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# HANKEL AND TOEPLITZ OPERATORS ON NONSEPARABLE HILBERT SPACES

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**Abstract.** We show that every operator that acts between two nonseparable Hilbert spaces can be "block diagonalized", where each diagonal block acts between two separable Hilbert spaces. Analogous results hold for operator-valued $\mathscr{H}^\infty$ functions and others.

Using these results, several theorems about representation, invertibility, factorization etc., which have previously been known only for separable Hilbert spaces, can now be generalized to arbitrary Hilbert spaces. We generalize several results often needed in systems and control theory, including the Lax–Halmos Theorem, Tolokonnikov's Lemma and the inner-outer factorization. We present our results both for the unit disc and for the half-plane.

## 1. Introduction

Let $\mathscr{X}$ and $\mathscr{Y}$ be arbitrary (possibly nonseparable) complex Hilbert spaces. If $\mathscr{T}$ is bounded and linear $\mathscr{X} \to \mathscr{Y}$ (i.e., $\mathscr{T} \in \mathscr{B}(\mathscr{X}, \mathscr{Y})$), then

$$\mathscr{T} = \begin{bmatrix} * & 0 & 0 & \cdots \\ 0 & * & 0 & \cdots \\ 0 & 0 & * & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{1}$$

where each $*$ stands for a bounded linear operator of the form $T \in \mathscr{B}(X, Y)$, where $X$ (resp., $Y$) is a closed separable subspace of $\mathscr{X}$ (resp., $\mathscr{Y}$), and all such subspaces $X$ (resp., $Y$) are orthogonal to each other and their (possibly uncountable) sum equals $\mathscr{X}$ (resp., $\mathscr{Y}$). The same holds if above $\mathscr{B}$ is replaced by, e.g., $\mathscr{H}^p$, $\mathscr{H}^p_{\text{strong}}$ or $\mathrm{L}^p_{\text{strong}}$, which shall be defined below. Analogous claims also hold when $\mathscr{B}(\mathscr{X}, \mathscr{Y})$ is replaced by $\mathscr{B}(\mathrm{L}^2(\mathscr{X}), \mathrm{L}^2(\mathscr{Y}))$ or similar. This "diagonalization method" is the main contribution of this article. Excluding holomorphicity, these results hold for real Hilbert spaces too, as explained in [Mik07], which is a supplement and extension to this article.

Standard interpolation results, such as the Nehari Theorem or the AAK Theorem, have been known for functions $\mathscr{T} \colon \mathbf{T} \to \mathscr{B}(\mathscr{X}, \mathscr{Y})$, where $\mathscr{X}$ and $\mathscr{Y}$ are separable. Such results can be extended to general $\mathscr{X}$ and $\mathscr{Y}$ by applying the known results to each $T$ to obtain an interpolant $U$ and then combining all $U$'s to a "block diagonal" function $\mathscr{U}$ that interpolates $\mathscr{T}$ in the same way. Similarly, if $\mathscr{T} \in \mathscr{H}^\infty(\mathscr{X}, \mathscr{Y})$ (a bounded holomorphic function $\mathbf{D} \to \mathscr{B}(\mathscr{X}, \mathscr{Y})$) is left-invertible in $\mathscr{H}^\infty$, then so is each $T$, hence then each $T$ can be extended to an

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invertible function $\begin{bmatrix} T & \tilde{T} \end{bmatrix} \in \mathscr{H}^\infty(X \times Z, Y)$, by Tolokonnikov's Lemma. By combining all such pairs we get an invertible extension $\begin{bmatrix} \mathscr{T} & \tilde{\mathscr{T}} \end{bmatrix} \in \mathscr{H}^\infty(\mathscr{X} \times \mathscr{Z}, \mathscr{Y})$ (Theorem 4.3). Similar claims also hold for other interpolation representation, factorization and [left] invertibility theorems. Practically the only limitation is that the interpolant, the representative, the factors or the [left] inverse must satisfy some norm estimate that does not depend on the particular subspaces involved. This condition is usually inherent in representation and interpolation results, hence nontrivial only in certain factorization and invertibility results.

In Section 2 we present our notation and introduce the space $\mathrm{L}^\infty_{\mathrm{strong}}(\mathscr{X}, \mathscr{Y})$ of functions $F\colon \mathbf{T} \to \mathscr{B}(\mathscr{X}, \mathscr{Y})$ such that $Fx \in \mathrm{L}^\infty(\mathscr{Y})$ for every $x \in \mathscr{X}$. It equals the set of "$\ell^2$ Fourier multipliers", i.e., of functions for which the map $f \mapsto Ff$ is bounded $\mathrm{L}^2(\mathscr{X}) \to \mathrm{L}^2(\mathscr{Y})$.

In Section 3 we present the diagonalization results mentioned above. These are then illustrated by the extended representation, characterization and invertibility theorems of Section 4, for functions on the unit disc and for their Hankel and Toeplitz operators. In the separable case these results are essentially known [Nik02], [Pel03], [RR85], [FF90], [Nik86].

For reference purposes, we present analogous results for the half-plane in Section 5. There we also show that one can use translations instead of the shift. The results on translation-invariant subspaces have previously been known in the scalar/finite-dimensional case [Lax59], the others in the separable case. Corresponding proofs and further details on the relations between the disc and the half-plane are presented in Section 6.

Naturally, our methods could be applied also to generalize similar existing separable-case results on several other sets in place of the unit disc or of the half-plane.

Section 7 contains historical notes. Auxiliary results and some technical proofs are presented in the appendices.

The main contribution of this article is the diagonalization method of Section 3. The examples in Sections 4 and 5 are chosen to cover the results most often used in the theory of well-posed linear systems, for reference purposes.

The third contribution of this article is the illustration of some pathological phenomena of $\mathrm{L}^\infty_{\mathrm{strong}}$, both within the main text and in Appendix C. Part of these appear in the separable case too. However, to bypass most technical difficulties, here we have restricted most of our results to $\mathscr{H}^\infty$ and left, e.g., all interpolation results to [Mik07], so a busy reader can ignore $\mathrm{L}^\infty_{\mathrm{strong}}$.

Many more representation, interpolation and other results are extended in the much more technical [Mik07], where, in addition, the methods of Section 3 are extended to further functions classes etc. Moreover, there also real Hilbert spaces are treated, some results of this article are extended (e.g., $\mathscr{H}^\infty$ is replaced by $\mathrm{L}^\infty_{\mathrm{strong}}$ or by $\mathscr{H}^2_{\mathrm{strong}}$, $\mathscr{H}^2$ is replaced by $\mathrm{L}^2$, or further equivalent conditions and additional results are given), and additional details are presented.

## 2. Notation and $\mathrm{L}^\infty_{\mathrm{strong}}$

In this section we present our (standard) notation. In the "continuous-time" sections 5 and 6 the notation is slightly different with, e.g., $\mathbf{R}$ in place of $\mathbf{T}$. We also present some properties of $\mathscr{H}^p$, $\mathrm{L}^p$, $\mathscr{H}^p_{\mathrm{strong}}$ and $\mathrm{L}^p_{\mathrm{strong}}$.

First we recall that a Hilbert space is isomorphic to $\ell^2(W)$, where $W$ is its orthonormal basis. The space is nonseparable iff $W$ is uncountable. An example of a nonseparable Hilbert space is the Besicovich space (the completion of the space of almost-periodic functions; it is equivalent to $\ell^2(\mathbf{R})$).

*Measurable* means Bochner-measurable. We set $\|f\|_B = \infty$ when $B$ is a Banach space and $f \notin B$. By $f[A]$ we denote the image $\{f(a) \mid a \in A\}$ of $A$. We set $\mathbf{Z} := \{0, \pm 1, \pm 2, \ldots\}$, $\mathbf{N} := \{0, 1, 2, \ldots\}$, $\mathbf{T} := \{z \in \mathbf{C} \mid |z| = 1\}$, $\mathbf{D} := \{z \in \mathbf{C} \mid |z| < 1\}$. By $[F]$ (or by $F$ when there is no risk of ambiguity) we denote the equivalence class of a function $F$ (in, e.g., $\mathrm{L}^p$ or in $\mathrm{L}^p_{\mathrm{strong}}$). By $M_F$ we denote the multiplication operator $f \mapsto Ff$ (i.e., $(M_F f)(z) := F(z)f(z)$ $(z \in \mathbf{D})$), for functions on $\mathbf{D}$.

The symbols of the form $V(A; B)$ usually stand for spaces of functions $A \to B$. When $A = \mathbf{T}$ or $A = \mathbf{D}$, we often omit "$A$;"; when also $B = \mathscr{B}(\mathscr{X}, \mathscr{Y})$, we often write $V(\mathscr{X}, \mathscr{Y})$ instead of $V(A; \mathscr{B}(\mathscr{X}, \mathscr{Y}))$. E.g., by $\mathrm{L}^\infty(B)$ we denote the space of (equivalence classes of) essentially bounded measurable functions $\mathbf{T} \to B$, when $B$ is a Banach space. When $1 \le p < \infty$, we define $\mathrm{L}^p(B)$ by setting

$$\|f\|_p^p := \|f\|_{\mathrm{L}^p(B)}^p := \frac{1}{2\pi} \int_0^{2\pi} \|f(\mathrm{e}^{it})\|_B^p \, dt \tag{2}$$

Let $1 \le p \le \infty$. We denote by $\mathrm{L}^p_{\mathrm{strong}}(\mathscr{X}, \mathscr{Y})$ the space of (equivalence classes of) functions $F \colon \mathbf{T} \to \mathscr{B}(\mathscr{X}, \mathscr{Y})$ for which $Fx \in \mathrm{L}^p(\mathscr{Y})$ for each $x \in \mathscr{X}$, with the norm

$$\|F\|_{\mathrm{L}^p_{\mathrm{strong}}} := \sup\{\|Fx\|_{\mathrm{L}^p(\mathscr{Y})} \mid \|x\|_{\mathscr{X}} \le 1\}. \tag{3}$$

By $\mathscr{H}^p(B)$ we denote the holomorphic functions $\mathbf{D} \to B$, where $\mathbf{D} := \{z \in \mathbf{C} \mid |z| < 1\}$ is the unit disc, with the norm

$$\|f\|_{\mathscr{H}^p} := \sup_{r<1} \|f(r\cdot)\|_{\mathrm{L}^p} < \infty. \tag{4}$$

By $\mathscr{H}^p_{\mathrm{strong}}(\mathscr{X}, \mathscr{Y})$ we denote the functions $F \colon \mathbf{D} \to \mathscr{B}(\mathscr{X}, \mathscr{Y})$ for which $Fx \in \mathscr{H}^p(\mathscr{Y})$ for each $x \in \mathscr{X}$. It follows that $F$ is holomorphic [HP57, Theorem 3.10.1] and

$$\|F\|_{\mathscr{H}^p_{\mathrm{strong}}} := \sup\{\|Fx\|_{\mathscr{H}^p(\mathscr{Y})} \mid \|x\|_{\mathscr{X}} \le 1\} < \infty. \tag{5}$$

The spaces $\mathrm{L}^p(B)$, $\mathscr{H}^p(B)$, $\mathrm{L}^\infty_{\mathrm{strong}}(\mathscr{X}, \mathscr{Y})$ and $\mathscr{H}^p_{\mathrm{strong}}(\mathscr{X}, \mathscr{Y})$ are Banach spaces, and $\mathscr{H}^\infty_{\mathrm{strong}} = \mathscr{H}^\infty$ (by the Uniform Boundedness Theorem). However, $\mathrm{L}^p_{\mathrm{strong}}$ is an incomplete subspace of $\mathscr{B}(\mathscr{X}, \mathrm{L}^p(\mathscr{Y}))$ whenever $\mathscr{X}$ and $\mathscr{Y}$ are infinite-dimensional and $p < \infty$ [Mik08, below Theorem 2.5], [Mik06, Example 4.3].

We mention below some basic properties of $\mathrm{L}^\infty_{\mathrm{strong}}$.

**Remarks 2.1.** If $\dim \mathscr{X} < \infty$, then $\mathrm{L}^\infty_{\mathrm{strong}}(\mathscr{X}, \mathscr{Y}) = \mathrm{L}^\infty(\mathscr{B}(\mathscr{X}, \mathscr{Y}))$ isometrically. Even in the infinite-dimensional case, the two norms coincide on $\mathrm{L}^\infty$ [Mik07], i.e., $\mathrm{L}^\infty(\mathscr{B}(\mathscr{X}, \mathscr{Y}))$ is a closed subspace of $\mathrm{L}^\infty_{\mathrm{strong}}(\mathscr{X}, \mathscr{Y})$. Nevertheless, in the nonseparable case we may have a (non-Bochner-measurable) function $F \colon \mathbf{T} \to \mathscr{B}(\mathscr{X}, \mathscr{Y})$ with $\|F\|_{\mathrm{L}^\infty_{\mathrm{strong}}} = 0$ (i.e., $[F] = [0]$) even though $\|F(z)\|_{\mathscr{B}(\mathscr{X}, \mathscr{Y})} = 1$ for each $z \in \mathbf{T}$, as shown in Example C.1 below.

If $\mathscr{X}$ and $\mathscr{Y}$ are separable, then $\mathrm{L}^\infty_{\mathrm{strong}}$ coincides with the space of essentially bounded weakly measurable functions $\mathbf{T} \to \mathscr{B}(\mathscr{X}, \mathscr{Y})$ [Mik07]. The latter description is often [Pel03, p. 66], [RR85, pp. 81–82] used as the definition of "$\mathrm{L}^\infty$", (actually, of $\mathrm{L}^\infty_{\mathrm{strong}} = \mathrm{L}^\infty_{\mathrm{weak}} \supsetneq \mathrm{L}^\infty$) in the separable case.

By [Mik08] (and Lemma 6.1 or [Mik07, Theorem 4.3]), the space $\mathrm{L}^\infty_{\mathrm{strong}}$ is exactly the space of "Fourier multipliers" $\ell^2(\mathbf{Z};\mathscr{X}) \to \ell^2(\mathbf{Z};\mathscr{Y})$, where $\ell^2(\mathbf{Z};\mathscr{X})$ denotes the space of square-summable functions $\mathbf{Z} \to \mathscr{X}$. Analogously, the space $\mathrm{L}^\infty_{\mathrm{strong}}(\mathbf{R};\mathscr{B}(\mathscr{X},\mathscr{Y}))$ is the space of Fourier multipliers $\mathrm{L}^2(\mathbf{R};\mathscr{X}) \to \mathrm{L}^2(\mathbf{R};\mathscr{Y})$ (Theorem 5.2).

Now we recall [Mik08, Theorem 2.5] (through Lemma 6.1), which shows that any bounded linear operator $\mathscr{X} \to \mathrm{L}^\infty(\mathscr{Y})$ is determined by a $\mathrm{L}^\infty_{\mathrm{strong}}(\mathscr{X},\mathscr{Y})$ function (class) and that any $\mathrm{L}^\infty_{\mathrm{strong}}(\mathscr{X},\mathscr{Y})$ function can be redefined so as not to exceed its norm:

**Proposition 2.2.** *We have* $\mathrm{L}^\infty_{\mathrm{strong}}(\mathscr{X},\mathscr{Y}) = \mathscr{B}(\mathscr{X},\mathrm{L}^\infty(\mathscr{Y}))$, *isometrically. Moreover, for each* $T \in \mathscr{B}(\mathscr{X},\mathrm{L}^\infty(\mathscr{Y}))$, *there exists a function* $F\colon \mathbf{T} \to \mathscr{B}(\mathscr{X},\mathscr{Y})$ *such that* $T_F\colon x \mapsto [Fx]$ *equals* $T$ *and* $\sup_{\mathbf{T}} \|F\|_{\mathscr{B}(\mathscr{X},\mathscr{Y})} = \|T\|$.

Note that if $[F],[\tilde{F}] \in \mathrm{L}^\infty_{\mathrm{strong}}(\mathscr{X},\mathscr{Y})$, then $T_{\tilde{F}} = T_F$ iff $\|F-\tilde{F}\|_{\mathrm{L}^\infty_{\mathrm{strong}}} = 0$, although we may have $\operatorname{ess\,sup} \|\tilde{F}\|_{\mathscr{B}(\mathscr{X},\mathscr{Y})} = \infty$ when $\mathscr{X}$ is nonseparable, by Example C.1(c).

The *Poisson integral* of $f$ is defined as

$$(6) \qquad f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta-t)+r^2} f(e^{it})\,dt \qquad (r \geq 0,\ \theta \in [0,2\pi)).$$

Any $\mathscr{H}^p(\mathscr{X})$ function is the Poisson integral of an $\mathrm{L}^p(\mathscr{X})$ function:

**Proposition 2.3.** ($\mathscr{H}^p \subset \mathrm{L}^p$) *Let* $f \in \mathscr{H}^p(\mathscr{X})$, $1 \leq p \leq \infty$. *Then* $f$ *has a* boundary function $f_0 \in \mathrm{L}^p(\mathscr{X})$ *such that* $f(rz) \to f_0(z)$ *for a.e.* $z \in \mathbf{T}$, *as* $r \to 1-$. *Moreover,* $\|f_0\|_p = \|f\|_{\mathscr{H}^p} = \lim_{r\to 1-} \|f(r\cdot)\|_p$, *and* $f$ *is the Poisson integral of* $f_0$. *If* $p < \infty$, *then* $\|f(r\cdot) - f_0\|_p \to 0$ *as* $r \to 1-$.

*Proof.* Since $\mathbf{D}$ is separable, so is $f[\mathbf{D}]$. Therefore, we may assume that $\mathscr{X}$ is separable. Consequently, the proposition follows from [RR85, pp. 84 & 88–89]. □

By $\mathscr{H}^\infty(\mathscr{X},\mathscr{Y})$ we denote the Banach space of bounded holomorphic functions $\mathbf{D} \to \mathscr{B}(\mathscr{X},\mathscr{Y})$ with the supremum norm. It is a closed subspace of $\mathrm{L}^\infty_{\mathrm{strong}}$:

**Proposition 2.4.** ($\mathscr{H}^\infty \subset \mathrm{L}^\infty_{\mathrm{strong}}$) *Let* $F \in \mathscr{H}^\infty(\mathscr{X},\mathscr{Y})$. *Then there exists a unique* boundary function $[F_0] \in \mathrm{L}^\infty_{\mathrm{strong}}(\mathscr{X},\mathscr{Y})$ *such that for each* $x \in \mathscr{X}$ *there is a null set* $N_x \subset \mathbf{T}$ *for which* $F(rz)x \to F_0(z)x$ *in* $\mathscr{Y}$, *as* $r \to 1-$, *for each* $z \in \mathbf{T} \setminus N_x$.

*If* $f \in \mathscr{H}^p(\mathscr{X})$ *and* $G \in \mathscr{H}^\infty(\mathscr{Y},\mathscr{Z})$ ($1 \leq p \leq \infty$), *where also* $\mathscr{Z}$ *is a Hilbert space, with boundary functions* $f_0$ *and* $G_0$, *respectively, then the boundary functions of* $Ff$ *and* $GF$ *equal* $F_0 f_0$ *and* $G_0 F_0$, *respectively.*

(This follows from [Mik08, Theorem 1.5] through Lemma 6.1.)

We have $\|F(rz) - F(z)\|_{\mathscr{B}(\mathscr{X},\mathscr{Y})} \to 0$ for a.e. $z \in \mathbf{T}$ iff $F_0 \in \mathrm{L}^\infty(\mathscr{B}(\mathscr{X},\mathscr{Y}))$, or equivalently, iff $F_0$ is Bochner-measurable (use the Poisson integral formula or see [Mik07, Lemma A.8]). Nevertheless, $Fx$ is the Poisson integral of $F_0 x$ for each $x \in \mathscr{X}$.

We identify a function $F \in \mathscr{H}^\infty(\mathscr{X},\mathscr{Y})$ (or $F \in \mathscr{H}^p(\mathscr{X})$) by its boundary function (equivalence class) $F_0$, thus extending it to the boundary $\mathbf{T}$, even though the extension is unique only as a class. Therefore, $\mathscr{H}^p(\mathscr{X})$ (resp., $\mathscr{H}^\infty(\mathscr{X},\mathscr{Y})$) is considered as a subspace of $\mathrm{L}^p(\mathscr{X})$ (resp., of $\mathrm{L}^\infty_{\mathrm{strong}}(\mathscr{X},\mathscr{Y})$). Note that $\mathrm{L}^2(\mathscr{X})$ and

$\mathscr{H}^2(\mathscr{X})$ are Hilbert spaces. We consider $\mathscr{B}$ as the subspace of constant functions (in $\mathrm{L}^p$, $\mathrm{L}^p_{\mathrm{strong}}$, $\mathscr{H}^p$, $\mathscr{H}^p_{\mathrm{strong}}$ or similar).

In Section 3 we define $P_X$, $\tilde{P}_X$, $F_{X,Y}$ and $\mathscr{V}$, in Section 4 we define $F^*F$, $H^2_-$, $P_+$, $P_-$, $S$ and $S^*$, and in Section 5 we define $\mathbf{C}^+$, $\tau^t$, TI and TIC (and redefine $\mathscr{H}^p$, $\mathrm{L}^p$, $\mathscr{H}^p_{\mathrm{strong}}$, $\mathrm{L}^p_{\mathrm{strong}}$, $P_+$, $P_-$, $\Gamma_{\mathscr{E}}$, $\widehat{f}$, $\mathscr{F}$ etc. for Sections 5–6).

## 3. Diagonalization

In Theorem 3.2 we shall present in detail the diagonalization process explained around equation (1). Before that, in Theorem 3.1, we show how to combine such "diagonal blocks" "$T$" to an operator "$\mathscr{T}$".

Recall first that if the vectors $x_\alpha \in \mathscr{X}$ are orthogonal for each $\alpha \in \mathscr{A}$, then $x := \sum_{\alpha\in\mathscr{A}} x_\alpha$ converges in $\mathscr{X}$ iff $R := \sum_{\alpha\in\mathscr{A}} \|x_\alpha\|^2 < \infty$ (in particular, at most countably many $x_\alpha$ may be nonzero). If $R < \infty$, then $\|x\|^2 = R$. [Rud74, Theorem 12.6]

If $X$ (resp., $Y$) is a closed subspace of $\mathscr{X}$ (resp., $\mathscr{Y}$), then we denote the orthogonal projection $\mathscr{X} \to X$ by $P_X$ (resp., $\mathscr{Y} \to Y$ by $P_Y$). Thus, $P_Y^* \in \mathscr{B}(Y, \mathscr{Y})$ is the canonical isometric embedding $Y \to \mathscr{Y}$. By $\tilde{P}_X = \tilde{P}_X^* \in \mathscr{B}(\mathscr{X})$ we denote the zero extension of $P_X$ (similarly for $P_Y$).

We go on with some fairly obvious facts on "diagonal" operators (cf. (1)):

**Theorem 3.1.** ($\{F_{X,Y}\} \mapsto F$) *Let $\mathscr{V}$ a collection of pairs $(X,Y)$, where the spaces $X$ (resp., $Y$) are pairwise orthogonal closed subspaces of $\mathscr{X}$ (resp., $\mathscr{Y}$).*

*If $F_{X,Y} \in \mathscr{B}(X,Y)$ for all $(X,Y) \in \mathscr{V}$ and $M := \sup_{(X,Y)\in\mathscr{V}} \|F_{X,Y}\|_{\mathscr{B}(X,Y)} < \infty$, then $F := \sum_{(X,Y)\in\mathscr{V}} P_Y^* F_{X,Y} P_X$ satisfies*[1]

$$F \in \mathscr{B}(\mathscr{X},\mathscr{Y}), \qquad \|F\|_{\mathscr{B}(\mathscr{X},\mathscr{Y})} = M, \tag{7}$$

$$P_Y^* F_{X,Y} P_X = \tilde{P}_Y^* F \tilde{P}_X = \tilde{P}_Y^* F = F \tilde{P}_X \tag{8}$$

$$\langle y, Fx\rangle_{\mathscr{Y}} = \sum_{(X,Y)\in\mathscr{V}} \langle \tilde{P}_Y y, F\tilde{P}_X x\rangle_Y \tag{9}$$

*for all $x \in \mathscr{X}$, $y \in \mathscr{Y}$. Moreover, the map $(F_{X,Y})_{(X,Y)\in\mathscr{V}} \mapsto F$ is linear.*

(a1) *If $\sum_{(X,Y)\in\mathscr{V}} X = \mathscr{X}$, and $g \in \mathscr{X}$ or $g \in \mathscr{H}^2(\mathscr{X})$, then $g = \sum \tilde{P}_X g$, $\|g\|^2 = \sum \|P_X g\|^2$, and $Fg = \sum P_Y^* F_{X,Y} P_X g$. In particular, then $P_X g = 0$ for all but (at most) countably many $(X,Y) \in \mathscr{V}$. Conversely, if $g_X \in \mathscr{H}^2(X)$ for each $(X,Y) \in \mathscr{V}$ and $R := \sum \|g_X\|_2^2 < \infty$, then $g := \sum P_X^* g_X \in \mathscr{H}^2(\mathscr{X})$ and $\|g\|_2^2 = R$.*

(a2) *Let $\mathscr{Z}$ be a Hilbert space, let $Z_Y$ be pairwise orthogonal closed subspaces of $\mathscr{Z}$, and let $G_{Y,Z_Y} \in \mathscr{B}(Y,Z)$ for each $(X,Y) \in \mathscr{V}$, and $\sup_{(X,Y)\in\mathscr{V}} \|G_{Y,Z_Y}\| < \infty$, then $G := \sum P_{Z_Y}^* G_{Y,Z_Y} P_Y \in \mathscr{B}(\mathscr{Y},\mathscr{Z})$ and $GF = \sum P_{Z_Y}^* G_{Y,Z_Y} F_{X,Y} P_X$.*

(b1) *All of the above in this theorem also holds with $\mathscr{H}^\infty$ in place of $\mathscr{B}$.*

*Here the sum $F(z)x := \sum P_Y^* F_{X,Y}(z) P_X x$ converges for each $z \in \mathbf{D}$ and each $x \in \mathscr{X}$. Similarly, (9) holds pointwise everywhere on $\mathbf{D}$.*

Above and below all sums run over $\mathscr{V}$. By $\mathscr{X} = \sum X$ in (a1) above we mean that $x = \sum P_X x$ for each $x \in \mathscr{X}$ (so $\sum \|P_X x\|^2 = \|x\|^2 < \infty$), equivalently, that $\cap_{(X,Y)\in\mathscr{V}} X^\perp = \{0\}$.

*Proof of Theorem 3.1.* We start by proving all but (a1) and (a2) (i.e., just the initial claims for both $\mathscr{B}$ and $\mathscr{H}^\infty$). Note first that the linearity claim is obvious in

[1] The definition means that $Fx := \sum P_Y^* F_{X,Y} P_X x$ for each $x \in \mathscr{X}$.

all settings. Without loss of generality, we assume that $\sum_{(X,Y)\in\mathscr{V}} X = \mathscr{X}$ (otherwise we may replace $\mathscr{X}$ by the sum).

1° Let $x \in \mathscr{X}$ be arbitrary. From [Rud74, Theorem 12.6] we conclude that $\|x\|_{\mathscr{X}}^2 = \sum_{(X,Y)\in\mathscr{V}} \|P_X x\|_X^2$, and that $P_X x = 0$ for all but countably many $(X,Y) \in \mathscr{V}$. Since $\|P_Y^* F_{X,Y} P_X x\| \le M\|P_X x\|$, the sum $\sum_{(X,Y)\in\mathscr{V}} P_Y^* F_{X,Y} P_X x$ converges (so $F$ is well defined) and

$$\|Fx\|_{\mathscr{Y}}^2 = \sum_{(X,Y)\in\mathscr{V}} \|P_Y Fx\|_Y^2 \le \sum_{(X,Y)\in\mathscr{V}} M^2 \|P_X x\|_X^2 = M^2\|x\|_{\mathscr{X}}^2. \tag{10}$$

Clearly $F$ is also linear. Thus, $F \in \mathscr{B}$ and $\|F\|_{\mathscr{B}} \le M$; obviously, also $\|F\|_{\mathscr{B}} \ge M$.

Equation (8) is obvious. It follows that $\langle y, Fx\rangle_{\mathscr{Y}} = \sum_{\mathscr{V}} \langle y, P_Y^* F_{X,Y} P_X x\rangle = \sum_{\mathscr{V}} \langle y, \tilde{P}_Y^* F \tilde{P}_X x\rangle$, so also (9) holds.

2° *Case $\mathscr{H}^\infty$ in place of $\mathscr{B}$:* From 1° it follows that now $F$ is a function $\mathbf{D} \to \mathscr{B}(\mathscr{X},\mathscr{Y})$, bounded by $M$. If $(X,Y) \in \mathscr{V}$ and $x \in X$, then $\langle y, Fx\rangle_{\mathscr{Y}} = \langle P_Y y, F_{X,Y} x\rangle_Y$ is holomorphic for each $y \in \mathscr{Y}$, hence $F$ is holomorphic [HP57, Theorem 3.10.1] (because the span of such $x$ is dense in $\mathscr{X}$). Equations (8) and (9) obviously follow from 1° (applied to $F(z)$ for each $z \in \mathbf{D}$).

(a1) We prove the case $\mathscr{H}^\infty$ below; the other cases are analogous or easier.

For the case $g \in \mathscr{X}$, the claims were established in 1° (except the third one, which follows by definition). Assume then $g \in \mathscr{H}^2(\mathscr{X})$. Since (use Proposition 2.4)

$$\|g\|_2^2 = \frac{1}{2\pi}\int_{\mathbf{T}} \|g\|_{\mathscr{X}}^2\, dm = \frac{1}{2\pi}\int_{\mathbf{T}} \sum_{\mathscr{V}} \|P_X g\|_{\mathscr{X}}^2\, dm = \sum_{\mathscr{V}} \|P_X g\|_2^2, \tag{11}$$

we have $P_X g = 0$ for all but countably many $(X,Y) \in \mathscr{V}$.

From this, [Rud74, Theorem 12.6] and 1°–2° we now get $g = \sum \tilde{P}_X g$ and $Fg = \sum P_Y^* F_{X,Y} P_X g$ both in $\mathscr{H}^2$ and pointwise. The same holds for the converse claims.

(a2) The first claim was given in (7). If $x \in \mathscr{X}$, then

$$GFx = G\sum P_Y^* F_{X,Y} P_X x = \sum P_{Z_Y}^* G_{Y,Z_Y} F_{X,Y} P_X x. \tag{12}$$ □

Now we establish the diagonalization (1), i.e., the converse to Theorem 3.1:

**Theorem 3.2.** ($F \mapsto \{F_{X,Y}\}$) *Let $F\colon \mathbf{D} \to \mathscr{B}(\mathscr{X},\mathscr{Y})$ be continuous. Then there exists a collection $\mathscr{V}$ of pairs $(X,Y)$, where the spaces $X$ (resp., $Y$) are pairwise orthogonal closed separable subspaces of $\mathscr{X}$ (resp., $\mathscr{Y}$) such that $\tilde{P}_Y F \tilde{P}_X = \tilde{P}_Y F = F\tilde{P}_X$ for each $(X,Y) \in \mathscr{V}$, and $\mathscr{X} = \sum_{(X,Y)\in\mathscr{V}} X$, $\mathscr{Y} = \sum_{(X,Y)\in\mathscr{V}} Y$. If $\mathscr{X} = \mathscr{Y}$, then we can, in addition, have $X = Y$ for every $(X,Y) \in \mathscr{V}$.*

*Set $F_{X,Y} := P_Y F P_X^*$ for each $(X,Y) \in \mathscr{V}$. Then $F = \sum_{(X,Y)\in\mathscr{V}} P_Y^* F_{X,Y} P_X$.*

(The proof is given in Appendix B.)

Thus, if $F$ is a constant or if $F \in \mathscr{H}^\infty$, then Theorem 3.1 can be applied. The classes $\mathrm{L}^\infty_{\mathrm{strong}}$, $\mathscr{H}^2_{\mathrm{strong}}$, $\mathrm{L}^1_{\mathrm{strong}}$, $\mathscr{H}$, $\mathscr{B}(\mathrm{L}^2(\mathscr{X}), \mathrm{L}^2(\mathscr{Y}))$ and others are treated in [Mik07].

## 4. Results for the unit disc

In this section we extend to the nonseparable case some standard facts on the Hankel and Toeplitz operators of operator-valued $\mathscr{H}^\infty$ functions. We work on the unit disc (or circle); corresponding results for the half-plane (or real line) are given in Section 5.

As above, $\mathscr{X}$ and $\mathscr{Y}$ denote arbitrary (possibly nonseparable) Hilbert spaces. We start with a useful concept.

We call $F \in \mathscr{H}^\infty(\mathscr{X},\mathscr{Y})$ *inner* and write $F^*F = I$ if (i) below holds (recall Proposition 2.4). (One could show that $F^* \in \mathrm{L}^\infty_{\mathrm{strong}}$ etc. [Mik07], but we omit this.) There are several equivalent conditions for a function being inner:

**Theorem 4.1.** (Inner) *Let $F \in \mathscr{H}^\infty(\mathscr{X},\mathscr{Y})$. Then the claims (i)–(iii') below are equivalent:*

(i) $\|Fx\|_{\mathscr{Y}} = \|x\|_{\mathscr{X}}$ *a.e. on* $\mathbf{T}$ *for every* $x \in \mathscr{X}$*;*
(ii) $\langle Fx', Fx\rangle = \langle x', x\rangle$ *a.e. on* $\mathbf{T}$ *for every* $x, x' \in \mathscr{X}$*;*
(iii) $\|Ff\|_2 = \|f\|_2$ *for every* $f \in \mathscr{H}^2(\mathscr{X})$*;*
(iii') $\|Ff\|_2 = \|f\|_2$ *for every* $f \in \mathrm{L}^2(\mathscr{X})$*.*

*Moreover, the following holds:*

(a) *If $F$ and $\mathscr{V}$ are as in Theorem 3.1(b1), then $F^*F = I$ iff $F^*_{X,Y}F_{X,Y} = I$ for every $(X,Y) \in \mathscr{V}$.*

*Proof.* 1° Pick $\mathscr{V}$ as in Theorem 3.2. If $(X,Y) \in \mathscr{V}$ and $x \in X$, then $\|Fx\| = \|F_{X,Y}x\|$, so we get "only if". But if $\|F_{X,Y}x\| = \|x\|$ a.e. (i.e., $\|Fx\| = \|x\|$ a.e.) for every $x \in X$ for every $(X,Y) \in \mathscr{V}$, then $\|Fx\| = \|x\|$ for every $x \in \mathscr{X}$, by orthogonality. Indeed, with $x = \sum_k \alpha_k x_k$, $x_k \in X_k$, $(X_k, Y_k) \in \mathscr{V}$ for every $k$, we have

$$\|Fx\|^2_{\mathscr{Y}} = \|F\sum_k \alpha_k x_k\|^2 = \|\sum_k \alpha_k F x_k\|^2 \tag{13}$$

$$= \sum_k |\alpha_k|^2 \|Fx_k\|^2 = \sum_k |\alpha_k|^2 \|x_k\|^2 = \|x\|^2_{\mathscr{X}} \tag{14}$$

a.e. (see Theorem 3.1(a1)) when the sum is finite, hence always, by continuity. Thus, (a) holds.

2° Obviously, we have (ii)⇒(i)⇒(iii')⇒(iii). Assume then ¬(ii), i.e., that $F^*Fx \neq x$ on $E \subset \mathbf{T}$, where $m(E) > 0$. We may assume that $x \in X$, $(X,Y) \in \mathscr{V}$ as in (a), so ¬(iii) follows from the (well-known) separable case. □

In Example C.2 we construct an inner $F \in \mathscr{H}^\infty(\ell^2(\mathbf{T}), \ell^2(\mathbf{T}))$ for which there exists a unique "natural" boundary function $F_0 \in \mathrm{L}^\infty_{\mathrm{strong}}$ of $F$, namely the strong limit of $F$ everywhere on $\mathbf{T}$, and that function has $F_0(z)^*F_0(z) \neq I$ for every $z \in \mathbf{T}$. That cannot happen in the separable case.

It is well known that square-integrable functions on the unit circle $\mathbf{T}$ are exactly those with $\ell^2$ Laurent series coefficients:

$$\mathrm{L}^2(\mathscr{X}) = \{f = \sum_{k=-\infty}^{\infty} z^k x_k \mid \|f\|_2^2 := \sum_k \|x_k\|^2_{\mathscr{X}} < \infty\}. \tag{15}$$

Moreover, $\mathscr{H}^2$ (resp., $\mathscr{H}^2_-$) consists of those series where $x_k = 0$ for all $k < 0$ (resp., $k \geq 0$); cf. Proposition 2.3.

**Definition 4.2.** ($P_+$, $P_-$) By $P_+ : \sum_{k=-\infty}^{\infty} z^k x_k \mapsto \sum_{k=0}^{\infty} z^k x_k$ we denote the orthogonal projection $\mathrm{L}^2 \to \mathscr{H}^2$, and we set $P_- := I - P_+$.

The Corona Theorem says that if(f) $F \in \mathscr{H}^\infty(\mathscr{X},\mathscr{Y})$ and $F(z)^*F(z) \geq \varepsilon I$ for all $z \in \mathbf{D}$, then $F$ is left-invertible in $\mathscr{H}^\infty$. Unfortunately, the "if" part holds only when $\mathscr{X}$ is finite-dimensional [Tre89] (or trivially when $\dim \mathscr{Y} < \dim \mathscr{X}$). However, the coercivity of the anti-Toeplitz operator is always a sufficient and necessary condition for left-invertibility. Moreover, a related result, Tolokonnikov's Lemma, says that a left-invertible $\mathscr{H}^\infty$ function (as in (ii) below) can be complemented to an invertible one (as in (iii)):

**Theorem 4.3.** (Tolokonnikov) *Let $F \in \mathscr{H}^\infty(\mathscr{X},\mathscr{Y})$. Then the following are equivalent:*

(i) *The anti-Toeplitz operator $P_- F P_-$ is* coercive, *i.e., there exists $\varepsilon > 0$ such that for every $g \in \mathscr{H}^2_-(\mathscr{X})$ we have*

$$\|P_- F P_- g\|_2 \geq \varepsilon \|g\|_2. \tag{16}$$

(i') *The multiplication operator $M_{F^{\mathrm{d}}}$ by $F^{\mathrm{d}} := F(\bar{\cdot})^*$ maps $\mathscr{H}^2(\mathtt{Y})$ onto $\mathscr{H}^2(\mathtt{X})$.*

(ii) *$GF = I$ for some $G \in \mathscr{H}^\infty(\mathscr{Y},\mathscr{X})$.*

(iii) *There exist a closed subspace $\mathscr{Z} \subset \mathscr{Y}$ and a function $\tilde{F} \in \mathscr{H}^\infty(\mathscr{Z},\mathscr{X})$ such that $\begin{bmatrix} F & \tilde{F} \end{bmatrix} \in \mathscr{H}^\infty(\mathscr{X}\times\mathscr{Z},\mathscr{Y})$ is invertible in $\mathscr{H}^\infty$.*

*Assume (i). Then the best possible norm of a left-inverse $G$ in (ii) is $1/\varepsilon$ for the maximal $\varepsilon$ in (16). Set $M := \|F\|$. In (iii), (if $\mathscr{X} \neq \{0\}$) we can have $\|\tilde{F}\| = 1$, $\| \begin{bmatrix} F & \tilde{F} \end{bmatrix} \| \leq \sqrt{M^2+1}$, and*

$$\| \begin{bmatrix} F & \tilde{F} \end{bmatrix}^{-1} \|_{\mathscr{H}^\infty} \leq \frac{M}{\varepsilon}\sqrt{1+\varepsilon^{-2}}. \tag{17}$$

By $GF = I$ in (ii) we mean that $G(z)F(z) = I$ for each $z \in \mathbf{D}$, or equivalently, that $GFx = x$ a.e. on $\mathbf{T}$ for each $x \in \mathscr{X}$ (Proposition 2.4).

*Proof.* Observe first that (i') is equivalent to (i).

1° For the separable case, the equivalence of (i)–(iii) and the fact that we can have $\|G\| = 1/\varepsilon$ were established in Theorems 1.2 and 2.1 of [Tre04] (if we drop "$\subset \mathscr{Y}$"). We explain below the norm estimates for (iii) in the separable case.

By the proof of Lemma 6.1 of [Tre04], we have $\|\mathscr{P}\| \leq M/\varepsilon$. Therefore, $\|I - \mathscr{P}\| \leq M/\varepsilon$, by Lemma A.4 (if $\mathscr{X} \neq 0$). Since (in the middle of that proof) $\Theta$ is inner, we have $\|\Theta\| = 1$ and $\|R\| = \|\mathscr{Q}\| = \|I - \mathscr{P}\| \leq M/\varepsilon$ and $\|\tilde{F}\| \leq 1$ (near the end of the proof), hence $\| \begin{bmatrix} F & \tilde{F} \end{bmatrix} \| \leq \sqrt{M^2+1}$. As mentioned above, we can have $\|G\| = 1/\varepsilon$, which leads to

$$\|\tilde{G}\| = \| \begin{bmatrix} G\mathscr{P} \\ R \end{bmatrix} \| \leq \sqrt{(\varepsilon^{-1}\cdot\varepsilon^{-1}M)^2 + (\varepsilon^{-1}M)^2} = \varepsilon^{-1}M\sqrt{1+\varepsilon^{-2}}. \tag{18}$$

2° Since the implications (iii)⇒(ii)⇒(i) are obvious (take $\varepsilon := 1/\|P_- G P_-\|$ and note that $P_- G = P_- G P_-$, because $P_- G P_+ = 0$), we assume (i). Apply Theorem 3.2 to $F$, and then find (by 1° above), for each $(X,Y) \in \mathscr{V}$, a Hilbert space $Z_X$ and functions $\tilde{F}_{X,Y} \in \mathscr{H}^\infty(Z_X, Y)$, $K_{X,Y} \in \mathscr{H}^\infty(Y, X \times Z_X)$ such that

$$K_{X,Y} \begin{bmatrix} F_{X,Y} & \tilde{F}_{X,Y} \end{bmatrix} = I, \qquad \begin{bmatrix} F_{X,Y} & \tilde{F}_{X,Y} \end{bmatrix} K_{X,Y} = I, \tag{19}$$

$\|K_{X,Y}\| \leq \varepsilon^{-1}M\sqrt{1+\varepsilon^{-2}}$, and $\|\tilde{F}_{X,Y}\| \leq 1$. Let $\mathscr{Z} \subset \prod_{(X,Y)\in\mathscr{V}} Z_X$ be as in Lemma A.3, $\tilde{F} := \sum P_Y^* \tilde{F}_{X,Y} P_{Z_X}$ (Theorem 3.1(b1)), $K := \sum P_{X\times Z_X}^* K_{X,Y} P_Y$ to have

$$\|K\| \leq \varepsilon^{-1}M\sqrt{1+\varepsilon^{-2}}, \quad \|\tilde{F}\| \leq 1, \quad K \begin{bmatrix} F & \tilde{F} \end{bmatrix} = I, \quad \text{and} \quad \begin{bmatrix} F & \tilde{F} \end{bmatrix} K = I, \tag{20}$$

by Theorem 3.1(a2).

Now we have established (iii) except for $\mathscr{Z}$ being a subspace of $\mathscr{Y}$. But $\dim(\mathscr{X} \times \mathscr{Z}) = \dim \mathscr{Y}$, by Lemma A.2, so $\mathscr{Z}$ is isometrically isometric to a subspace, say $\tilde{\mathscr{Y}}$, of $\mathscr{Y}$. Let $T \in \mathscr{B}(\mathscr{Z}, \tilde{\mathscr{Y}})$ be such an isometry and replace $\tilde{F}$ by $\tilde{F}T$ and $\mathscr{Z}$ by $\tilde{\mathscr{Y}}$ to complete (iii).

3° *The estimate in (ii):* By Theorem 1.2 of [Tre04], we can have $\|G_{X,Y}\| \le 1/\varepsilon$ for a left inverse $G_{X,Y} \in \mathscr{H}^\infty(Y, X)$ of $F_{X,Y}$ (see 2°), for each $(X, Y) \in \mathscr{V}$. Apply Theorem 3.1(b1)&(a2) to obtain $G \in \mathscr{H}^\infty(\mathscr{Y}, \mathscr{X})$ such that $GF = I$ and $\|G\| \le 1/\varepsilon$. Obviously, $\varepsilon \ge 1/\|P_- G P_-\| \ge 1/\|G\|$, hence $\|G\| = 1/\varepsilon$ is the minimal norm of a left inverse. □

When $F^*F = I$, one more equivalent condition in Theorem 4.3 is that the Hankel norm of $F$ is less than one:

**Theorem 4.4.** *Let $F \in \mathscr{H}^\infty(\mathscr{X}, \mathscr{Y})$ be inner. Then the following are equivalent:*

(i) *The anti-Toeplitz operator $P_- F P_-$ is coercive (see* (16)*).*
(ii) $\|P_+ F P_-\| < 1$.

*If (ii) holds, then the best possible norm for a left inverse $G \in \mathscr{H}^\infty(\mathscr{Y}, \mathscr{X})$ of $F$ is given by $\|G\|^{-2} = 1 - \|P_+ F P_-\|^2$.*

(The Nehari Theorem [Mik07] says that (i) holds iff $d(F, \mathscr{H}^\infty_-) < 1$. Note that $P_+ F P_-$ is called the *Hankel operator* of $F$. The equivalence of (i) and (ii) actually holds for any $F \in \mathrm{L}^\infty_{\text{strong}}$ such that $F^*F = I$.)

*Proof.* 1° Since $F^*F = I$, we have

$$\|g\|^2 = \|Fg\|^2 = \|P_- F P_- g\|^2 + \|P_+ F P_- g\|^2 \tag{21}$$

for each $g \in \mathscr{H}^2_-(\mathscr{X})$, hence (i) is equivalent to (ii).

2° Assume (ii). By (21), we have $\varepsilon^2 = 1 - \|P_+ F P_-\|^2$ for the maximal $\varepsilon$ in (16), so the last claim follows from Theorem 4.3. □

We record an important special case of the last claim in Theorem 4.4:

**Corollary 4.5.** (Coprime) *Let $\mathscr{X}$, $\mathscr{Y}_1$, and $\mathscr{Y}_2$ be Hilbert spaces. If $\left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right] \in \mathscr{H}^\infty(\mathscr{X}, \mathscr{Y}_1 \times \mathscr{Y}_2)$ is inner, then $F$ and $G$ are right coprime iff $\|P_+ \left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right] P_-\| < 1$.*

Functions $F$ and $G$ being *right coprime* means that $\tilde{F}F + \tilde{G}G = I$ on $\mathbf{D}$ for some $\begin{bmatrix} \tilde{F} & \tilde{G} \end{bmatrix} \in \mathscr{H}^\infty(\mathscr{Y}_1 \times \mathscr{Y}_2, \mathscr{X})$. In systems theory, a "right fraction" $FG^{-1}$ is called "normalized" iff $\left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right]$ is inner [CO06]. Recall that $\|P_+ \left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right] P_-\|$ is the Hankel norm of $\left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right]$.

The *shift* $S$ is defined by $(Sf)(z) = zf(z)$. Recall that for $F \in \mathscr{H}^\infty(\mathscr{X}, \mathscr{Y})$, $f \in \mathscr{H}^2(\mathscr{X})$ we have set $(M_F f)(z) := F(z)f(z)$ for $z \in \mathbf{D}$, hence for a.e. $z \in \mathbf{T}$ too, by Proposition 2.4.

The first, third and fourth of the following "well-known" results are often phrased as "$\mathscr{H}^\infty$ consists of the causal maps on $\mathscr{H}^2$", "causal and anti-causal operators are (multiplications by) constants", and "inner-outer means constant":

**Proposition 4.6.** (Causal, anti-causal and inner-outer) *Let $T \in \mathscr{B}(\mathscr{H}^2(\mathscr{X}), \mathscr{H}^2(\mathscr{Y}))$. Then $T = M_F$ for some $F \in \mathscr{H}^\infty(\mathscr{X}, \mathscr{Y})$ iff $ST = TS$. Let $F \in \mathscr{H}^\infty(\mathscr{X}, \mathscr{Y})$. Then $F^{-1} \in \mathscr{H}^\infty(\mathscr{Y}, \mathscr{X})$ iff $M_F$ is invertible $\mathscr{H}^2(\mathscr{X}) \to \mathscr{H}^2(\mathscr{Y})$. If $F^* \in \mathscr{H}^\infty$, then $F \in \mathscr{B}(\mathscr{X}, \mathscr{Y})$. If $F$ is inner and $\overline{M_F[\mathscr{H}^2(\mathscr{X})]} = \mathscr{H}^2(\mathscr{Y})$, then $F = F^{-*} \in \mathscr{B}(\mathscr{X}, \mathscr{Y})$.*

The condition "$F^* \in \mathscr{H}^\infty$" can be interpreted as that the map $\mathbf{D} \ni z \mapsto F(z)^*$ is in $\mathscr{H}^\infty(\mathscr{Y},\mathscr{X})$, or equivalently, that $M_F^* = M_G$ for some $G \in \mathscr{H}^\infty$. Recall that $\mathscr{B}$ is identified with the subspace of constant functions.

*Proof of Proposition 4.6.* 1° The first claim is from Theorem 1.15B, p. 15 of [RR85]. There $\mathscr{X} = \mathscr{Y}$ is assumed, but one can consider $T$ (with zero extension) as an operator $\mathscr{H}^2(\mathscr{X}\times\mathscr{Y}) \to \mathscr{H}^2(\mathscr{X}\times\mathscr{Y})$ and then remove the zero extension of $F$.

2° If $F$ is invertible in $\mathscr{H}^\infty$, then, obviously, $M_{F^{-1}} = M_F^{-1}$, so assume that $M_F^{-1}$ exists. Now $M_F^{-1}Sf = M_F^{-1}SM_FM_F^{-1}f = M_F^{-1}M_FSM_F^{-1}f = SM_F^{-1}f\ \forall f \in \mathscr{H}^2(\mathscr{Y})$, hence $M_F^{-1} = M_G$ for some $G \in \mathscr{H}^\infty(\mathscr{Y},\mathscr{X})$, by 1°. Obviously, $G(z) = F(z)^{-1}$ ($z \in \mathbf{D}$).

3° The third claim follows from Theorem 1.15B, p. 15 of [RR85].

4° Assume now that $F$ is inner (so $M_F^*M_F = I$). Then $M_F[\mathscr{H}^2(\mathscr{X})]$ is closed, hence then $\overline{M_F[\mathscr{H}^2(\mathscr{X})]} = \mathscr{H}^2(\mathscr{Y})$ implies that $M_F$ is invertible (hence unitary), so $F^{-1} \in \mathscr{H}^\infty$, by 2°. But $M_{F^{-1}} = M_F^*M_FM_{F^{-1}} = M_F^*$, so $F \in \mathscr{B}$, by 3°, hence $F^* = F^{-1}$. □

"Has smaller range than" means "is divisible by":

**Theorem 4.7.** (Divisor) *Assume that $F \in \mathscr{H}^\infty(\mathscr{X},\mathscr{Y})$, $G \in \mathscr{H}^\infty(\mathscr{Z},\mathscr{Y})$ for some Hilbert space $\mathscr{Z}$, and $G$ is inner. Then $M_F[\mathscr{H}^2(\mathscr{X})] \subset M_G[\mathscr{H}^2(\mathscr{Z})]$ iff $G$ is a left divisor of $F$.*

The latter means that $F = GK$ for some $K \in \mathscr{H}^\infty(\mathscr{X},\mathscr{Z})$. If also $F$ is inner, then so is $K$.

*Proof.* "If" is obvious, so assume that $M_F[\mathscr{H}^2(\mathscr{X})] \subset M_G[\mathscr{H}^2(\mathscr{Z})]$. Then, for each $f \in \mathscr{H}^2(\mathscr{X})$ there exists a unique $g_f \in \mathscr{H}^2(\mathscr{Z})$ such that $M_Gg_f = M_Ff$; left $T$ denote the map $f \mapsto g_f$. Then $M_F = M_GT$, $\|T\| \le \|F\|$, $T\colon \mathscr{H}^2(\mathscr{X}) \to \mathscr{H}^2(\mathscr{Z})$ is linear, and $M_FSf = SM_Ff = SM_GTf = M_GSTf$, hence $g_{Sf} = STf$, i.e., $TSf = STf$, for every $f \in \mathscr{H}^2(\mathscr{X})$. By Proposition 4.6, $T \in \mathscr{H}^\infty(\mathscr{X},\mathscr{Y})$. □

We call $\mathscr{M} \subset \mathscr{H}^2(\mathscr{X})$ *shift-invariant* if $S\mathscr{M} = \mathscr{M}$, where $(Sf)(z) := zf(z)$. Such subspaces are ranges of "unique" inner functions:

**Theorem 4.8.** (Lax–Halmos) *A closed subspace $\mathscr{M}$ of $\mathscr{H}^2(\mathscr{X})$ is shift-invariant iff we have $\mathscr{M} = M_F[\mathscr{H}^2(\mathscr{X}_0)]$ for some closed subspace $\mathscr{X}_0 \subset \mathscr{X}$ and some inner $F \in \mathscr{H}^\infty(\mathscr{X}_0,\mathscr{X})$.*

*If also $\mathscr{M} = M_G[\mathscr{H}^2(\mathscr{X}_1)]$ for some Hilbert space $\mathscr{X}_1$ and some inner $G \in \mathscr{H}^\infty(\mathscr{X}_1,\mathscr{X})$, then $G = FT$ for some $T = T^{-*} \in \mathscr{B}(\mathscr{X}_1,\mathscr{X}_0)$.*

*Proof.* 1° *Existence:* "If" is obvious ($SM_Fg = M_FSg \in M_F[\mathscr{H}^2(\mathscr{X}_0)]$ for each $g \in \mathscr{H}^2(\mathscr{X}_0)$), so we only prove "only if". Let $\mathscr{V}$ be as in Theorem 3.2 (with $0\colon \mathbf{D} \to \mathscr{B}(\mathscr{X},\mathscr{X})$ in place of $F$, because we just need a complete collection of separable orthogonal subspaces). For each $(X,X) \in \mathscr{V}$, the subspace $\mathscr{M}_X := \mathscr{M} \cap \mathscr{H}^2(X)$ is closed and shift-invariant, hence $\mathscr{M}_X = F_X[\mathscr{H}^2(\mathscr{X}_X)]$ for some closed $\mathscr{X}_X \subset X$ and some inner $F_X \in \mathscr{H}^\infty(\mathscr{X}_X, X)$, by the separable case of this theorem (e.g., p. 17 of [Nik86]).

Define $\mathscr{Z} \subset \prod_{(X,X)\in\mathscr{V}} \mathscr{X}_X$ as in Lemma A.3. Then $F := \sum P_X^*F_XP_{\mathscr{X}_X} \in \mathscr{H}^\infty(\mathscr{Z},\mathscr{X})$ is inner, by Theorem 3.1(a2) (we have a priori $\|F\|_{\mathscr{H}^\infty} \le 1 < \infty$ because $\|F_X\|_{\mathscr{H}^\infty} \le 1$, for each $X$, because $F_X$ is inner, hence (a2) is applicable). Given $g \in \mathscr{M}$ and $(X,X) \in \mathscr{V}$, we have $P_Xg \in \mathscr{M}_X$, hence $P_Xg = F_Xf_X$ for some

$f_X \in \mathscr{H}^2(\mathscr{X}_X)$. But $\|f_X\|_2 = \|P_X g\|_2$, because $F_X$ is inner. By Theorem 3.1(a1), $f := \sum P_X^* f_X \in \mathscr{H}^2(\mathscr{X})$, $\|f\|_2 = \|g\|_2$, and

$$Ff = \sum P_X^* F_X f = \sum P_X^* P_X g = \sum \tilde{P}_X g = g. \tag{22}$$

Thus, $\mathscr{M} \subset F[\mathscr{H}^2(\mathscr{Z})]$. Conversely, given $f \in \mathscr{H}^2(\mathscr{Z})$, from Theorem 3.1(a1) we see that $Ff = \sum P_X^* F_X P_X f \in \mathscr{M}$, hence $\mathscr{M} = F[\mathscr{H}^2(\mathscr{Z})]$.

By Lemma A.2, $\mathscr{Z}$ is unitarily equivalent to a closed subspace $\mathscr{X}_0$ of $\mathscr{X}$, so we can replace $F$ by $FT^{-1}$, where $T = T^{-*} \in \mathscr{B}(\mathscr{Z}, \mathscr{X}_0)$, because $FT^{-1}[\mathscr{H}^2(\mathscr{X}_0)] = F[\mathscr{H}^2(\mathscr{Z})] = \mathscr{M}$.

2° *Uniqueness:* By Theorem 4.7, $G = FT$, where $T \in \mathscr{H}^\infty(\mathscr{X}_1, \mathscr{X}_0)$. But $T$ is inner and $M_T$ is onto, hence $T = T^{-*} \in \mathscr{B}(\mathscr{X}_1, \mathscr{X}_0)$, by Proposition 4.6. □

A map $F \in \mathscr{H}^2_{\text{strong}}(\mathscr{X}, \mathscr{Y})$ is called *outer* if the set $\{Fp \mid p \in \mathscr{P}(\mathscr{X})\}$ is dense in $\mathscr{H}^2(\mathscr{Y})$; here $\mathscr{P}(\mathscr{X})$ stands for the polynomials, i.e., for functions of the form $\sum_{k=0}^n z^k x_k$. If $F \in \mathscr{H}^\infty(\mathscr{X}, \mathscr{Y})$, then, obviously, $F$ is outer iff $M_F[\mathscr{H}^2(\mathscr{X})]$ is dense in $\mathscr{H}^2(\mathscr{Y})$.

**Theorem 4.9.** (Inner-Outer Factorization) *Every $F \in \mathscr{H}^2_{\text{strong}}(\mathscr{X}, \mathscr{Y})$ can be expressed as $F = F_i F_o$, where $F_o \in \mathscr{H}^2_{\text{strong}}(\mathscr{X}, \mathscr{Y}_0)$ is outer and $F_i \in \mathscr{H}^\infty(\mathscr{Y}_0, \mathscr{Y})$ is inner, $\mathscr{Y}_0$ being a closed subspace of $\mathscr{Y}$. Moreover, $\|F_o\|_{\mathscr{H}^2_{\text{strong}}} = \|F\|_{\mathscr{H}^2_{\text{strong}}}$ and $\|F_o\|_{\mathscr{H}^\infty} = \|F\|_{\mathscr{H}^\infty} \le \infty$.*

*If also $F = F_i' F_o'$, where $F_o' \in \mathscr{H}^2_{\text{strong}}(\mathscr{X}, \mathscr{Z}')$ is outer and $F_i' \in \mathscr{H}^\infty(\mathscr{Z}', \mathscr{Y})$ is inner, $\mathscr{Z}'$ being a Hilbert space, then there exists $T = T^{-*} \in \mathscr{B}(\mathscr{Z}', \mathscr{Y}_0)$ such that $F_i' = F_i T$ and $F_o' = T^* F_o$.*

(Because $F_i$ is inner, we have $\|F_o\| = \|F\|$ for almost any reasonable norm.)

*Proof.* Also this could be deduced from the separable case. However, we shall deduce this from Theorem 4.8.

Since $M_F[\mathscr{P}(\mathscr{X})]$ is a shift-invariant subspace of $\mathscr{H}^2(\mathscr{Y})$, so is its closure, which equals $M_{F_i}[\mathscr{H}^2(\mathscr{Y}_0)]$ for some closed subspace $\mathscr{Y}_0 \subset \mathscr{Y}$ and some inner $F_i \in \mathscr{H}^\infty(\mathscr{Y}_0, \mathscr{X})$, by Theorem 4.8. For each $x \in \mathscr{X}$, there exists a unique $f_x \in \mathscr{H}^2(\mathscr{Y}_0)$ such that $Fx = M_{F_i} f_x$. The map $T\colon x \mapsto f_x$ is linear, hence so is $F_o(z)\colon x \mapsto f_x(z)$, for each $z \in \mathbf{D}$. By [HP57, Theorem 3.10.1], $F_o\colon \mathbf{D} \to \mathscr{B}(\mathscr{X}, \mathscr{Y}_0)$ is holomorphic. Obviously, $\|f_x\|_2 = \|Fx\|_2$ for every $x$, hence $\|F_o\|_{\mathscr{H}^2_{\text{strong}}} = \|F\|_{\mathscr{H}^2_{\text{strong}}}$. By the continuity of $M_{F_i}$, we have

$$\overline{M_{F_i}\overline{M_{F_o}[\mathscr{P}(\mathscr{X})]}}, = \overline{M_{F_i} M_{F_o}[\mathscr{P}(\mathscr{X})]} = \overline{M_F[\mathscr{P}(\mathscr{X})]} = M_{F_i}[\mathscr{H}^2(\mathscr{Y}_0)] \tag{23}$$

hence the function $F_o$ must be outer.

By Theorem 4.8, $F_i' = F_i T$ for any other inner-outer factorization $F = F_i' F_o'$ of $F$. But then, for $z \in \mathbf{D}$, we have $T^* F_o(z) = T^* F_i(z)^* F(z) = (F_i'(z))^* F(z) = F_o'(z)$. □

Assume that $F$, $F_o$ and $F_i$ are as above and $F \in \mathscr{H}^\infty$. Then $F^*F = F_o^* F_o$ in $\mathrm{L}^\infty_{\text{strong}}$ in the sense that $\|Ff\|_{\mathscr{Y}} = \|F_o f\|_{\mathscr{Y}_0}$ a.e. on $\mathbf{T}$ for every $f \in \mathscr{H}^2(\mathscr{X})$ (hence for every $f \in \mathscr{X}$). Moreover, $M_F^* M_F \ge \varepsilon I$ for some $\varepsilon > 0$ (i.e., $F$ is left-invertible in $\mathrm{L}^\infty_{\text{strong}}$) iff $F_o$ is invertible in $\mathscr{H}^\infty$. If it is, then $F_o$ is called a (invertible) *spectral factor* of $F^*F$. (All this is well known and the claims follow easily from the above.)

## 5. Results for the half-plane

In this section we present results analogous to those in the previous sections but with the real line $\mathbf{R}$ (resp., half-plane $\mathbf{C}^+ := \{z \in \mathbf{C} \mid \operatorname{Im} z > 0\}$) in place of the unit circle $\mathbf{T}$ (resp., disc $\mathbf{D}$). The main difference is that we want to use the translations $\tau^t \colon f \mapsto f(t + \cdot)$ instead of the right-shift $S$. The *half-plane notation* used in this section differs from the *disc notation* of the previous sections.

We first state that the results in the previous sections hold with this notation too (Lemma 5.1). Then we rewrite those corresponding to Section 4 to their "time-domain" forms, using the "Fourier multiplier" result that the elements of $\mathrm{L}^\infty_{\text{strong}}$ (resp., $\mathscr{H}^\infty$) correspond isometrically to the time-invariant (resp., causal) operators $\mathrm{L}^2(\mathscr{X}) \to \mathrm{L}^2(\mathscr{Y})$. One easily verifies that such "time-domain" forms could be used in discrete time too, on operators $\ell^2(\mathscr{X}) \to \ell^2(\mathscr{Y})$ (cf. Remarks 2.1). Most comments and explaining text in Section 4 applies here too. The proofs are given in Section 6.

We start by presenting some of this half-plane notation. Let $B$ be a Banach space and let $1 \le p \le \infty$. By $\mathrm{L}^p(B)$ we denote the $\mathrm{L}^p$ space of functions $\mathbf{R} \to B$. By $\mathscr{H}^p(B)$ we denote the Banach space of holomorphic functions $f \colon \mathbf{C}^+ \to B$ for which

$$\|f\|_{\mathscr{H}^p} := \sup_{r>0} \|f(\cdot + ir)\|_{\mathrm{L}^p} < \infty. \tag{24}$$

By $P_+$ we denote the orthogonal projection $\mathrm{L}^2 \to \mathscr{H}^2$ for any Hilbert space $H$. Again $P_- := I - P_+$, $\mathscr{H}^2_- := P_-[\mathrm{L}^2]$. The $\mathrm{L}^p_{\text{strong}}$ and $\mathscr{H}^p_{\text{strong}}$ spaces are defined as before (now on $\mathbf{R}$ and on $\mathbf{C}^+$, respectively).

We now record the fact that all above results hold with this half-plane notation too (the remaining definitions will follow).

**Lemma 5.1.** *Propositions 2.2–2.4 hold with this notation too except that we must replace* $1-$ *(resp.,* $r\cdot$*,* $rz$*) by* $0+$ *(resp.,* $\cdot + ir$*,* $z + ir$*), and that the Poisson integral is different* [RR85], [Mik08]. *Also the results in Section 3 hold with this notation.*

*The results in Section 4 hold with this notation too except that we reformulate Proposition 4.6 and Theorems 4.8–4.9 as given below in Proposition 5.8 and in Theorems 5.10–5.11, and that* $\mathscr{P}$ *must be replaced by* $\tilde{\mathscr{P}} := \Diamond^{-1}\mathscr{P}$*, which will be defined below Remark 6.2.*

*In all above results, we assume that* $\mathbf{T}$ *(resp.,* $\mathbf{D}$*) has been replaced by* $\mathbf{R}$ *(resp.,* $\mathbf{C}^+$*).*

(The proof is given in Lemma 6.3. Alternative explicit versions of the results in Section 4 are given below.)

Next we present the *time-domain* concepts corresponding to those above. Analogous concepts also exist in discrete time (corresponding to the "disc notation"), but they are more useful in continuous time, since the translations $\tau^t$ do not have nice "frequency-domain" (Fourier/Laplace side) equivalents like the multiplication by $z$ corresponding to the discrete-time shift $S$.

By $\mathrm{TI}(\mathscr{X}, \mathscr{Y})$ we denote the operators $\mathscr{E} \colon \mathrm{L}^2(\mathscr{X}) \to \mathrm{L}^2(\mathscr{Y})$ that are *translation-invariant*: $\mathscr{E}\tau^t = \tau^t\mathscr{E}$ for every $t \in \mathbf{R}$.

We set $\pi_+ f := \begin{cases} f(t), & t \ge 0; \\ 0, & t < 0 \end{cases}$, $\pi_- := I - \pi_+$. We identify any function $f$ defined on $\mathbf{R}_+ := [0, \infty)$ with its zero extension to $\mathbf{R}$. By $\mathrm{L}^2_+(\mathscr{X})$ we denote the Hilbert space of $\mathrm{L}^2(\mathscr{X})$ functions supported on $\mathbf{R}_+$, and by $\mathrm{L}^2_-(\mathscr{X})$ the orthogonal complement of $\mathrm{L}^2(\mathscr{X})$.

By $\mathrm{TIC}(\mathscr{X},\mathscr{Y})$ we denote the operators $\mathscr{D} \in \mathrm{TI}(\mathscr{X},\mathscr{Y})$ that are *causal*: $\pi_-\mathscr{D}\pi_+ = 0$. Both $\mathrm{TI}(\mathscr{X},\mathscr{Y})$ and $\mathrm{TIC}(\mathscr{X},\mathscr{Y})$ are obviously closed subspaces of $\mathscr{B}(\mathrm{L}^2(\mathscr{X}), \mathrm{L}^2(\mathscr{Y}))$; whose norm we use.

By $\mathscr{F}f := \widehat{f}(s) := \int_{-\infty}^{\infty} \mathrm{e}^{-ist} f(t)\, dt$ we denote the Fourier–Laplace transform of a function for those $s \in \mathbf{C}$ for which $\widehat{f}(s)$ converges absolutely. If $f \in \mathrm{L}^1$, then $\widehat{f} \in \mathrm{L}^\infty$. This extends to a unitary map $\mathrm{L}^2(\mathscr{X}) \to \mathrm{L}^2(\mathscr{X})$ satisfying $\mathscr{F}\pi_+ = P_+\mathscr{F}$. Thus, if $f \in \mathrm{L}^2_+(\mathscr{X})$, then $\widehat{f} \in \mathscr{H}^2(\mathscr{X})$, and $\widehat{f}_{|\mathbf{R}}$ coincides with the boundary function of $\widehat{f}_{|\mathbf{C}^+}$.

Now we recall the extension to general Hilbert spaces of the standard $\mathrm{L}^2$ Fourier multiplier result [Mik08, Theorem 1.2].

**Theorem 5.2.** ($\widehat{\mathrm{TI}} = \mathrm{L}^\infty_{\mathrm{strong}}$) *For each $\mathscr{E} \in \mathrm{TI}(\mathscr{X},\mathscr{Y})$ there exists a unique function (equivalence class) $\hat{\mathscr{E}} \in \mathrm{L}^\infty_{\mathrm{strong}}(\mathscr{X},\mathscr{Y})$ such that $\hat{\mathscr{E}}\widehat{f} = \widehat{\mathscr{E}f}$ a.e. on $\mathbf{R}$ for every $f \in \mathrm{L}^2(\mathscr{X})$. Moreover, $\|\hat{\mathscr{E}}\|_{\mathrm{L}^\infty_{\mathrm{strong}}} = \|\mathscr{E}\|_{\mathscr{B}(\mathrm{L}^2(\mathscr{X}),\mathrm{L}^2(\mathscr{Y}))}$, and every $\hat{\mathscr{E}} \in \mathrm{L}^\infty_{\mathrm{strong}}(\mathscr{X},\mathscr{Y})$ is of this form.*

The following is well known [Wei91]:

**Proposition 5.3.** ($\widehat{\mathrm{TIC}} = \mathscr{H}^\infty$) *For any $\mathscr{D} \in \mathrm{TIC}(\mathscr{X},\mathscr{Y})$ there exists a unique function $\hat{\mathscr{D}} \in \mathscr{H}^\infty(\mathscr{X},\mathscr{Y})$ such that $(\widehat{\mathscr{D}f})(z) = \hat{\mathscr{D}}(z)\widehat{f}(z)$ for all $z \in \mathbf{C}^+$ and all $f \in \mathrm{L}^2_+(\mathscr{X})$.*

*Moreover, this identification is an isometric isomorphism of* TIC *onto $\mathscr{H}^\infty$.*

Naturally, the strong boundary function $\lim_{r\to 0+} \hat{\mathscr{D}}(\cdot + ir)$ equals that given by Theorem 5.2, analogously to Proposition 2.4. We identify $\hat{\mathscr{D}}_{|\mathbf{C}^+}$ with $\hat{\mathscr{D}}_{|\mathbf{R}}$.

We call $\mathscr{D} \in \mathrm{TIC}(\mathscr{X},\mathscr{Y})$ *inner* if $\mathscr{D}^*\mathscr{D} = I$ (on $\mathrm{L}^2$, or equivalently, on $\mathrm{L}^2_+$).

**Theorem 5.4.** (Inner) *Let $\mathscr{D} \in \mathrm{TIC}(\mathscr{X},\mathscr{Y})$. Then the claims (i)–(iii') below are equivalent:*

(i) $\|\hat{\mathscr{D}}x\|_{\mathscr{Y}} = \|x\|_{\mathscr{X}}$ *a.e. on $\mathbf{R}$ for every $x \in \mathscr{X}$;*
(ii) $\langle\hat{\mathscr{D}}x', \hat{\mathscr{D}}x\rangle = \langle x', x\rangle$ *a.e. on $\mathbf{R}$ for every $x, x' \in \mathscr{X}$;*
(iii) $\|\mathscr{D}f\|_2 = \|f\|_2$ *for every $f \in \mathrm{L}^2_+(\mathscr{X})$;*
(iii') $\mathscr{D}^*\mathscr{D} = I$ *(on $\mathrm{L}^2(\mathscr{X})$).*

(Recall that further equivalent conditions are given in Theorem 4.1, by Lemma 5.1, with $F := \hat{\mathscr{D}}$ and with $\mathbf{R}$ in place of $\mathbf{T}$.)

The reflection $\mathscr{R}$ is defined by $(\mathscr{R}f)(t) := f(-t)$. On (i') below note that $\mathscr{E}^{\mathrm{d}} \in \mathrm{TI}$, $\mathscr{D}^{\mathrm{d}} \in \mathrm{TIC}$, $(\mathscr{E}^{\mathrm{d}})^{\mathrm{d}} = \mathscr{E}$ and $\mathscr{F}(\mathscr{E}^{\mathrm{d}}) = \hat{\mathscr{E}}(\bar{\cdot})^*$ for every $\mathscr{E} \in \mathrm{TI}$, $\mathscr{D} \in \mathrm{TIC}$.

**Theorem 5.5.** (Tolokonnikov) *Let $\mathscr{D} \in \mathrm{TIC}(\mathscr{X},\mathscr{Y})$. Then the following are equivalent:*

(i) *The anti-Toeplitz operator $\pi_-\mathscr{D}\pi_-$ is* coercive, *i.e., there exists $\varepsilon > 0$ such that for each $g \in \mathrm{L}^2_-(\mathscr{X})$ we have*

$$\|\pi_-\mathscr{D}\pi_-g\|_2 \geq \varepsilon\|g\|_2. \tag{25}$$

(i') *The "dual" map $\mathscr{D}^{\mathrm{d}} := \mathscr{R}\mathscr{D}^*\mathscr{R}$ maps $\mathrm{L}^2_+(\mathtt{Y})$ onto $\mathrm{L}^2_+(\mathtt{X})$.*
(ii) *$\mathscr{G}\mathscr{D} = I$ for some $\mathscr{G} \in \mathrm{TIC}(\mathscr{Y},\mathscr{X})$.*

(iii) *There exist a closed subspace* $\mathscr{Z} \subset \mathscr{Y}$ *and a map* $\tilde{\mathscr{D}} \in \mathrm{TIC}(\mathscr{Z}, \mathscr{X})$ *such that* $\begin{bmatrix} \mathscr{D} & \tilde{\mathscr{D}} \end{bmatrix} \in \mathrm{TIC}(\mathscr{X} \times \mathscr{Z}, \mathscr{Y})$ *is invertible.*

*Assume (i). Then the best possible norm of a left-inverse* $\mathscr{G}$ *in (ii) is* $1/\varepsilon$ *for the maximal* $\varepsilon$ *in (25). Set* $M := \|\mathscr{D}\|$. *In (iii), we can have* $\|\tilde{\mathscr{D}}\| = 1$, $\| \begin{bmatrix} \mathscr{D} & \tilde{\mathscr{D}} \end{bmatrix} \| \leq \sqrt{M^2+1}$, *and (if* $\mathscr{X} \neq \{0\}$*)*

$$\| \begin{bmatrix} \mathscr{D} & \tilde{\mathscr{D}} \end{bmatrix}^{-1} \|_{\mathscr{H}^\infty} \leq \frac{M}{\varepsilon}\sqrt{1+\varepsilon^{-2}}. \tag{26}$$

If, in Theorem 5.5, we have $\mathscr{D}^*\mathscr{D} = I$, then one more equivalent condition is that the Hankel norm of $\mathscr{D}$ is less than one.

**Theorem 5.6.** *Let* $\mathscr{E} \in \mathrm{TI}(\mathscr{X}, \mathscr{Y})$ *and* $\mathscr{E}^*\mathscr{E} = I$. *Then the following are equivalent:*

(i) *The anti-Toeplitz operator* $\pi_-\mathscr{E}\pi_-$ *is coercive (see* (25)*).*
(ii) $\|\pi_+\mathscr{E}\pi_-\| < 1$.

*If* $\mathscr{E} \in \mathrm{TIC}(\mathscr{X}, \mathscr{Y})$, $\mathscr{E}^*\mathscr{E} = I$, *and (ii) holds, then the best possible norm for a left inverse* $\mathscr{G} \in \mathrm{TIC}(\mathscr{Y}, \mathscr{X})$ *of* $\mathscr{E}$ *is given by* $\|\mathscr{G}\|^{-2} = 1 - \|\pi_+\mathscr{E}\pi_-\|^2$.

*Right coprime* means below that $\mathscr{P}\mathscr{M} + \mathscr{Q}\mathscr{N} = I$ for some $\mathscr{P}, \mathscr{Q} \in \mathrm{TIC}$, i.e., that $\hat{\mathscr{P}}\hat{\mathscr{M}} + \hat{\mathscr{Q}}\hat{\mathscr{N}} \equiv I$ on $\mathbf{C}^+$ for some $\hat{\mathscr{P}}, \hat{\mathscr{Q}} \in \mathscr{H}^\infty$.

**Corollary 5.7.** (Coprime) *Let* $\mathscr{X}, \mathscr{Y}_1, \mathscr{Y}_2$ *be Hilbert spaces. If a map* $\left[\begin{smallmatrix} \mathscr{N} \\ \mathscr{M} \end{smallmatrix}\right] \in \mathrm{TIC}(\mathscr{X}, \mathscr{Y}_1 \times \mathscr{Y}_2)$ *is inner, then* $\mathscr{N}$ *and* $\mathscr{M}$ *are right coprime iff* $\|\pi_+ \left[\begin{smallmatrix} \mathscr{N} \\ \mathscr{M} \end{smallmatrix}\right] \pi_-\| < 1$.

An operator $\mathscr{D} \in \mathrm{TI}$ is uniquely determined by its Toeplitz operator $\pi_+\mathscr{D}\pi_+$ (or by $P_+\hat{\mathscr{D}}P_+$). Moreover, the following hold.

**Proposition 5.8.** (Causal, anti-causal and inner-outer) *Let* $\mathscr{D} \in \mathscr{B}(\mathrm{L}^2(\mathscr{X}), \mathrm{L}^2(\mathscr{Y}))$. *Then* $\mathscr{D} \in \mathrm{TIC}(\mathscr{X}, \mathscr{Y})$ *iff* $\pi_+\tau^t\mathscr{D}\pi_+ = \pi_+\mathscr{D}\tau^t\pi_+$ *for every* $t < 0$. *Let* $\mathscr{D} \in \mathrm{TIC}(\mathscr{X}, \mathscr{Y})$. *Then* $\mathscr{D}^{-1} \in \mathrm{TIC}(\mathscr{Y}, \mathscr{X})$ *iff* $\pi_+\mathscr{D}\pi_+$ *is invertible* $\mathrm{L}^2_+(\mathscr{X}) \to \mathrm{L}^2_+(\mathscr{Y})$. *Moreover, if* $\mathscr{D}, \mathscr{D}^* \in \mathrm{TIC}$, *then* $\mathscr{D} \in \mathscr{B}(\mathscr{X}, \mathscr{Y})$. *If* $\mathscr{D} \in \mathrm{TIC}(\mathscr{X}, \mathscr{Y})$, $\mathscr{D}^*\mathscr{D} = I$ *and* $\overline{\mathscr{D}[\mathrm{L}^2_+(\mathscr{X})]} = \mathrm{L}^2_+(\mathscr{Y})$, *then* $\mathscr{D} = \mathscr{D}^{-*} \in \mathscr{B}(\mathscr{X}, \mathscr{Y})$.

**Theorem 5.9.** (Divisor) *Assume that* $\mathscr{D} \in \mathrm{TIC}(\mathscr{X}, \mathscr{Y})$, $\mathscr{G} \in \mathrm{TIC}(\mathscr{Z}, \mathscr{Y})$ *for some Hilbert space* $\mathscr{Z}$, *and* $\mathscr{G}^*\mathscr{G} = I$. *Then* $\mathscr{D}[\mathrm{L}^2_+(\mathscr{X})] \subset \mathscr{G}[\mathrm{L}^2_+(\mathscr{Z})]$ *iff* $\mathscr{G}$ *is a left divisor of* $\mathscr{D}$.

The latter means that $\mathscr{D} = \mathscr{G}\mathscr{K}$ for some $\mathscr{K} \in \mathrm{TIC}(\mathscr{X}, \mathscr{Z})$. If $\mathscr{D}^*\mathscr{D} = I$, then $\mathscr{K}^*\mathscr{K} = I$.

We call $\mathscr{M} \subset \mathrm{L}^2_+(\mathscr{X})$ *translation-invariant* if $\tau^t\mathscr{M} = \mathscr{M}$ $(t < 0)$.

**Theorem 5.10.** (Lax–Halmos) *A closed subspace* $\mathscr{M}$ *of* $\mathrm{L}^2_+(\mathscr{X})$ *is translation-invariant iff* $\mathscr{M} = \mathscr{D}[\mathrm{L}^2_+(\mathscr{X}_0)]$ *for some closed subspace* $\mathscr{X}_0 \subset \mathscr{X}$ *and some inner* $\mathscr{D} \in \mathrm{TIC}(\mathscr{X}_0, \mathscr{X})$.

*If also* $\mathscr{M} = \mathscr{G}[\mathrm{L}^2_+(\mathscr{X}_1)]$ *for some Hilbert space* $\mathscr{X}_1$ *and some inner* $\mathscr{G} \in \mathrm{TIC}(\mathscr{X}_1, \mathscr{X})$, *then* $\mathscr{G} = \mathscr{D}T$ *for some* $T = T^{-*} \in \mathscr{B}(\mathscr{X}_1, \mathscr{X}_0)$.

A map $\mathscr{D} \in \mathrm{TIC}(\mathscr{X}, \mathscr{Y})$ is called *outer* if $\mathscr{D}[\mathrm{L}^2_+(\mathscr{X})]$ is dense in $\mathrm{L}^2_+(\mathscr{Y})$.

**Theorem 5.11.** (Inner-Outer Factorization) *Every* $\mathscr{D} \in \mathrm{TIC}(\mathscr{X}, \mathscr{Y})$ *can be expressed as* $\mathscr{D} = \mathscr{D}_i\mathscr{D}_o$, *where* $\mathscr{D}_o \in \mathrm{TIC}(\mathscr{X}, \mathscr{Y}_0)$ *is outer and* $\mathscr{D}_i \in \mathrm{TIC}(\mathscr{Y}_0, \mathscr{Y})$ *is inner,* $\mathscr{Y}_0$ *being a closed subspace of* $\mathscr{Y}$. *Moreover,* $\|\mathscr{D}_o\|_{\mathrm{TIC}} = \|\mathscr{D}\|_{\mathrm{TIC}}$.

*If also $\mathscr{D} = \mathscr{D}_i'\mathscr{D}_o'$, where $\mathscr{D}_o' \in \mathrm{TIC}(\mathscr{X}, \mathscr{Z}')$ is outer and $\mathscr{D}_i' \in \mathrm{TIC}(\mathscr{Z}', \mathscr{Y})$ is inner, $\mathscr{Z}'$ being a Hilbert space, then there exists $T = T^{-*} \in \mathscr{B}(\mathscr{Z}', \mathscr{Y}_0)$ such that $\mathscr{D}_i' = \mathscr{D}_i T$ and $\mathscr{D}_o' = T^*\mathscr{D}_o$.*

If $\mathscr{D}$, $\mathscr{D}_o$ and $\mathscr{D}_i$ are as above, then $\mathscr{D}^*\mathscr{D} \geq \varepsilon I$ for some $\varepsilon > 0$ (i.e., $\mathscr{D}$ is left-invertible in TI) iff $\mathscr{D}_o$ is invertible in TIC. If it is, then $\mathscr{D}_o$ is called a (invertible) *spectral factor* of $\mathscr{D}^*\mathscr{D}$, because $\mathscr{D}_o^*\mathscr{D}_o = \mathscr{D}^*\mathscr{D}$.

## 6. Proofs for the half-plane

In this technical section we prove the results of Section 5.

The proofs in Section 4 could be rewritten for Section 5 except that on some results there are no separable versions in the literature for the half-plane, so the use of the Cayley Transform is the easiest way to prove these results.

In that setting, the shift $S$ is mapped to the Laguerre shift $S_{\mathrm{Lag}}$ that maps $f$ to $z \mapsto f(z)i(1-z)/(1+z)$ and $\mathscr{H}^2(\mathscr{Z})$ onto the $\mathscr{H}^2$ space $\mathbf{C}^+ \to \mathscr{Z}$, for any Hilbert space $\mathscr{Z}$ [RR85, p. 59], [Sta05, Theorem 12.3.1].

Since in applications on $\mathbf{R}$ one usually wants to use and translations instead of the shift, we have also established the results given in Section 5, sometimes with non-straight-forward proofs, given in Lemma 6.3 below. The symbols $B$ and $B_2$ stand for arbitrary Banach spaces.

All results on the unit disc can easily be converted for the half-plane (to their $S_{\mathrm{Lag}}$ form, some to the standard form too) by using the (extensions of the) well-known properties of the *Cayley Transform*

$$\phi\colon z \mapsto i\frac{1-z}{1+z} \qquad \text{and its inverse} \qquad \phi^{-1}\colon s \mapsto \frac{1-is}{1+is} \tag{27}$$

that are listed in Lemma 6.1 below. Here we sometimes write the domains and target spaces explicitly; e.g., $\mathrm{L}^p(\mathbf{T}; B)$ stands for $\mathrm{L}^p$ functions $\mathbf{T} \to B$; otherwise we refer to the "disc notation" of Sections 1–4.

**Lemma 6.1.** (Cayley Transform) *The Cayley Transform $\phi$ maps $\mathbf{D} \to \mathbf{C}^+$ and $\mathbf{T} \to \mathbf{R} \cup \{\infty\}$ one-to-one and onto. Measurable (resp., null) sets (and only they) are mapped to measurable (resp., null) sets.*

*The corresponding composite map $\cdot \circ \phi$ maps $\mathscr{H}^\infty(\mathbf{C}^+; B) \to \mathscr{H}^\infty(B)$,*

$$\mathrm{L}^\infty(\mathbf{R}; B) \to \mathrm{L}^\infty(B), \quad \textit{and} \quad \mathrm{L}^\infty_{\mathrm{strong}}(\mathbf{R}; \mathscr{B}(B, B_2)) \to \mathrm{L}^\infty_{\mathrm{strong}}(B, B_2) \tag{28}$$

*isometrically onto. Measurable functions (and only they) are mapped to measurable functions.*

*The map $\diamondsuit f \mapsto \gamma \cdot (f \circ \phi)$ is an isometric isomorphism of $\mathrm{L}^2(\mathbf{R}; B)$ onto $\mathrm{L}^2(\mathbf{T}; B)$ and of $\mathscr{H}^2(\mathbf{C}^+; B)$ onto $\mathscr{H}^2(\mathbf{D}; B)$, where $\gamma(z) := 2\sqrt{\pi}/(1+z)$.*

*Therefore, $\heartsuit\colon T \mapsto \diamondsuit T \diamondsuit^{-1}$ maps*

$$\mathscr{B}(\mathrm{L}^p(\mathbf{R}; B), \mathrm{L}^p(\mathbf{R}; B_2)) \quad \textit{onto} \quad \mathscr{B}(\mathrm{L}^p(\mathbf{T}; B), \mathrm{L}^p(\mathbf{T}; B_2)) \tag{29}$$

*isometrically. Moreover, for every $F \in \mathrm{L}^\infty_{\mathrm{strong}}(\mathbf{R}; \mathscr{B}(B, B_2))$, we have $\heartsuit_p M_F = M_{F\circ\phi}$. Finally, $\heartsuit_p$ commutes with $P_+$ and $P_-$ as well as with adjoints and valid compositions of operators.*

*Proof.* The scalar version of this lemma is essentially given in [Hof88, pp. 128–131], and essentially the same proofs apply in the general case too. The details can also be found in [Mik02, Section 13.2]. (Note that in this article we have the additional constant $(2\pi)^{1/p}$ due to our normalized measure on $\mathbf{T}$.) $\square$

Thus, the inverse of $\heartsuit$ maps $\mathscr{H}^\infty$ on $\mathbf{D}$ onto bounded holomorphic functions on $\mathbf{C}^+$, and $\mathrm{L}^\infty_{\mathrm{strong}}(B, B_2)$ onto $\mathrm{L}^\infty_{\mathrm{strong}}(\mathbf{R}; \mathscr{B}(B, B_2))$, isometrically.

By simply Cayley Transforming all sets and operators in Section 4 we observe the following:

**Remark 6.2.** All results in Section 4 hold also in their $S_{\mathrm{Lag}}$ forms.

Note that here, e.g., the conditions (ii) and (iii) of Theorem 4.3 remain unchanged (except that $\mathscr{H}^\infty$ on $\mathbf{D}$ is mapped to $\mathscr{H}^\infty$ on $\mathbf{C}^+$) but, in (i'), $\mathscr{H}^2$ becomes its Cayley transform, a weighted $\mathscr{H}^2$ space on $\mathbf{C}^+$, and $P_-$ in (i) undergoes an analogous change.

Note also that $\mathscr{P}(\mathscr{X})$ is thus replaced by $\tilde{\mathscr{P}}(\mathscr{X}) := \diamondsuit^{-1}\mathscr{P}(\mathscr{X}) \subset \mathscr{H}^2(\mathbf{C}^+; \mathscr{X})$, which consists of certain rational functions.

This is not satisfactory for the results explicitly involving $\mathscr{H}^2$ or $S$, so we establish the results of Section 5 here.

**Lemma 6.3.** *The results in Section 5 hold. Moreover, Lemmata A.1 and A.2 hold with the half-plane notation of Section 5 too.*

*Proof.* The results in Sections 2 and 3 (cf. Lemma 5.1) and in the appendices follow from the same proofs, mutatis mutandis.

For the rest, the claims follow from Lemma 6.1 except for the results concerning the shift; we shall treat them below.

1° *Proposition 5.8:* The first and the third claim are well known; see, e.g., [Wei91] and Lemma 2.1.7 of [Mik02]. For the others, the original proof will do, mutatis mutandis.

2° *Theorem 5.9:* The original proof (of Theorem 4.7) will do.

3° *Theorem 5.10:* We only prove “only if”, since the rest follows as in the original proof. As in the proof of the latter lemma on p. 106 of [Hof88], we observe that $\mathscr{M}$ is invariant under the multiplication by any $\mathscr{H}^\infty(\mathbf{C})$ function, hence so is $\diamondsuit[\mathscr{M}]$, hence $S[\diamondsuit\mathscr{M}] \subset \diamondsuit\mathscr{M}$, hence $\diamondsuit\mathscr{M} = M_{\tilde{F}}[\mathscr{H}^2(\mathscr{Y}_0)]$ for some closed subspace $\mathscr{Y}_0 \subset \mathscr{Y}$ and some inner $\tilde{F} \in \mathscr{H}^\infty(\mathscr{Y}_0, \mathscr{X})$, by Theorem 4.8.

But we have $\diamondsuit^{-1} M_{\tilde{F}} \diamondsuit = \heartsuit^{-1} M_{\tilde{F}} = M_F$, where $F := \tilde{F} \circ \phi^{-1}$. Therefore, the function $F \in \mathscr{H}^\infty(\mathbf{C}^+; \mathscr{B}(\mathscr{Y}_0, \mathscr{X}))$ is inner, and

$$\mathscr{M} = \diamondsuit^{-1} M_{\tilde{F}}[\mathscr{H}^2(\mathscr{Y}_0)] = M_F \diamondsuit^{-1}[\mathscr{H}^2(\mathscr{Y}_0)] = M_F[\mathscr{H}^2(\mathbf{C}^+; \mathscr{Y}_0)]. \tag{30}$$ $\square$

## 7. Notes

The contents of Section 2 are mostly from [Mik08] and [Mik02, Appendix F] (or older in the separable case). Section 3 seems to be new.

The results in Section 4 seem to be new in the nonseparable case except Proposition 4.6, as explained in its proof. However, probably none of those results is new in the separable case. Most of them can be found in [Nik02], [Pel03], [Nik86], [RR85] or in other similar monographs, as explained in Section 4 and below. These monographs also record the history of the results.

The operator-valued version of Theorem 4.3 was established in [Tre04] but the equivalence of (i) and (ii) was given already in [Arv75] and [SF76]. The estimates for $\tilde{F}$ and $\begin{bmatrix} F & \tilde{F} \end{bmatrix}^{-1}$ in Theorem 4.3 are due to Sergei Treil.

Theorem 4.4 is essentially given on p. 203 of [Nik86] in the scalar case. Corollary 4.5 has been established at least in [CO06] (in the separable case), with a constructive proof.

Our proof of Theorem 4.7 is from p. 240 of [FF90]. The history of the Beurling–Lax–Halmos Theorem 4.8 (resp., inner-outer factorization 4.9) is explained on p. 21 (resp., 107–108) of [RR85]. The shift-invariant subspaces of $\mathrm{L}^2(\mu)$ can be found in [Nik86, pp. 14–17] (the separable case). For the Nevanlinna class N the inner-outer factorization was given on p. 100 of [RR85] (note that their "inner" allows also partial isometries and that $\mathrm{N} \not\subset \mathscr{H}^2_{\mathrm{strong}}$ and $\mathscr{H}^2_{\mathrm{strong}} \not\subset \mathrm{N}$), but our version is from [Nik86] and [FF90].

A finite-dimensional version of Theorem 5.10 and a scalar version of Theorem 5.11 are given in [Lax59].

Lemma A.3 is known. The statement of Lemma A.4 is due to Sergei Treil.

As explained in the introduction, further details, extensions, generalizations and further similar results are given in [Mik07]. Further related results for (possibly) nonseparable Hilbert spaces can be found in Chapters 1–3 of [RR85], in [Mik08], and in [Mik02], particularly in Sections 13.1, 6.4, 6.5, Chapters 2–5, and in Appendix F.

## Appendix A. Auxiliary results

In this appendix we list some results on Hilbert spaces.

By $\dim \mathscr{X}$ we denote the cardinality of an arbitrary orthonormal basis of $\mathscr{X}$ (it is independent of the basis [Mik02, Lemma A.3.1(a1)]). Thus, "$\dim \mathscr{X} \le \dim \mathscr{Y}$" means that there exists a one-to-one map of an orthonormal basis of $\mathscr{X}$ into an orthonormal basis of $\mathscr{Y}$. We need the following facts.

**Lemma A.1.** (dim) (a) *If* $T \in \mathscr{B}(\mathscr{X}, \mathscr{Y})$ *and* $T^*T \ge \varepsilon I$, *then we have* $\dim \mathscr{X} = \dim T[\mathscr{X}] \le \dim \mathscr{Y}$.

(b) *If* $\mathscr{X}$ *is infinite-dimensional, then* $\dim \mathrm{L}^2(\mathscr{X}) = \dim \mathscr{H}^2(\mathscr{X}) = \dim \mathscr{X}$.

(The elementary proof is given in [Mik07].)

In the case of an inner function, the output space cannot have a smaller dimension than the input space (here $M_F \colon \mathscr{H}^2(\mathscr{X}) \to \mathscr{H}^2(\mathscr{Y})$ refers to the operator $f \mapsto Ff$):

**Lemma A.2.** (dim) *Assume that* $F \in \mathrm{L}^\infty_{\mathrm{strong}}(\mathscr{X}, \mathscr{Y})$ *is satisfies* $M_F^* M_F \ge \varepsilon I$ *for some* $\varepsilon > 0$. *Then* $\dim \mathscr{X} \le \dim \mathscr{Y}$ *and hence* $\mathscr{X}$ *is isometrically isomorphic to a closed subspace, say* $\tilde{\mathscr{Y}}$, *of* $\mathscr{Y}$, *i.e.,* $T\mathscr{X} = \tilde{\mathscr{Y}}$ *for some* $T = T^{-*} \in \mathscr{B}(\mathscr{X}, \tilde{\mathscr{Y}})$.

*Proof.* 1° It is well-known that if $\mathscr{X}$ is separable, then $F(z)^*F(z) \ge \varepsilon I$ for a.e. $z \in \mathbf{T}$, so then $\dim \mathscr{X} \le \dim \mathscr{Y}$, by Lemma A.1(a).

2° Assume that $\mathscr{X}$ is nonseparable. By Lemma A.1(b)&(a), we have $\dim \mathscr{X} = \dim \mathscr{H}^2(\mathscr{X}) \le \dim \mathscr{H}^2(\mathscr{Y})$. Therefore, $\mathscr{H}^2(\mathscr{Y})$ is nonseparable, hence so is $\mathscr{Y}$. Consequently, $\dim \mathscr{Y} = \dim \mathscr{H}^2(\mathscr{Y}) \ge \dim \mathscr{X}$. □

Sometimes we need to build a Hilbert space as the direct sum of a collection of (not necessarily disjoint) Hilbert spaces. The following is obvious:

**Lemma A.3.** (Direct sum) *If* $Z_X$ *is a Hilbert space for each* $X \in \mathscr{Q}$, *and we set* $\|z\|^2_{\mathscr{Z}} := \sum_{X \in \mathscr{Q}} \|z(X)\|^2_{Z_X}$, *then* $\mathscr{Z} := \{z \in \prod_{X \in \mathscr{Q}} Z_X \bigm| \|z\|_{\mathscr{Z}} < \infty\}$ *becomes a Hilbert space with the inner product* $\langle z, w \rangle_{\mathscr{Z}} := \sum_{X \in \mathscr{Q}} \langle z(X), w(X) \rangle_{Z_X}$.

In a Hilbert space, a projection has the same norm as its complementary projection:

**Lemma A.4.** *If $P = P^2 \in \mathscr{B}(\mathscr{X})$ and $0 \neq P \neq I$, then $\|P\| = \|I - P\| \geq 1$.*

*Proof.* Set $Q := I - P$, $U_P := \{Px \mid \|Px\| = 1\}$, $U_Q := \{Qx \mid \|Qx\| = 1\}$. Let $Q'$ denote the orthogonal projection $\mathscr{X} \to Q[\mathscr{X}]$, $Q^\perp := 1 - Q'$. for each $x \in \mathscr{X}$. Since $\|P\| = \sup_{x \neq 0} \|Px\|/\|x\| = \sup_{\|Px\|=1} \|x\|^{-1}$, we have

$$\|P\|^{-2} = \inf_{\|Px\|=1} \|x\|^2 = \inf_{\|Px\|=1} \|Px + Qx\|^2 \tag{31}$$

$$= \inf_{p \in U_P,\ q \in Q[\mathscr{X}]} \|p + q\|^2 = \inf_{p \in U_P,\ q \in Q[\mathscr{X}]} \|Q^\perp p + Q'p + q\|^2 \tag{32}$$

$$= \inf_{p \in U_P} \|Q^\perp p\|^2 = \inf_{p \in U_P} (1 - \|Q'p\|^2) \tag{33}$$

$$= \inf_{p \in U_P} (1 - \sup_{q \in U_Q} |\langle p, q\rangle|^2) = 1 - \inf_{p \in U_P,\ q \in U_Q} |\langle p, q\rangle|^2 \tag{34}$$

$= \|Q\|^{-2}$ (exchange the roles of $P$ and $Q$ for this last equality). □

## Appendix B. Proof of Theorem 3.2

In this section we prove Theorem 3.2. We start with an auxiliary result. It says that if we study the effects of $F\colon \mathbf{D} \to \mathscr{B}(\mathscr{X}, \mathscr{Y})$ and $G\colon \mathbf{D} \to \mathscr{B}(\mathscr{Y}, \mathscr{X})$ on separable sets $X_0 \subset \mathscr{X}$ and $Y_0 \subset \mathscr{Y}$, we can without loss of generality assume that $\mathscr{X}$ and $\mathscr{Y}$ are separable.

**Lemma B.1.** *Let $X_0 \subset \mathscr{X}$ and $Y_0 \subset \mathscr{Y}$ be separable subsets, and let $F\colon \mathbf{D} \to \mathscr{B}(\mathscr{X}, \mathscr{Y})$ and $G\colon \mathbf{D} \to \mathscr{B}(\mathscr{Y}, \mathscr{X})$ be continuous functions.*

*Then there are closed, separable subspaces $\tilde{X} \subset \mathscr{X}$ and $\tilde{Y} \subset \mathscr{Y}$ that satisfy $X_0 \subset \tilde{X}$, $Y_0 \subset \tilde{Y}$, $F(z)x \in \tilde{Y}$ and $G(z)y \in \tilde{X}$ for every $x \in \tilde{X}$, $y \in \tilde{Y}$ and $z \in \mathbf{D}$.*

*Proof.* Let $\mathscr{J}$ (resp., $\mathscr{K}$) denote the collection of closed, separable subspaces of $\mathscr{X}$ (resp., $\mathscr{Y}$). Set $X_1 := \overline{\operatorname{span} X_0}$. Let $\mathbf{D}' \subset \mathbf{D}$ be dense and countable.

1° *Finding $Y_k$:* Given any $k \in \{1, 2, \ldots\}$ and any $X_k \in \mathscr{J}$ with $X_0 \subset X_k$, choose a countable dense subset $S_k \subset X_k$. For each $x \in S_k$, choose $Y_x \in \mathscr{K}$ such that $\{F(z)x \mid z \in \mathbf{D}'\} \cup Y_0 \subset Y_x$. Then $Y_k' := Y_0 \cup (\cup_{x \in S_k} Y_x)$ is separable, hence contained in some $Y_k \in \mathscr{K}$. Obviously, $F(z)x \in Y_k$ for every $z \in \mathbf{D}$ and every $x \in X_k$.

2° *Finding $X_{k+1}$:* Similarly, given any $k \in \{1, 2, \ldots\}$ and $Y_k \in \mathscr{K}$ with $Y_0 \subset Y_k$, we find, as in 1°, a space $X_{k+1} \in \mathscr{J}$ such that $X_0 \subset X_{k+1}$ and $G(z)y \in X_{k+1}$ for every $z \in \mathbf{D}$ and every $y \in Y_k$.

3° Given any sequences of subspaces $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$, chosen as above, set $\tilde{X} := \overline{\operatorname{span}(\cup_k X_k)} \in \mathscr{J}$, $\tilde{Y} := \overline{\operatorname{span}(\cup_k Y_k)} \in \mathscr{K}$. □

*Proof of Theorem 3.2.* (a) In 3° below we shall obtain $\mathscr{V}$ by Hausdorff's Maximality Theorem using the fact that any nonmaximal collection $\tilde{\mathscr{V}}$ of the form specified in 1° can be extended, as will be shown in 2°.

1° *Requirements on $\tilde{\mathscr{V}}$:* We require that $\tilde{\mathscr{V}}$ satisfies the theorem in place of $\mathscr{V}$ except that $\tilde{\mathscr{X}} := \sum_{(X,Y) \in \mathscr{V}} X$ and $\tilde{\mathscr{Y}} := \sum_{(X,Y) \in \mathscr{V}} X$ need not equal $\mathscr{X}$ and $\mathscr{Y}$, respectively. We also require that

$$F\tilde{P}_{\tilde{\mathscr{X}}} = \tilde{P}_{\tilde{\mathscr{Y}}} F \tilde{P}_{\tilde{\mathscr{X}}} = \tilde{P}_{\tilde{\mathscr{Y}}} F. \tag{35}$$

2° Assume that $\tilde{\mathscr{V}}$ is as in 1° (e.g., $\mathscr{V} = \{(\{0\}, \{0\})\}$). Assume also that $\tilde{\mathscr{Y}} \neq \mathscr{Y}$ or $\tilde{\mathscr{X}} \neq \mathscr{X}$ (otherwise $\mathscr{V} := \tilde{\mathscr{V}}$ will do). In 2.1°–2.3° we shall construct closed separable subspaces $X \subset \tilde{\mathscr{X}}^\perp$ and $Y \subset \tilde{\mathscr{Y}}^\perp$, so that $\tilde{\mathscr{V}}' := \tilde{\mathscr{V}} \cup \{(X,Y)\}$ is as in 1° and $X \neq \{0\}$ or $Y \neq \{0\}$.

2.1° *Case* $\tilde{\mathscr{X}} = \mathscr{X}$*:* Pick some $y \in \tilde{\mathscr{Y}}^\perp$ and set $Y := \mathbf{C}y$, $X := \{0\}$ (by (35), $F = F\tilde{P}_{\mathscr{X}} = F\tilde{P}_{\tilde{\mathscr{X}}} = \tilde{P}_{\tilde{\mathscr{Y}}}F$, hence $(I - \tilde{P}_{\tilde{\mathscr{Y}}})F = 0$, hence $\tilde{P}_Y F = 0 = F\tilde{P}_X = \tilde{P}_Y F \tilde{P}_X$).

2.2° *Case* $\tilde{\mathscr{Y}} = \mathscr{Y}$*:* Analogously, pick some $x \in \tilde{\mathscr{X}}^\perp$ and set $X := \mathbf{C}x$, $Y := \{0\}$.

2.3° *Case* $\tilde{\mathscr{X}} \neq \mathscr{X}$ *and* $\tilde{\mathscr{Y}} \neq \mathscr{Y}$*:* Pick some nonempty separable $X_0 \subset \tilde{\mathscr{X}}^\perp$ and $Y_0 \subset \tilde{\mathscr{Y}}^\perp$. Choose $X$ and $Y$ as in Lemma B.1 but with $\tilde{\mathscr{X}}^\perp$, $\tilde{\mathscr{Y}}^\perp$ and $F^*$ in place of $\mathscr{X}$, $\mathscr{Y}$ and $G$, respectively. Then $FP_X = P_Y^* F P_X$ and $F^* P_Y = P_X^* F^* P_Y$, so

$$F\tilde{P}_X = \tilde{P}_Y^* F \tilde{P}_X = \tilde{P}_Y^* F. \tag{36}$$

Therefore, the requirements in 1° are satisfied for $\tilde{\mathscr{V}}' := \tilde{\mathscr{V}} \cup \{(X,Y)\}$ in place of $\tilde{\mathscr{V}}$:

$$F\tilde{P}_{\tilde{\mathscr{X}}'} = F\tilde{P}_{\tilde{\mathscr{X}}} + F\tilde{P}_X = \tilde{P}_{\tilde{\mathscr{Y}}}F + \tilde{P}_Y F = \tilde{P}_{\tilde{\mathscr{Y}}'}F, \tag{37}$$

hence $\tilde{P}_{\tilde{\mathscr{Y}}'} F \tilde{P}_{\tilde{\mathscr{X}}'} = \tilde{P}_{\tilde{\mathscr{Y}}'} \tilde{P}_{\tilde{\mathscr{Y}}'} F = \tilde{P}_{\tilde{\mathscr{Y}}'} F$.

3° Now we obtain $\mathscr{V}$ by a standard application of Hausdorff's Maximality Theorem. Indeed, let $\mathscr{A}$ the collection of all sets $\tilde{\mathscr{V}}$ that satisfy 1°. Let $\mathscr{A}' \subset \mathscr{A}$ be a maximal subchain and set $\tilde{\mathscr{V}} := \cup \mathscr{A}'$. Then we must have $\tilde{\mathscr{X}} = \mathscr{X}$ and $\tilde{\mathscr{Y}} = \mathscr{Y}$, by maximality (and 2°). Clearly $\mathscr{V} := \tilde{\mathscr{V}}$ satisfies (a).

(b) Let $z \in \mathbf{D}$. Obviously, $\sum_{\mathscr{V}} P_Y^* F_{X,Y}(z) P_X x = F(z)x$ when $x \in X$ for some $(X,Y) \in \mathscr{V}$, hence for any $x \in \mathscr{X}$ (because $\sum_{\mathscr{V}} P_Y^* F_{X,Y}(z) P_X \in \mathscr{B}(\mathscr{X},\mathscr{Y})$, by Theorem 3.1). □

## Appendix C. $\mathrm{L}^\infty_{\mathrm{strong}}$ and inner functions

In this section we illustrate some pathologies of $\mathrm{L}^\infty_{\mathrm{strong}}$ over nonseparable Hilbert spaces in two examples.

Firstly, we may have $[F] = 0 \in \mathrm{L}^\infty_{\mathrm{strong}}$ even if $F(z) \neq 0 \in \mathscr{B}(\mathscr{X},\mathscr{Y})$ for each $z \in \mathbf{T}$:

**Example C.1.** (a) Assume that $\mathscr{X} = \ell^2(\mathbf{T};\mathbf{C})$ and $\mathscr{Y} \neq \{0\}$. Pick $y_0 \in Y$ such that $\|y_0\| = 1$. For each $z \in \mathbf{T}$, define $\Lambda_z \in \mathscr{X}^*$ by $\Lambda_z x := x(z)$ ($x \in \mathscr{X}$) and $F(z) \in \mathscr{B}(\mathscr{X},\mathscr{Y})$ by

$$F(z)x := z y_0 \Lambda_z x = z x(z) y_0. \tag{38}$$

(Note that $\|F(z)\| = |z| = 1$ for each $z \in \mathbf{T}$.) Given $x \in \mathscr{X}$, we have $F(z)x = 0$ a.e., hence $\|F\|_{\mathrm{L}^\infty_{\mathrm{strong}}} = 0$ even though $F(z) \neq 0$ for each $z \in \mathbf{T}$.

(b) Note also that $F(z)^* y_0 = \bar{z} e_z$, where $\langle x, e_z \rangle = x(z)$. Therefore, $F(z)^* y_0$ is not measurable (not being almost separably-valued), so $F^*$ is not strongly measurable (hence not $\mathrm{L}^\infty_{\mathrm{strong}}$).

(c) If we replace $y_0$ by $(\operatorname{Re} z)^{-1} y_0$ in (38), then $\|F(z)\|_{\mathscr{B}(\mathscr{X},Y)} = |\operatorname{Re} z|^{-1}$, hence then we have $\operatorname{ess\,sup} \|F\|_{\mathscr{B}} = \infty$ even though still $[F] = [0] \in \mathrm{L}^\infty_{\mathrm{strong}}$.

Even worse, we may have $F \in \mathscr{H}^\infty$ inner with boundary function $F_0 \in \mathrm{L}^\infty_{\mathrm{strong}}$ such that $Fx \to F_0 x$ nontangentially at every point of $\mathbf{T}$, for every $x \in \mathscr{X}$ and yet $F_0(z)^* F_0(z) \neq I$ for each $z \in \mathbf{T}$. Indeed the boundary function of the function $h \in \mathscr{H}^\infty$ given by $h(z) := \mathrm{e}^{(z+1)/(z-1)}$ satisfies $h(1) = 0$ and $|h(z)| = 1$ for $z \in \mathbf{T} \setminus \{1\}$ (by Lemma 6.1). By rotating $h$ by all possible angles and combining these uncountably many rotated copies to a function $F \in \mathscr{H}^\infty(\mathscr{X})$, this function has the properties explained above:

**Example C.2.** Define $h\colon \mathbf{D} \to \mathbf{C}$ by $h(z) := \mathrm{e}^{(z+1)/(z-1)}$. Then $h \in \mathscr{H}^\infty$ is inner,

$$h(z) = \exp(-2i \operatorname{Im} z/|z-1|^2) \in \mathbf{T} \quad \text{for} \quad z \in \mathbf{T} \setminus \{1\}, \tag{39}$$

and $h(1) = 0$ (and all these limits are nontangential).

Set $\mathscr{X} := \ell^2(\mathbf{T}; \mathbf{C})$, and define $F\colon \overline{\mathbf{D}} \to \mathscr{B}(\mathscr{X})$ by $(Fe_s)(z) := h(\bar{s}z)e_s$ for each $s \in \overline{\mathbf{D}}$, where $e_s := \chi_{\{s\}}$ (the functions $e_s$ form the natural orthonormal basis of $\mathscr{X}$). Then $F$ is inner, by Theorem 4.1(a) (set $\mathscr{V} := \{(X_s, X_s) \mid s \in \mathbf{T}\}$, where $X_s := \mathbf{C}e_s$, so that $F_{X,X}x = hx$ $(x \in X)$ for each $(X, X) \in \mathscr{V}$).

Moreover, $F_0 := F_{|\mathbf{T}}$ is the unique function $\mathbf{T} \to \mathscr{B}(\mathscr{X})$ for which $F_0 x$ is the (nontangential) limit of $F_{|\mathbf{D}}x$ for each $x \in \mathscr{X}$. Nevertheless, $F(z)^*F(z) = I - P_z \neq I$ for each $z \in \mathbf{T}$, where $P_z$ is the orthogonal projection $\mathscr{X} \to \mathbf{C}e_z$.

Even in the above example, we could redefine $F_0$ (within the same class in $\mathrm{L}^\infty_{\text{strong}}$) so that $F_0(z)^*F_0(z) = I$ for each $z \in \mathbf{T}$ (e.g., by setting above $h(1) := 1$). However, then $F_0(z)e_z = e_z$ would no longer be equal to the nontangential limit 0 of $Fe_z$ at $z$, for any $z \in \mathbf{T}$. Thus, the fact that $F_0(z)^*F_0(z) \neq I$ everywhere is inherent in the inner function $F$, not a consequence of an artificial choice of $F_0$ within $[F_0]$.

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