)a})(1 + 2^j|x-y|)^a,$$

we have

$$\sup_{y \in \mathbf{R}^n} \frac{|\psi_j * \lambda_k * \varphi_k * f(y)|}{(1 + 2^j|x-y|)^a} \lesssim \varphi^*_{k,a} f(x) \times \begin{cases} 2^{(k-j)(S+1)} & \text{if } k \le j, \\ 2^{(j-k)(S+1)} & \text{if } k \ge j. \end{cases}$$

Note that for $k = 1$, we do not use the condition $D^\tau \widehat{\lambda}(0) = 0$ in the above proof of the last estimate, so by replacing λ_1 and φ_1 with Λ and Φ, respectively, we have an analogous estimate

$$\sup_{y\in\mathbf{R}^n} \frac{|\psi_j * \Lambda * \Phi * f(y)|}{(1+2^j|x-y|)^a} \lesssim \Phi_a^* f(x) 2^{-j(S+1)}.$$

Thus we obtain

$$\psi_{j,a}^* f(x) \lesssim \Phi_a^* f(x) 2^{-j(S+1)} + \sum_{k=1}^\infty \varphi_{k,a}^* f(x) \times \begin{cases} 2^{(k-j)(S+1)} & \text{if } k \le j, \\ 2^{(j-k)(S+1)} & \text{if } k \ge j. \end{cases}$$

Hence with $\delta = \min(1, S+1-s) > 0$ for all $f \in \mathscr{S}'$, $x \in \mathbf{R}^n$, $j \in \mathbf{N}$,

$$2^{js}\psi_{j,a}^* f(x) \lesssim \Phi_a^* f(x) 2^{-j\delta} + \sum_{k=1}^\infty 2^{ks}\varphi_{k,a}^* f(x) 2^{-|k-j|\delta}. \tag{12}$$

Again, for $j = 1$ we did not use (2) to get this estimate, so we can replace ψ_1 with Ψ to have

$$2^{js}\Psi_a^* f(x) \lesssim \Phi_a^* f(x) 2^{-j\delta} + \sum_{k=1}^\infty 2^{ks}\varphi_{k,a}^* f(x) 2^{-j\delta}. \tag{13}$$

The desired estimates (10) and (11) follow from (12), (13) and Lemma 3.

Step 2. In this step we will show the following estimates. In the conditions of (4), for all $f \in \mathscr{S}'(\mathbf{R})$

$$\|\Psi_a^* f\|_{L^{p(\cdot)}} + \|\{2^{sj}\psi_{j,a}^* f\}_1^\infty\|_{\ell_q(L^{p(\cdot)})} \lesssim \|\Psi * f\|_{L^{p(\cdot)}} + \|\{2^{js}\psi_j * f\}_1^\infty\|_{\ell_q(L^{p(\cdot)})}. \tag{14}$$

In the conditions of (5), for all $f \in \mathscr{S}'(\mathbf{R}^n)$

$$\|\Psi_a^* f\|_{L^{p(\cdot)}} + \|\{2^{sj}\psi_{j,a}^* f\}_1^\infty\|_{L^{p(\cdot)}(\ell_q)} \lesssim \|\Psi * f\|_{L^{p(\cdot)}} + \|\{2^{js}\psi_j * f\}_1^\infty\|_{L^{p(\cdot)}(\ell_q)}. \tag{15}$$

For all $f \in \mathscr{S}'(\mathbf{R}^n)$, from the identity

$$f = \Lambda * \Phi * f + \sum_{k=1}^\infty \lambda_k * \psi_k * f,$$

by replacing f with $f(2^{-j}\cdot)$, $j \in \mathbf{N}$, and dilating we get

$$f = \Lambda_j * \Phi_j * f + \sum_{k=j+1}^\infty \lambda_k * \psi_k * f.$$

We convolve both sides with ψ_j and use the commutativity of convolution to derive

$$\psi_j * f = (\Lambda_j * \Phi_j) * (\psi_j * f) + \sum_{k=j+1}^\infty (\psi_j * \lambda_k) * (\psi_k * f). \tag{16}$$

By Lemma 1, the estimate

$$|\psi_j * \lambda_k(z)| \le C_N \frac{2^{jn}2^{(j-k)N}}{(1+2^j|z|)^a}, \quad z \in \mathbf{R}^n, \tag{17}$$

holds for $k \geq j$ with arbitrarily large $N > 0$, and C_N is a constant depending on N. The estimate

$$|\Phi_j * \Lambda_j(z)| \leq C \frac{2^{jn}}{(1+2^j|z|)^a}, \quad z \in \mathbf{R}^n, \tag{18}$$

is obvious. By putting the last two estimates (17) and (18) into (16), we get for all $f \in \mathscr{S}'(\mathbf{R}^n)$, $y \in \mathbf{R}^n$, and $j \in \mathbf{N}$,

$$|\psi_j * f(y)| \leq C_N \sum_{k=j}^{\infty} 2^{jn} 2^{(j-k)N} \int_{\mathbf{R}^n} |\psi_k * f(z)|\, dz. \tag{19}$$

For any $r \in (0,1]$, divide both sides of (19) by $(1+2^j|x-y|)^a$, then in the left hand side taking the supremum over $y \in \mathbf{R}^n$, in the right hand side making use of the following inequalities

$$\begin{aligned} |\psi_k * f(z)| &\leq |\psi_k * f(z)|^r [\psi_{k,a}^* f(x)]^{1-r} (1+2^k|x-z|)^{a(1-r)}, \\ \frac{(1+2^k|x-z|)^{a(1-r)}}{(1+2^j|x-z|)^a} &\leq \frac{2^{(k-j)a}}{(1+2^k|x-z|)^{ar}}, \end{aligned} \tag{20}$$

we obtain that for all $f \in \mathscr{S}'(\mathbf{R}^n)$, $x \in \mathbf{R}^n$ and $j \in \mathbf{N}$, the estimate

$$\psi_{j,a}^* f(x) \leq C_N \sum_{k=j}^{\infty} 2^{(j-k)N'} \int_{\mathbf{R}^n} \frac{2^{kn}|\psi_k * f(z)|^r}{(1+2^k|x-z|)^{ar}}\, dz [\psi_{k,a}^* f(x)]^{1-r}, \tag{21}$$

where $N' = N - a + n$ can be taken arbitrarily large.

Similarly, we can prove that for all $f \in \mathscr{S}'(\mathbf{R})$ the estimate

$$\begin{aligned} \Psi_a^* f(x) \leq C_N \Bigg(&\int_{\mathbf{R}^n} \frac{|\Psi * f(z)|^r}{(1+|x-z|)^{ar}}\, dz [\Psi_a^* f(x)]^{1-r} \\ &+ \sum_{k=1}^{\infty} 2^{-kN'} \int_{\mathbf{R}^n} \frac{2^{kn}|\psi_k * f(z)|^r}{(1+2^k|x-z|)^{ar}}\, dz [\psi_{k,a}^* f(x)]^{1-r} \Bigg). \end{aligned} \tag{22}$$

We fix now any $x \in \mathbf{R}^n$ and apply Lemma 5 with

$$\begin{aligned} &d_j = \psi_{j,a}^* f(x), \quad j \in \mathbf{N}, \quad d_0 = \Psi_a^* f(x), \\ &b_j = \int_{\mathbf{R}^n} \frac{2^{kn}|\psi_k * f(z)|^r}{(1+2^k|x-z|)^{ar}}\, dz, \quad j \in \mathbf{N}, \quad b_0 = \int_{\mathbf{R}^n} \frac{|\Psi * f(z)|^r}{(1+|x-z|)^{ar}}\, dz. \end{aligned}$$

We have the estimate

$$[\psi_{j,a}^* f(x)]^r \leq C_N' \sum_{k=j}^{\infty} 2^{(j-k)Nr} \int_{\mathbf{R}^n} \frac{2^{kn}|\psi_k * f(z)|^r}{(1+2^k|x-z|)^{ar}} dz, \tag{23}$$

where $C_N' = C_{N+a-n}$. Moreover, (23) is true also for $r > 1$. Indeed, it suffices to take (19) with $a+n$ instead of a, apply Hölder's inequality in k and z, and finally the inequality (20).

Now we choose r such that $n/a < r$, thus the function $\frac{1}{(1+|z|)^{ar}} \in L_1$, and by the majorant property of the Hardy–Littlewood maximal operator $\mathscr{M}$ (see [20], (3.9) in Chapter 2), it follows from (23) that

$$[\psi_{j,a}^* f(x)]^r \leq C_N' \sum_{k=j}^{\infty} 2^{(j-k)Nr} \mathscr{M}(|\psi_k * f|^r)(x), \tag{24}$$

together with the corresponding estimate for $\Psi_a^* f(x)$.

We now choose and fix $N > \max(-s, 0)$ and apply Lemma 3 with

$$g_j = 2^{jsr} \mathscr{M}(|\psi_k * f|^r), \quad j \in \mathbf{N}, \quad g_0 = \mathscr{M}(|\Psi * f|^r)$$

in the spaces $L^{p(\cdot)}(\ell_q)$ and $\ell_q(L^{p(\cdot)})$. It follows from (24) that for all $f \in \mathscr{S}'(\mathbf{R}^n)$

$$\begin{aligned} &\|\Psi_a^* f\|_{L^{p(\cdot)}} + \|\{2^{sj}\psi_{j,a}^* f\}_1^\infty\|_{\ell_q(L^{p(\cdot)})} \\ &\lesssim \|\mathscr{M}_r(\Psi * f)\|_{L^{p(\cdot)}} + \|\{2^{js}\mathscr{M}_r(\psi_j * f)\}_1^\infty\|_{\ell_q(L^{p(\cdot)})} \end{aligned} \tag{25}$$

and

$$\begin{aligned} &\|\Psi_a^* f\|_{L^{p(\cdot)}} + \|\{2^{sj}\psi_{j,a}^* f\}_1^\infty\|_{L^{p(\cdot)}(\ell_q)} \\ &\lesssim \|\mathscr{M}_r(\Psi * f)\|_{L^{p(\cdot)}} + \|\{2^{js}\mathscr{M}_r(\psi_j * f)\}_1^\infty\|_{L^{p(\cdot)}(\ell_q)}, \end{aligned} \tag{26}$$

where we use the notation $\mathscr{M}_r(g) = (\mathscr{M}(|g|^r))^{1/r}$.

For (25), by the definition of the variable Lebesgue space, we have (14), because by Theorem 1.2 of [4] we can choose r so that $n/a < r < p_0$ and $p(\cdot)/r \in \mathscr{B}(\mathbf{R}^n)$.

For (26), we choose r so that $n/a < r < \min(q, p_0.)$ By Lemma 4 and again $p(\cdot)/r \in \mathscr{B}(\mathbf{R}^n)$, we have (15), because

$$\begin{aligned} \|\{2^{js}\mathscr{M}_r(\psi_j * f)\}_1^\infty\|_{L^{p(\cdot)}(\ell_q)} &= \|\{2^{js}\mathscr{M}|\psi_j * f|^r\}_1^\infty\|_{L^{p(\cdot)/r}(\ell_{q/r})}^r \\ &\leq C\|\{2^{js}|\psi_j * f|^r\}_1^\infty\|_{L^{p(\cdot)/r}(\ell_q)}^r \\ &= C\|\{2^{js}|\psi_j * f|\}_1^\infty\|_{L^{p(\cdot)}(\ell_q)}. \end{aligned}$$

Step 3. We will check that (4) and (5) follow from (10), (11), (14) and (15). For instance, let us do it for (4). The left inequality in (4) is proved by the chain of estimates

$$\text{the left side of (4)} \lesssim \|A_a^* f\|_{L^{p(\cdot)}} + \|\{2^{js}\theta_{j,a}^* * f\}\|_{\ell_q(L^{p(\cdot)})} \lesssim \|f\|_{B_{p(\cdot),q}^s},$$

here we first used (10) with $\Phi = A$, $\varphi = \theta$, and then applied (14) with $\Psi = A$, $\psi = \theta$.

The right inequality in (4) is proved by another chain

$$\begin{aligned} \|f\|_{B_{p(\cdot),q}^s} &\lesssim \|A_a^* f\|_{L^{p(\cdot)}} + \|\{2^{js}\theta_{j,a}^* * f\}\|_{\ell(L^{p(\cdot)})} \\ &\lesssim \|\Psi_a^* f\|_{L^{p(\cdot)}} + \|\{2^{js}\psi_{j,a}^* f\}\|_{\ell_q(L^{p(\cdot)})} \lesssim RHS(4), \end{aligned}$$

here the first inequality is obvious, the second is (10) with $\Phi = \Psi$, $\varphi = \psi$, and A and θ instead of Ψ and ψ in the left hand side. Finally, the third inequality is (14). This finishes the proof. □

Remark 2. The author learned from the referee and Professor Hästö that Diening, Hästö and Roudenko have recently studied Triebel–Lizorkin spaces with variable indices independently. Their method is different, and applies to variable s and q, but not negative s; for their results, see [9].

Remark 3. Almeida and Samko in [2] and, independently, Gurka, Harjulehto and Nekvinda in [12] have introduced Bessel potential spaces with variable exponents. These spaces are special cases covered by those of this paper, for the proof, see [25].

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