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PLANAR BEURLING TRANSFORM AND GRUNSKY INEQUALITIES

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Abstract. In recent work with Baranov, it was explained how to view the classical Grunsky inequalities in terms of an operator identity, involving a transferred Beurling operator induced by the conformal mapping. The main property used is the fact that the Beurling operator is unitary on $L^2(\mathbf{C})$. As the Beurling operator is also bounded on $L^p(\mathbf{C})$ for $1 < p < +\infty$ (with so far unknown norm), an analogous operator identity was found which produces a generalization of the Grunsky inequalities to the L^p setting. Here, we consider weighted Hilbert spaces $L^2_\theta(\mathbf{C})$ with weight $|z|^{2\theta}$, for $0 \le \theta \le 1$, and find that the Beurling operator perturbed by adding a Cauchy-type operator acts unitarily on $L^2_\theta(\mathbf{C})$. After transferring to the unit disk $\mathbf{D}$ with the conformal mapping, we find a generalization of the Grunsky inequalities in the setting of the space $L^2_\theta(\mathbf{D})$; this generalization seems to be essentially known, but the formulation is new. As a special case, the generalization of the Grunsky inequalities contains the Prawitz theorem used in a recent paper with Shimorin. We also mention an application to quasiconformal maps.

1. Introduction

Beurling and Fourier transforms. In this note, we shall study a perturbation of the Beurling transform in the complex plane $\mathbf{C}$. The *Fourier transform* of an appropriately area-integrable function f is

$$\mathfrak{F}[f](\xi) = \int_{\mathbf{C}} \mathrm{e}^{-2\mathrm{i}\,\mathrm{Re}[z\bar\xi]}\, f(z)\,\mathrm{d}A(z), \quad \xi \in \mathbf{C},$$

while the *Beurling transform* is the singular integral operator

$$\mathfrak{B}_{\mathbf{C}}[f](z) = \mathrm{pv}\int_{\mathbf{C}} \frac{f(w)}{(w-z)^2}\,\mathrm{d}A(w), \quad z \in \mathbf{C};$$

here "pv" stands for "principal value", and

$$\mathrm{d}A(z) = \frac{\mathrm{d}x\mathrm{d}y}{\pi}, \quad z = x + \mathrm{i}y,$$

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is normalized area measure. The two transforms are connected via

$$\mathfrak{F}\mathfrak{B}_{\mathbf{C}}[f](\xi) = -\frac{\bar{\xi}}{\xi}\,\mathfrak{F}[f](\xi), \quad \xi \in \mathbf{C}.$$

By the Plancherel identity, $\mathfrak{F}$ is a unitary transformation on $L^2(\mathbf{C})$, which is supplied with the standard norm

$$\|f\|^2_{L^2(\mathbf{C})} = \int_{\mathbf{C}} |f(z)|^2 \, \mathrm{d}A(z).$$

It is clear from this and the above relationship that $\mathfrak{B}_{\mathbf{C}}$ is unitary on $L^2(\mathbf{C})$ as well. We recall that an operator T acting on a complex Hilbert space $\mathscr{H}$ is unitary if $T^*T = TT^* = \mathrm{id}$, where T^* is the adjoint and "id" is the identity operator. Expressed differently, that T is unitary means that T is a surjective isometry.

The Cauchy transform. The *Cauchy transform* $\mathfrak{C}_{\mathbf{C}}$ is the integral transform

$$\mathfrak{C}_{\mathbf{C}}[f](z) = \int_{\mathbf{C}} \frac{f(w)}{w-z} \, \mathrm{d}A(w),$$

defined for appropriately integrable functions. It is related to Beurling transform $\mathfrak{B}_{\mathbf{C}}$ via

$$\mathfrak{B}_{\mathbf{C}}[f](z) = \partial_z \mathfrak{C}_{\mathbf{C}}[f](z),$$

where both sides are understood in the sense of distribution theory. Here, we use the notation

$$\partial_z = \frac{1}{2}\bigg(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\bigg), \quad \bar{\partial}_z = \frac{1}{2}\bigg(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\bigg).$$

The perturbed Beurling transform. For real θ, let $L^2_\theta(\mathbf{C})$ denote the Hilbert space of square integrable functions on $\mathbf{C}$ with norm

$$\|f\|^2_{L^2_\theta(\mathbf{C})} = \int_{\mathbf{D}} |f(z)|^2 |z|^{2\theta} \mathrm{d}A(z) < +\infty.$$

Moreover, let $\mathfrak{T}_{\mathbf{C}}$ denote the operator

$$\mathfrak{T}_{\mathbf{C}}[h](z) = \frac{1}{z}\,\mathfrak{C}_{\mathbf{C}}[h](z),$$

for suitably integrable functions h. It turns out that it is enough to require that $h \in L^2_\theta(\mathbf{C})$ for some positive θ for $\mathfrak{T}_{\mathbf{C}}[h]$ to be well-defined. We also need the operator $\mathfrak{T}'_{\mathbf{C}}$, as defined by

$$\mathfrak{T}'_{\mathbf{C}}[h](z) = \mathfrak{C}_{\mathbf{C}}\bigg[\frac{h}{z}\bigg](z).$$

We introduce, for $0 \le \theta \le 1$, the perturbed Beurling transform

$$\mathfrak{B}^\theta_{\mathbf{C}} = \mathfrak{B}_{\mathbf{C}} + \theta\,\mathfrak{T}_{\mathbf{C}}, \tag{1.1}$$

while for $-1 \le \theta \le 0$, we instead write

$$\mathfrak{B}^\theta_{\mathbf{C}} = \mathfrak{B}_{\mathbf{C}} + \theta\,\mathfrak{T}'_{\mathbf{C}}. \tag{1.2}$$

Theorem 1.1. *For $-1 \le \theta \le 1$, the operator $\mathfrak{B}^\theta_{\mathbf{C}}$ acts unitarily on $L^2_\theta(\mathbf{C})$.*

The proof of this theorem is supplied in the next section.

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2. The perturbed Beurling transform

For $N = 1, 2, 3, \ldots$, let $\mathscr{A}_N$ denote the N-th roots of unity, that is, the collection of all $\alpha \in \mathbf{C}$ with $\alpha^N = 1$. For $n = 1, \ldots, N$, we consider the closed subspace $L^2_{n,N}(\mathbf{C})$ of $L^2(\mathbf{C})$ consisting of functions f having the invariance property

$$f(\alpha z) = \alpha^n f(z), \quad z \in \mathbf{C},\ \alpha \in \mathscr{A}_N. \tag{2.1}$$

It is easy to see that $f \in L^2_{n,N}(\mathbf{C})$ if and only if $f \in L^2(\mathbf{C})$ is of the form

$$f(z) = z^n\, g(z^N), \quad z \in \mathbf{C}, \tag{2.2}$$

where g some other complex-valued function.

We shall now study the Beurling transform on the subspaces $L^2_{n,N}(\mathbf{C})$.

The Beurling transform and root-of-unity invariance. Fix an $N = 1, 2, 3, \ldots$ and an $n = 1, \ldots, N$. We suppose $f \in L^2_{n,N}(\mathbf{C})$. Then, by the change of variables formula,

$$\begin{aligned}\mathfrak{B}_{\mathbf{C}}[f](z) &= \operatorname{pv}\int_{\mathbf{C}} \frac{f(w)}{(w-z)^2}\,\mathrm{d}A(w) = \operatorname{pv}\int_{\mathbf{C}} \frac{\alpha^n}{(\alpha w - z)^2}\, f(w)\,\mathrm{d}A(w) \\ &= \alpha^{n-2}\,\mathfrak{B}_{\mathbf{C}}[f](\bar\alpha z), \quad z \in \mathbf{C},\end{aligned}$$

for $\alpha \in \mathscr{A}_N$. Taking the average over $\mathscr{A}_N$, we get the identity

$$\mathfrak{B}_{\mathbf{C}}[f](z) = \frac{1}{N}\operatorname{pv}\int_{\mathbf{C}} \sum_{\alpha\in\mathscr{A}_N} \frac{\alpha^n}{(\alpha w - z)^2}\, f(w)\,\mathrm{d}A(w), \quad z \in \mathbf{C}.$$

A symmetric sum. Next, we study the sum

$$F(z) = \frac{1}{N}\sum_{\alpha\in\mathscr{A}_N} \frac{\alpha^n}{1-\alpha z}.$$

This sum has the symmetry property

$$F(\beta z) = \bar\beta^n\, F(z), \quad \beta \in \mathscr{A}_N,$$

which means that F has the form

$$F(z) = z^{N-n}\, G(z^N).$$

The function G then has a simple pole at 1, and is analytic everywhere else in the complex plane. Moreover, F vanishes at infinity, so G vanishes there, too. This leaves us but one possibility, that G has the form

$$G(z) = \frac{C}{1-z},$$

where C is a constant. It is easily established that $C=1$. It follows that

$$F(z)=\frac{1}{N}\sum_{\alpha\in\mathscr{A}_N}\frac{\alpha^n}{1-\alpha z}=\frac{z^{N-n}}{1-z^N},\quad z\in\mathbf{C}. \tag{2.3}$$

As a consequence, we get that

$$\begin{aligned}H(z)=F(z)+zF'(z)=[zF(z)]'&=\frac{1}{N}\sum_{\alpha\in\mathscr{A}_N}\frac{\alpha^n}{(1-\alpha z)^2}\\&=z^{N-n}\biggl\{\frac{N}{(1-z^N)^2}-\frac{n-1}{1-z^N}\biggr\},\end{aligned}$$

where the left hand side identity is used to define the function $H(z)$. This allows us to compute the sum we need:

$$\frac{1}{N}\sum_{\alpha\in\mathscr{A}_N}\frac{\alpha^n}{(\alpha w-z)^2}=\frac{1}{z^2}\,H\biggl(\frac{w}{z}\biggr)=z^{n-2}w^{N-n}\biggl\{\frac{Nz^N}{(z^N-w^N)^2}-\frac{n-1}{z^N-w^N}\biggr\}.$$

For $f\in L^2_{n,N}(\mathbf{C})$, we thus get the representation

$$\mathfrak{B}_{\mathbf{C}}[f](z)=z^{n-2}\ \mathrm{pv}\int_{\mathbf{C}}\biggl\{\frac{Nz^N}{(z^N-w^N)^2}-\frac{n-1}{z^N-w^N}\biggr\}w^{N-n}\,f(w)\,\mathrm{d}A(w),\quad z\in\mathbf{C}.$$

Let f and g be connected via (2.2), and implement this relationship into the above formula:

$$\mathfrak{B}_{\mathbf{C}}[f](z)=z^{n-2}\ \mathrm{pv}\int_{\mathbf{C}}\biggl\{\frac{Nz^N}{(z^N-w^N)^2}-\frac{n-1}{z^N-w^N}\biggr\}w^{N}\,g(w^N)\,\mathrm{d}A(w),\ \ z\in\mathbf{C}. \tag{2.4}$$

A similar expression may be found for the Cauchy transform as well:

$$\mathfrak{C}_{\mathbf{C}}[f](z)=z^{n-N-1}\int_{\mathbf{C}}\frac{w^N}{w^N-z^N}\,g(w^N)\,\mathrm{d}A(w),\quad z\in\mathbf{C}. \tag{2.5}$$

It is easy to check that with

$$h(z)=\frac{z\,g(z)}{|z|^{2-2/N}},$$

where g is connected to f via (2.2), we have

$$\mathfrak{B}_{\mathbf{C}}[f](z)=z^{N+n-2}\,\mathfrak{B}_{\mathbf{C}}^{(n-1)/N}[h](z^N),\quad z\in\mathbf{C}.$$

The fact that $\mathfrak{B}_{\mathbf{C}}$ is an isometry becomes the norm identity

$$\int_{\mathbf{C}}|h(z)|^2\,|z|^{2\theta}\,\mathrm{d}A(z)=\int_{\mathbf{C}}\bigl|\mathfrak{B}^\theta_{\mathbf{C}}[h](z)\bigr|^2\,|z|^{2\theta}\,\mathrm{d}A(z), \tag{2.6}$$

where we suppose that $\theta=(n-1)/N$. However, fractions of this type are dense in the interval $[0,1]$, so that (2.6) extends to all θ with $0\le\theta\le1$. In other words, for $0\le\theta\le1$, *the operator* $\mathfrak{B}^\theta_{\mathbf{C}}$ *is unitary on the space* $L^2_\theta(\mathbf{C})$, which was defined earlier. But then, considering that

$$\mathfrak{B}^\theta_{\mathbf{C}}=\mathfrak{M}_z\mathfrak{B}^{\theta+1}_{\mathbf{C}}\mathfrak{M}_z^{-1},\quad -1\le\theta\le0,$$

which follows immediately from the fact that

$$\frac{1}{(w-z)^2}+\frac{\theta}{w(w-z)}=\frac{z}{w}\left\{\frac{1}{(w-z)^2}+\frac{\theta+1}{z(w-z)}\right\},$$

we conclude that $\mathfrak{B}_{\mathbf{C}}^{\theta}$ is unitary on $L^2_\theta(\mathbf{C})$ for $-1\le\theta\le 0$ as well.

This completes the proof of Theorem 1.1.

Remark 2.1. It is known [8] that $\mathfrak{B}_{\mathbf{C}}$ is a bounded operator on $L^2_\theta(\mathbf{C})$ for $-1<\theta<1$ (but not for $\theta=\pm 1$). This means that for $-1<\theta<1$, both terms in (1.1) are bounded operators on $L_\theta(\mathbf{C})$. We suspect that the second term in (1.1), the operator $\mathfrak{T}_{\mathbf{C}}$, is compact on $L^2_\theta(\mathbf{C})$ with small spectrum for $0<\theta<1$. The analogous statement for $\mathfrak{T}'_{\mathbf{C}}$ is essentially equivalent.

Extension to real θ. We first note that $\mathfrak{M}_z$, multiplication by the independent variable, is an isometric isomorphism $L^2_{\theta+1}(\mathbf{C})\to L^2_\theta(\mathbf{C})$ for all real θ. Therefore, for integers k and $0\le\theta\le 1$, the operator

$$\mathfrak{B}_{\mathbf{C}}^{\theta+k}=\mathfrak{M}_z^{-k}\mathfrak{B}_{\mathbf{C}}^{\theta}\mathfrak{M}_z^{k}$$

is unitary on $L^2_{\theta+k}(\mathbf{C})$. It supplies an extension of $\mathfrak{B}_{\mathbf{C}}^{\theta}$ to all real θ which coincides with the previously defined notion for $-1\le\theta\le 1$.

3. Applications of Beurling transforms to conformal mapping

Grunsky identity and inequalities. It was shown in [1] that if $\varphi\colon\mathbf{D}\to\Omega$ is a conformal mapping where $\Omega=\varphi(\mathbf{D})\subset\mathbf{C}$, then

$$\mathfrak{B}_\varphi[f](z)=\mathrm{pv}\int_{\mathbf{D}}\frac{\varphi'(z)\varphi'(w)}{(\varphi(w)-\varphi(z))^2}\,f(w)\,\mathrm{d}A(w),\quad z\in\mathbf{D},$$

is a contraction on $L^2(\mathbf{D})$; as a matter of fact, this follows from the fact that $\mathfrak{B}_{\mathbf{C}}$ is unitary on $L^2(\mathbf{C})$. Moreover, it was shown that if e denotes the function $e(z)=z$, so that

$$\mathfrak{B}_e[f](z)=\mathrm{pv}\int_{\mathbf{D}}\frac{1}{(w-z)^2}\,f(w)\,\mathrm{d}A(w),\quad z\in\mathbf{D},$$

we have the *Grunsky identity*

$$\mathfrak{B}_\varphi-\mathfrak{B}_e=\mathfrak{P}\mathfrak{B}_\varphi=\mathfrak{B}_\varphi\bar{\mathfrak{P}}=\mathfrak{P}\mathfrak{B}_\varphi\bar{\mathfrak{P}},\tag{3.1}$$

where $\mathfrak{P}$ and $\bar{\mathfrak{P}}$ are the associated Bergman projections

$$\mathfrak{P}[f](z)=\int_{\mathbf{D}}\frac{f(w)}{(1-z\bar{w})^2}\,\mathrm{d}A(w),\quad z\in\mathbf{D},$$

and

$$\bar{\mathfrak{P}}[f](z)=\int_{\mathbf{D}}\frac{f(w)}{(1-\bar{z}w)^2}\,\mathrm{d}A(w),\quad z\in\mathbf{D}.$$

As $\mathfrak{P}$ and $\bar{\mathfrak{P}}$ are contractions on $L^2(\mathbf{D})$, we find that

$$\big\|(\mathfrak{B}_\varphi-\mathfrak{B}_e)[f]\big\|_{L^2(\mathbf{D})}\le\|f\|_{L^2(\mathbf{D})},\quad f\in L^2(\mathbf{D}).\tag{3.2}$$

In [1], it is explained how (3.2) expresses the Grunsky inequalities in a compact manner.

We shall now try to carry out the same considerations in the weighted situation.

Transfer to the unit disk. We need to introduce some general notation. Let $\mathfrak{M}_F$ denote the operator of multiplication by the function F. We also need the Hilbert space $L^2_\theta(X)$ with the norm

$$\|h\|^2_{L^2_\theta(X)} = \int_X |h(z)|^2 \, |z|^{2\theta} \, \mathrm{d}A(z),$$

where X is some Borel measurable subset of $\mathbf{C}$ with positive area. In the sequel, we restrict θ to the interval $0 \le \theta \le 1$. Fix a simply connected domain Ω in $\mathbf{C}$, which contains the origin and is not the whole plane, and let $\varphi\colon \mathbf{D} \to \Omega$ denote the conformal mapping with $\varphi(0) = 0$ and $\varphi'(0) > 0$. Let $f \in L^2(\Omega)$, and extend it to the whole complex plane so that it vanishes on $\mathbf{C} \setminus \Omega$. Let $\mathfrak{B}_\Omega[f]$ denote the restriction to Ω of $\mathfrak{B}_{\mathbf{C}}[f]$, and do likewise to define the operators $\mathfrak{C}_\Omega$, $\mathfrak{T}_\Omega$, $\mathfrak{T}'_\Omega$, $\mathfrak{B}^\theta_\Omega$, as well as $\mathfrak{B}^{-\theta}_\Omega$. We introduce transferred operators on spaces over the unit disk in the following fashion. First, we suppose $f \in L^2_\theta(\Omega)$. Then the associated function

$$g(z) = \bar{\varphi}'(z) \left[\frac{\varphi(z)}{z}\right]^\theta f \circ \varphi(z), \quad z \in \mathbf{D}, \tag{3.3}$$

belongs to $L^2_\theta(\mathbf{D})$, with equality of norms:

$$\|g\|_{L^2_\theta(\mathbf{D})} = \|f\|_{L^2_\theta(\Omega)}.$$

The transferred Cauchy transform is defined as follows:

$$\mathfrak{C}^\theta_\varphi[g](z) = \left[\frac{\varphi(z)}{z}\right]^\theta \mathfrak{C}_\Omega[f] \circ \varphi(z) = \int_{\mathbf{D}} \left[\frac{w\,\varphi(z)}{z\,\varphi(w)}\right]^\theta \frac{\varphi'(w)}{\varphi(w) - \varphi(z)} \, g(w) \, \mathrm{d}A(w). \tag{3.4}$$

The transferred perturbed Beurling transform is defined analogously:

$$\begin{aligned}
\mathfrak{B}^\theta_\varphi[g](z) &= \varphi'(z) \left[\frac{\varphi(z)}{z}\right]^\theta \mathfrak{B}^\theta_\Omega[f] \circ \varphi(z) \\
&= \varphi'(z) \left[\frac{\varphi(z)}{z}\right]^\theta \left\{ \mathfrak{B}_\Omega[f] \circ \varphi(z) + \frac{\theta}{\varphi(z)} \, \mathfrak{C}_\Omega[f] \circ \varphi(z) \right\} \\
&= \mathfrak{B}^{\theta,0}_\varphi[g](z) + \theta \, \frac{\varphi'(z)}{\varphi(z)} \, \mathfrak{C}^\theta_\varphi[g](z),
\end{aligned}$$

where

$$\mathfrak{B}^{\theta,0}_\varphi[g](z) = \mathrm{pv} \int_{\mathbf{D}} \left[\frac{w\,\varphi(z)}{z\,\varphi(w)}\right]^\theta \frac{\varphi'(z)\varphi'(w)}{(\varphi(w) - \varphi(z))^2} \, g(w) \, \mathrm{d}A(w).$$

It is clear that $\mathfrak{B}^\theta_\varphi$ is a norm contraction on $L^2_\theta(\mathbf{D})$. Let $\mathfrak{P}_\theta$ be the integral operator

$$\mathfrak{P}_\theta[f](z) = \int_{\mathbf{D}} \left[\frac{1}{(1 - z\bar{w})^2} + \frac{\theta}{1 - z\bar{w}}\right] f(w) \, |w|^{2\theta} \mathrm{d}A(w);$$

it is the orthogonal projection to the subspace of analytic functions in $L^2_\theta(\mathbf{D})$. As both $\mathfrak{B}^\theta_\varphi$ and $\mathfrak{P}_\theta$ are contractions on $L^2_\theta(\mathbf{D})$, so is their product $\mathfrak{P}_\theta\mathfrak{B}^\theta_\varphi$. It remains to represent the operator $\mathfrak{P}_\theta\mathfrak{B}^\theta_\varphi$ in a reasonable fashion. The main observation is that

$$\left[\frac{w\,\varphi(z)}{z\,\varphi(w)}\right]^\theta \frac{\varphi'(z)\varphi'(w)}{(\varphi(w)-\varphi(z))^2} = \frac{1}{(w-z)^2} - \theta\left[\frac{\varphi'(z)}{\varphi(z)} - \frac{1}{z}\right]\frac{1}{w-z} + O(1)$$

near the diagonal $z = w$, so that

$$\begin{aligned} &\left[\frac{w\,\varphi(z)}{z\,\varphi(w)}\right]^\theta \frac{\varphi'(z)\varphi'(w)}{(\varphi(w)-\varphi(z))^2} + \theta\,\frac{\varphi'(z)}{\varphi(z)}\left[\frac{w\,\varphi(z)}{z\,\varphi(w)}\right]^\theta \frac{\varphi'(w)}{\varphi(w)-\varphi(z)} \\ &\quad = \frac{1}{(w-z)^2} + \frac{\theta}{z(w-z)} + O(1), \end{aligned} \tag{3.5}$$

again near the diagonal. We observe that in view of (3.5), we get the Grunsky-type identity

$$\mathfrak{P}_\theta\mathfrak{B}^\theta_\varphi = \mathfrak{B}^\theta_\varphi - \mathfrak{B}_{\mathbf{D}} + \mathfrak{P}_\theta\mathfrak{B}_{\mathbf{D}} + \theta\mathfrak{P}_\theta\mathfrak{T}_{\mathbf{D}} - \theta\mathfrak{T}_{\mathbf{D}}. \tag{3.6}$$

To make the involved operators $\mathfrak{P}_\theta\mathfrak{B}_{\mathbf{D}}$ and $\mathfrak{P}_\theta\mathfrak{T}_{\mathbf{D}}$ appearing in the right hand side of (3.6) more concrete, it is helpful to know that for $\lambda \in \mathbf{D}$,

$$\mathfrak{P}_\theta[f_\lambda](z) = \bar\lambda|\lambda|^{2\theta}\int_0^1 \left[\frac{1}{(1-t\bar\lambda z)^2} + \frac{\theta}{1-t\bar\lambda z}\right] t^\theta \mathrm{d}t, \quad f_\lambda(z) = \frac{1}{\lambda - z},$$

while

$$\mathfrak{P}_\theta[g_\lambda](z) = -\theta\,\bar\lambda^2|\lambda|^{2\theta-2}\int_0^1 \left[\frac{1}{(1-t\bar\lambda z)^2} + \frac{\theta}{1-t\bar\lambda z}\right] t^\theta \mathrm{d}t, \quad g_\lambda(z) = \frac{1}{(\lambda - z)^2}.$$

In view of these relations, we quickly verify that

$$\mathfrak{P}_\theta\mathfrak{B}_{\mathbf{D}} + \theta\mathfrak{P}_\theta\mathfrak{T}_{\mathbf{D}} = 0.$$

The Grunsky-type identity (3.6) thus simplifies a bit:

$$\mathfrak{P}_\theta\mathfrak{B}^\theta_\varphi = \mathfrak{B}^\theta_\varphi - \mathfrak{B}_{\mathbf{D}} - \theta\mathfrak{T}_{\mathbf{D}} = \mathfrak{B}^\theta_\varphi - \mathfrak{B}^\theta_{\mathbf{D}}. \tag{3.7}$$

The corresponding Grunsky-type inequality reads

$$\big\|\big(\mathfrak{B}^\theta_\varphi - \mathfrak{B}^\theta_{\mathbf{D}}\big)[f]\big\|_{L^2_\theta(\mathbf{D})} \le \|f\|_{L^2_\theta(\mathbf{D})}, \quad f \in L^2_\theta(\mathbf{D}). \tag{3.8}$$

To get a concrete example of how the Grunsky-type inequality works, we pick

$$f_\lambda(z) = |z|^{-2\theta}\left(\frac{1}{(1-\bar z\lambda)^2} - \frac{\theta}{1-\bar z\lambda}\right), \quad z \in \mathbf{D},$$

and compute

$$\begin{aligned} \big(\mathfrak{B}^\theta_\varphi - \mathfrak{B}^\theta_{\mathbf{D}}\big)[f](z) &= \left[\frac{\lambda\,\varphi(z)}{z\,\varphi(\lambda)}\right]^\theta \frac{\varphi'(z)\varphi'(\lambda)}{(\varphi(\lambda)-\varphi(z))^2} - \frac{1}{(\lambda-z)^2} \\ &\quad + \theta\,\frac{\varphi'(z)}{\varphi(z)}\left[\frac{\lambda\,\varphi(z)}{z\,\varphi(\lambda)}\right]^\theta \frac{\varphi'(\lambda)}{\varphi(\lambda)-\varphi(z)} - \frac{\theta}{z(\lambda-z)}. \end{aligned}$$

We see that (3.8) in this case assumes the form $(0\le\theta\le1)$

$$
\begin{aligned}
&\int_{\mathbf D}\bigg|\bigg[\frac{\lambda\,\varphi(z)}{z\,\varphi(\lambda)}\bigg]^{\theta}\frac{\varphi'(z)\varphi'(\lambda)}{(\varphi(\lambda)-\varphi(z))^2}-\frac{1}{(\lambda-z)^2}\\
&\qquad+\theta\,\frac{\varphi'(z)}{\varphi(z)}\bigg[\frac{\lambda\,\varphi(z)}{z\,\varphi(\lambda)}\bigg]^{\theta}\frac{\varphi'(\lambda)}{\varphi(\lambda)-\varphi(z)}-\frac{\theta}{z(\lambda-z)}\bigg|^2|z|^{2\theta}\mathrm{d}A(z)\\
&\le\int_{\mathbf D}|f_\lambda(z)|^2|z|^{2\theta}\mathrm{d}A(z)=\int_{\mathbf D}\bigg|\frac{1}{(1-\bar z\lambda)^2}-\frac{\theta}{1-\bar z\lambda}\bigg|^2|z|^{-2\theta}\mathrm{d}A(z)\\
&=\frac{1}{(1-|\lambda|^2)^2}-\frac{\theta}{1-|\lambda|^2}.
\end{aligned}\tag{3.9}
$$

The special case $\lambda=0$ gives us the inequality of Prawitz (see [6] and [7]; we assume $\varphi'(0)=1$):

$$\int_{\mathbf D}\bigg|\varphi'(z)\left[\frac{\varphi(z)}{z}\right]^{\theta-2}-1\bigg|^2|z|^{2\theta}\mathrm{d}A(z)\le\frac{1}{1-\theta}.$$

A dual version. We carry out the corresponding calculations on the basis of the fact that $\mathfrak{B}_{\mathbf C}^{-\theta}$ is unitary on $L^2_{-\theta}(\mathbf C)$ for $0\le\theta\le1$. In analogy with the above treatment, we connect two functions f,g via

$$g(z)=\bar\varphi'(z)\left[\frac{\varphi(z)}{z}\right]^{-\theta}f\circ\varphi(z),\quad z\in\mathbf D.\tag{3.10}$$

Then $f\in L^2_{-\theta}(\Omega)$ if and only if $g\in L^2_{-\theta}(\mathbf D)$, with equality of norms:

$$\|g\|_{L^2_\theta(\mathbf D)}=\|f\|_{L^2_\theta(\Omega)}.$$

The corresponding transferred Beurling transform assumes the form

$$
\begin{aligned}
\mathfrak{B}_\varphi^{-\theta}[g](z)&=\varphi'(z)\left[\frac{\varphi(z)}{z}\right]^{-\theta}\mathfrak{B}_\Omega^{-\theta}[f]\circ\varphi(z)\\
&=\varphi'(z)\left[\frac{\varphi(z)}{z}\right]^{-\theta}\left\{\mathfrak{B}_\Omega[f]\circ\varphi(z)-\theta\,\mathfrak{C}_\Omega\left[\frac{f}{z}\right]\circ\varphi(z)\right\}\\
&=\mathfrak{B}_\varphi^{-\theta,0}[g](z)-\theta\,\varphi'(z)\,\mathfrak{C}_\varphi^{-\theta}\left[\frac{g}{\varphi}\right](z),
\end{aligned}
$$

where $\mathfrak{B}_\varphi^{-\theta,0}$ and $\mathfrak{C}_\varphi^{-\theta}$ are as before (just plug in $-\theta$ in place of θ in the corresponding formulæ). It is clear that $\mathfrak{B}_\varphi^{-\theta}$ is a contraction on $L^2_{-\theta}(\mathbf D)$.

To cut a long story short, the Grunsky-type identity analogous to (3.7) reads

$$\mathfrak{P}_{-\theta}\mathfrak{B}_\varphi^{-\theta}=\mathfrak{B}_\varphi^{-\theta}-\mathfrak{B}_{\mathbf D}^{-\theta}.\tag{3.11}$$

Let $\bar{\mathfrak{P}}^*_{-\theta}$ be the operator

$$\bar{\mathfrak{P}}^*_{-\theta}[g](z)=|z|^{-2\theta}\int_{\mathbf D}\left(\frac{1}{(1-w\bar z)^2}-\frac{\theta}{1-w\bar z}\right)g(w)\,\mathrm{d}A(w);$$

it is a contraction on $L^2_\theta(\mathbf{D})$, which can be written

$$\bar{\mathfrak{P}}^*_{-\theta} = \mathfrak{M}_{|z|^{-2\theta}} \bar{\mathfrak{P}}_{-\theta} \mathfrak{M}_{|z|^{2\theta}},$$

where $\bar{\mathfrak{P}}_{-\theta}$ denotes the orthogonal projection onto the antiholomorphic functions in $L^2_{-\theta}(\mathbf{D})$. By forming adjoints, we find that (3.11) states that

$$\mathfrak{B}^\theta_\varphi \bar{\mathfrak{P}}^*_{-\theta} = \mathfrak{B}^\theta_\varphi - \mathfrak{B}^\theta_{\mathbf{D}}. \tag{3.12}$$

We now combine (3.7) with (3.12), and arrive at the following.

Theorem 3.1. $(0 \le \theta \le 1)$ *We have the Grunsky identity*

$$\mathfrak{B}^\theta_\varphi - \mathfrak{B}^\theta_{\mathbf{D}} = \mathfrak{P}_\theta \mathfrak{B}^\theta_\varphi = \mathfrak{B}^\theta_\varphi \bar{\mathfrak{P}}^*_{-\theta} = \mathfrak{P}_\theta \mathfrak{B}^\theta_\varphi \bar{\mathfrak{P}}^*_{-\theta}. \tag{3.13}$$

Moreover, we also have the Grunsky-type inequality

$$\big\| (\mathfrak{B}^\theta_\varphi - \mathfrak{B}^\theta_{\mathbf{D}})[f] \big\|_{L^2_\theta(\mathbf{D})} \le \|f\|_{L^2_\theta(\mathbf{D})}, \quad f \in L^2_\theta(\mathbf{D}),$$

with equality if and only if φ is a full mapping and $f(z)$ is of the form $|z|^{-2\theta}$ times an antianalytic function.

Remark 3.2. (a) It follows that (3.9) is an equality for full mappings.

(b) The above Grunsky-type inequality probably follows from the estimate mentioned by de Branges [2] as his point of departure for obtaining the more general results that led to the solution of the Bieberbach conjecture.

(c) It is possible to consider weighted L^p spaces of the type $L^p_\theta(\mathbf{C})$, and obtain norm estimates of perturbed Beurling transforms on such spaces from well-known estimates of the Beurling operator on $L^p(\mathbf{C})$. This then leads to appropriate Grunsky-type identities and inequalities in the weighted L^p setting.

4. Applications to quasiconformal maps

Quasiconformal maps. Here, we suppose that $\varphi\colon \mathbf{D} \to \Omega$ is quasiconformal, which means that it is a homoeomorphism which is one-to-one and onto, with

$$\bar\partial_z \varphi(z) = \mu(z)\, \partial_z \varphi(z), \quad z \in \mathbf{D}, \tag{4.1}$$

where μ is an Borel measurable function on $\mathbf{D}$ with

$$\|\mu\|_{L^\infty(\mathbf{C})} = \text{ess sup}\big\{ |\mu(z)| : z \in \mathbf{D} \big\} < 1.$$

As before, Ω is a simply connected domain in $\mathbf{C}$ other than $\mathbf{C}$ itself, which contains the origin. We assume that $\varphi(0) = 0$ and that μ vanishes on a (small) neighborhood of the origin. The function φ is then analytic near the origin. In the sequel, we shall think of the Beltrami coefficient μ as fixed. We plan to derive some information regarding the mapping φ.

The mapping $\phi = \phi_\mu$. We extend μ to all of $\mathbf{C}$ by declaring it to be

$$\mu(z) = \bar\mu\left(\frac{1}{\bar z}\right), \quad z \in \mathbf{D}_e,$$

where

$$\mathbf{D}_e = \big\{z \in \mathbf{C}:\ 1 < |z| < +\infty\big\}$$

is the (punctured) exterior disk, and by declaring it to vanish on the unit circle $\mathbf{T}$. Clearly, the extended μ has compact support.

The material mentioned here is largely a condensed version of Section 1.7 of [3]; we refer to that book for details. Let $F = F_\mu\colon \mathbf{C} \to \mathbf{C}$ solve the equation

$$(\mathrm{id} + \mathfrak{B}_{\mathbf{C}}\mathfrak{M}_\mu)[F] = \mathfrak{B}_{\mathbf{C}}[\mu];$$

A solution F exists and is unique, and it belongs to $L^p(\mathbf{C})$ for p in some open interval containing the point 2. We define

$$\Phi(z) = z + \bar{\mathfrak{C}}_{\mathbf{C}}[F](z) - \bar{\mathfrak{C}}_{\mathbf{C}}[F](0),$$

and obtain a quasiconformal map $\Phi = \Phi_\mu\colon \mathbf{C} \to \mathbf{C}$ which solves the Beltrami equation

$$\bar\partial_z \Phi(z) = \mu(z)\, \partial_z \Phi(z), \quad z \in \mathbf{C}.$$

Here, $\bar{\mathfrak{C}}_{\mathbf{C}}$ is the conjugate Cauchy transform

$$\bar{\mathfrak{C}}_{\mathbf{C}}[f](z) = \int_{\mathbf{C}} \frac{f(w)}{\bar w - \bar z}\, \mathrm{d}A(w), \quad z \in \mathbf{C}.$$

A calculation shows that the related mapping

$$\Psi(z) = \frac{1}{\bar\Phi\left(\frac{1}{\bar z}\right)}, \quad z \in \mathbf{C} \setminus \{0\},$$

solves the same Beltrami equation

$$\bar\partial_z \Psi(z) = \mu(z)\, \partial_z \Psi(z), \quad z \in \mathbf{C}.$$

As Ψ—like Φ—fixes the points 0 and ∞, it follows that

$$\Psi(z) = \lambda\, \Phi(z), \quad z \in \mathbf{C},$$

for some complex parameter λ. Since we must have

$$\frac{\Phi(z)}{\Psi(z)} = |\Phi(z)|^2 = \frac{1}{\lambda}, \quad z \in \mathbf{T},$$

it follows that $0 < \lambda < +\infty$. As a consequence, we have that

$$\phi(z) = \phi_\mu(z) = \sqrt{\lambda}\, \Phi(z), \quad z \in \mathbf{D},$$

maps $\mathbf{D}$ onto itself, and preserves the origin. Moreover, ϕ solves the same Beltrami equation (4.1) as does φ.

The induced transform. The parameter θ is assumed to be confined to the interval $0 \le \theta \le 1$. It is easy to see that it is possible to define a single-valued logarithm

$$\log \frac{\varphi(z)}{z}, \quad z \in \mathbf{D}.$$

One just checks that the associated differential is exact. This allows us to define real (and complex) powers of the function $\varphi(z)/z$. Next, we suppose $f \in L^2_\theta(\Omega)$, and associate to it the function g:

$$g(z) = (1 - |\mu(z)|^2)^{1/2}\, \bar\partial_z \bar\varphi(z) \left[\frac{\varphi(z)}{z}\right]^\theta f \circ \varphi(z), \quad z \in \mathbf{D}.$$

It is a consequence of the change-of-variables formula

$$\int_\Omega |F(z)|^2 \,\mathrm{d}A(z) = \int_{\mathbf{D}} |F \circ \varphi(z)|^2 \left(1 - |\mu(z)|^2\right) |\partial_z \varphi(z)|^2 \,\mathrm{d}A(z) \tag{4.2}$$

that

$$\|g\|_{L^2_\theta(\mathbf{D})} = \|f\|_{L^2_\theta(\Omega)}.$$

We *define the transferred Beurling transform* to be

$$\mathfrak{B}^{\theta,\mu}_\varphi[g](z) = (1 - |\mu(z)|^2)^{1/2}\, \partial_z \varphi(z) \left[\frac{\varphi(z)}{z}\right]^\theta \mathfrak{B}^\theta_\Omega[f] \circ \varphi(z), \quad z \in \mathbf{D},$$

so that $\mathfrak{B}^{\theta,\mu}_\varphi$ acts contractively on $L^2_\theta(\mathbf{D})$. In case $\theta = 0$, the formula simplifies pleasantly:

$$\mathfrak{B}^{0,\mu}_\varphi[g](z) = (1 - |\mu(z)|^2)^{1/2}\, \partial_z \int_{\mathbf{D}} \frac{(1 - |\mu(w)|^2)^{1/2} \partial_w \varphi(w)}{\varphi(w) - \varphi(z)}\, g(w) \,\mathrm{d}A(w), \quad z \in \mathbf{D}.$$

The differentiation is in the sense of distribution theory.

The Grunsky-type identity and inequality. Since φ and ϕ have the same Beltrami coefficient μ, there is a conformal mapping $\psi\colon \mathbf{D} \to \Omega$ fixing the origin such that $\varphi = \psi \circ \phi$. Next, we connect h and f via

$$h(z) = \bar\psi'(z) \left[\frac{\psi(z)}{z}\right]^\theta f \circ \psi(z), \quad z \in \mathbf{D},$$

so that

$$\|h\|_{L^2_\theta(\mathbf{D})} = \|f\|_{L^2_\theta(\Omega)} = \|g\|_{L^2_\theta(\mathbf{D})}$$

and

$$g(z) = (1 - |\mu(z)|^2)^{1/2}\, \bar\partial_z \bar\phi(z) \left[\frac{\phi(z)}{z}\right]^\theta h \circ \phi(z), \quad z \in \mathbf{D},$$

while

$$\mathfrak{B}^{\theta,\mu}_\varphi[g](z) = (1 - |\mu(z)|^2)^{1/2}\, \partial_z \phi(z) \left[\frac{\phi(z)}{z}\right]^\theta \mathfrak{B}^\theta_\psi[h] \circ \phi(z), \quad z \in \mathbf{D}.$$

To simplify the notation, let $\mathfrak{U}^{\theta,\mu}$ denote the unitary transformation on $L^2_\theta(\mathbf{D})$ given by

$$\mathfrak{U}^{\theta,\mu}[g](z) = (1 - |\mu(z)|^2)^{1/2}\, \partial_z \phi(z) \left[\frac{\phi(z)}{z}\right]^\theta g \circ \phi(z), \quad z \in \mathbf{D}.$$

so that $\mathfrak{B}_\varphi^{\theta,\mu} = \mathfrak{U}^{\theta,\mu}\mathfrak{B}_\psi^\theta$. Next, let the orthogonal projection $\mathfrak{P}_{\theta,\mu}$ on $L^2_\theta(\mathbf{D})$ be defined by

$$\mathfrak{P}_{\theta,\mu} = \mathfrak{U}^{\theta,\mu}\,\mathfrak{P}_\theta\left(\mathfrak{U}^{\theta,\mu}\right)^{-1}.$$

It now follows from the results of the previous section that

$$\mathfrak{P}_{\theta,\mu}\mathfrak{B}_\varphi^{\theta,\mu} = \mathfrak{B}_\varphi^{\theta,\mu} - \mathfrak{B}_\phi^{\theta,\mu}, \tag{4.3}$$

and since the left hand side is a contraction, we conclude that

$$\left\|\left(\mathfrak{B}_\varphi^{\theta,\mu} - \mathfrak{B}_\phi^{\theta,\mu}\right)[g]\right\|_{L^2_\theta(\mathbf{D})} \le \|g\|_{L^2_\theta(\mathbf{D})}, \quad g \in L^2_\theta(\mathbf{D}). \tag{4.4}$$

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