Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 34, 2009, 131–143

NEUMANN LAPLACIAN ON A DOMAIN WITH TANGENTIAL COMPONENTS IN THE BOUNDARY

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Abstract. We study the Neumann problem for the Poisson equation in a domain where two boundary components are tangential at a single point, such that a geometrical irregularity of the rotational cusp type is formed. We derive necessary and sufficient conditions for the existence of a solution with a finite Dirichlet integral.

1. Formulation of the problem

Since the pioneering paper [4] by Kondratiev much attention is devoted to elliptic boundary value problems in domains with irregular boundaries. An almost complete theory has been created in case of conical corner points and edges, see for example the key works [4], [11], [12], [19] and [5], [26], [13], [14], respectively, and also the monographs [2], [21], [6]. Other geometrical irregularities, common in everyday life and engineering practise, have been considered as well. The papers [10], [27] and [6] contain studies near cusp tips of peak-shaped domains, using reduction to conical domains and to differential operators with strongly perturbed coefficients.

There are nevertheless many types of irregularities, also interesting for applied sciences, which have not yet been deeply studied. We mention cuspidal edges and rotational cusps, the latter being the subject of the present paper. It is not difficult to imagine the applications of such geometrical forms in mechanics and even more applied problems in metal or woodworking machinery; think about drill tips, milling inserts, incisal surfaces etc. Evidently, the solvability and Fredholm properties of such elliptic problems are open, though there exist results on asymptotics of solutions (see e.g. [16]).

Let Ω be a domain in the Euclidean space $\mathbf{R}^n$, which is Lipschitz everywhere else except at the origin $\mathscr{O}$ of the Cartesian coordinate system $x = (y, z) \in \mathbf{R}^{n-1} \times \mathbf{R}^1$. We assume that in a cylindrical neighbourhood $\mathscr{U} = \mathbf{B}_R^{n-1} \times (-d, d)$ of $\mathscr{O}$, where $\mathbf{B}_R^{n-1} := \{y \in \mathbf{R}^{n-1} : |y| < R\}$, and $R > 0$, $d > 0$, a point $x = (y, z) \in \mathscr{U}$ belongs to

2000 Mathematics Subject Classification: Primary 35J25; Secondary 46E35, 35J20.

Key words: Poisson equation, Neumann problem, nonsmooth boundary, rotational cusp.

The research of S.A.N. was partially supported by the grant RFFI-06-01-257. The research of J.S. was partially supported by the Projet CPER Lorraine MISN: Analyse, optimisation et contrôle 3 in France, and the grant N51402132/3135 Ministerstwo Nauki i Szkolnictwa Wyzszego: Optymalizacja z wykorzystaniem pochodnej topologicznej dla przeplywow w osrodkach scisliwych in Poland.

the domain Ω, if and only if z satisfies the inequalities

$$-H_-(y) < z < H_+(y). \tag{1.1}$$

Here the functions $H_\pm$ are in the space $H^{1,\infty}(\mathbf{B}_R^{n-1})$, i.e., the gradients $\nabla_y H_\pm$ are bounded functions almost everywhere on the ball. In addition, $H := H_+ + H_- > 0$ for $r \in (0, R]$ and

$$H(y) = r^{1+\gamma}(H_0(\theta) + \mathscr{O}(r^\delta)), \quad r \to 0, \tag{1.2}$$

where H_0 is a smooth function of class C^2, positive on the unit sphere $\mathbf{S}^{n-2} \subset \mathbf{R}^{n-1}$, while $r = |y|$ and $\theta = r^{-1}y \in \mathbf{S}^{n-2}$ are spherical coordinates, and γ, δ are positive exponents.

Classical geometrical forms described by the relations (1.1), (1.2) are *balls kissing each other from inside or from outside* (see Figure 1), and also the well known hydrodynamical problem on the ellipsoid sitting on the halfspace, see Figure 2. In these cases we have $\gamma = 1$. There exist more complicated geometries: for example, the Figure 3 designates a surface of revolution, tangent to the cylindrical cone. If a ball is kissing a paraboloid from inside (see in Figure 4) such that the curvatures of both bodies coincide at the origin $\mathscr{O}$, the exponent γ turns out to be 3. Yet a different case, where γ may not be a constant at the origin, is designated in Figure 5. If the cylindrical surface is circular, then $\gamma = 1$ as before, but for a ball of radius $R_0 > 0$ and the parabolic cylinder with maximal curvature R_0^{-1} the domain is no more described by the relations (1.1), (1.2): one has $\gamma = 3$ along the trace (the generating parabola) and and $\gamma = 1$ along the axis (the line perpendicular to the traces), hence, there does not exist a positive function H_0. In the sequel γ is called the exponent of tangency of the surfaces $\Gamma_\pm = \{x \in \mathscr{U} : z = \pm H_\pm(y)\}$.

We consider the following Neumann problem for the Poisson equation in the domain Ω:

$$-\Delta_x u(x) = f(x),\ x \in \Omega,\ \partial_\nu u(x) = g(x),\ x \in \partial\Omega \setminus \mathscr{O}. \tag{1.3}$$

Here ∂_ν is the outward normal derivative, and the compability conditions on the data will be given later. It is clear that the following conditions, in an appropriate sense, should be satisfied:

$$\int_\Omega f(x)\,dx + \int_{\partial\Omega} g(x)\,ds_x = 0, \tag{1.4}$$

where ds_x is the $(n-1)$-dimensional surface measure. We shall study the question of the existence of solutions to problem (1.3) in the Sobolev space $H^1(\Omega)$. Also the following simplified problem appears often in applications: given a function G, continuous or smooth near the origin $\mathscr{O}$, find a solution to (1.3) with a finite Dirichlet integral, for the data

$$f = 0 \text{ on } \Omega \cap \mathscr{U},\ g = 0 \text{ on } \Gamma_-,\ g = G \text{ on } \Gamma_+. \tag{1.5}$$

It is known (see e.g., [8], [9], Chapter 14, and Section 6 of this paper), that the analogous Dirichlet problem in the two dimensional case

$$-\Delta_x u(x) = f(x),\ x \in \Omega,\ u(x) = g(x),\ x \in \partial\Omega \setminus \mathscr{O}, \tag{1.6}$$

has a negative answer, if $G(\mathscr{O}) \neq 0$. This is so since the inclusion $u \in H^1(\Omega)$ means that in a sense the difference $u(y, H_+(y)) - u(y, H_-(y))$ admits the null limit for $y \to 0$.

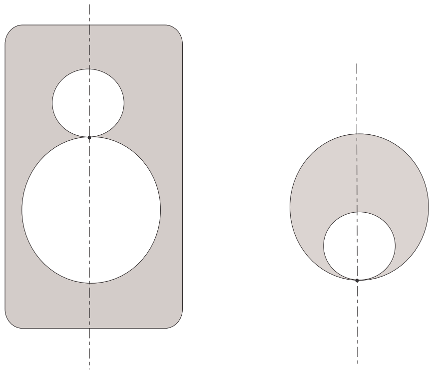

Figure 1.

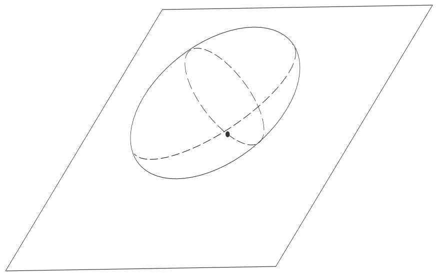

Figure 2.

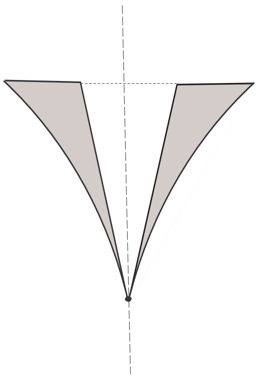

Figure 3.

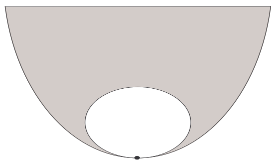

Figure 4.

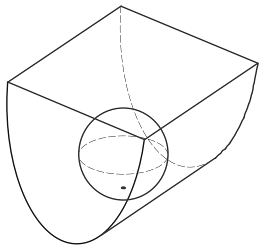

Figure 5.

In the paper we derive the criterium

$$\gamma < n \tag{1.7}$$

for the existence of bounded energy solutions to problem (1.3) under conditions (1.4), (1.5) and $G(\mathscr{O}) \neq 0$. If $\gamma \geq n$, then, for any solution of (1.3) with the right-hand sides given above, the Dirichlet integral diverges. As a consequence we find that the Neumann and Dirichlet problems (with (1.5)) behave in substantially different ways: for the geometrical forms of Figures 1 and 2 and for the most interesting dimensions $n = 2$ and $n = 3$, the solution of the Neumann problem enjoys a finite energy, but this is not true for the Dirichlet problem.

The results on the solvability of the Neumann problem, Proposition 3.1 in Section 3, follow from a weighted inequality given by Lemma 2.1. These results assure the sufficiency of the condition (1.7) in the case of data (1.5). The next two sections are devoted to the verification of the necessary condition. In Section 5 we choose appropriate test functions in the integral identity for problem (1.3), and this leads to the conclusion that for $G(\mathscr{O}) \neq 0$ there is no solution with the finite Dirichlet integral.

In the beginning of Section 4 we give explanations for the existence of the constraints (1.7), based on asymptotic constructions designed in [16].

In Section 6 we discuss possible generalizations and mention some open problems.

2. Weighted trace inequalities

The main result of this section is the verification of the following statement, with the scheme borrowed from [25]. The result is used for the analysis of the variational formulation of Neumann problem (1.3). We mention the book [15] for plenty of similar and much more general results on trace inequalities. We also mention that a result similar to ours in a more special case was recently obtained in [1].

Lemma 2.1. *The following weighted inequality is valid:*

$$c\|\rho^{-1}u; L^2(\Omega)\| + c\|\rho^{(\gamma-1)/2}u; L^2(\partial\Omega)\| \le \|u; H^1(\Omega)\|, \tag{2.1}$$

where $\rho(x) = |x|$, *and the constant* c *is independent of* $u \in H^1(\Omega)$.

Proof. Since $r \ge r_0 > 0$ on $\Omega \setminus \mathscr{U}$, we may assume without loss of generality that the support of the function u is included in the set $\overline{\Omega} \cap \mathscr{U}$. We represent the function u in the form

$$\begin{gathered} u(y,z) = \bar{u}(y) + u_\perp(y,z), \\ \int\limits_{-H_-(y)}^{H_+(y)} u_\perp(y,z)\,dz = 0 \text{ for almost all } y \in \mathbf{B}_R^{n-1}. \end{gathered} \tag{2.2}$$

By orthogonality condition (2.2), the Poincaré inequality holds on the interval $\Upsilon(y) = (-H_-(y), H_+(y))$ with the length $H(y)$. Denoting $\partial_z := \partial/\partial z$ we can thus write

$$\begin{aligned} \int\limits_{\Omega\cap\mathscr{U}} |\partial_z u(x)|^2\,dx &= \int\limits_{\mathbf{B}_R^{n-1}} \int\limits_{\Upsilon(y)} |\partial_z u_\perp(y,z)|^2\,dz\,dy \\ &\ge \pi^2 \int\limits_{\mathbf{B}_R^{n-1}} \int\limits_{\Upsilon(y)} H(y)^{-2}|u_\perp(y,z)|^2\,dz\,dy \\ &\ge C \int\limits_{\Omega\cap\mathscr{U}} r^{-2(1+\gamma)}|u_\perp(x)|^2 dx, \quad C > 0. \end{aligned} \tag{2.3}$$

The standard method of verification of the trace inequalities (see, e.g., [7]) leads to

$$\begin{aligned} I_0 &:= \sum_\pm \int\limits_{\Gamma_\pm} \rho^{-1-\gamma}|u_\perp(x)|^2\,ds_x \\ &\le c\sum_\pm \int\limits_{\mathbf{B}_R^{n-1}} H(y)^{-1}|u_\perp(y, \pm H_\pm(y)|^2\,dy \\ &\le c \int\limits_{\Omega\cap\mathscr{U}} (|\partial_z u_\perp(y,z)|^2 + H(y)^{-2}|u_\perp(y,z)|^2)\,dy\,dz. \end{aligned} \tag{2.4}$$

We remark that by our assumptions, the boundary Jacobian on the surfaces $\Gamma_\pm$ is bounded from above and below, and the quantities $r^{1+\gamma}$, $\rho^{1+\gamma}$, and $H(y)$ are infinitesimal and of the same order, when $x \to \mathscr{O}$ from inside of Ω.

We have

$$\int\limits_{\Omega\cap\mathscr{U}} |\nabla_y u(x)|^2\,dx = \int\limits_{\mathbf{B}_R^{n-1}} H(y)|\nabla_y \bar{u}(y)|^2\,dy + \int\limits_{\Omega\cap\mathscr{U}} |\nabla_y u_\perp(x)|^2\,dx \tag{2.5}$$
$$+\,2\int\limits_{\mathbf{B}_R^{n-1}} \nabla_y\bar{u}(y)\cdot\int\limits_{\Upsilon(y)} \nabla_y u_\perp(y,z)\,dz\,dy =: I_1+I_2+I_3.$$

The dot "$\cdot$" denotes the inner product in $\mathbf{R}^{n-1}$. We apply the following one dimensional Hardy inequality (see [3])

$$\int\limits_0^R r^{\alpha-1}|U(r)|^2\,dr \le \frac{4}{\alpha^2}\int\limits_0^R r^{\alpha+1}|\partial_r U(r)|^2\,dr \tag{2.6}$$

with the exponent $\alpha = n-2+\gamma>0$; it is valid, if $U(R)=0$. It follows that

$$I_1 \ge c\int\limits_{\mathbf{B}_R^{n-1}} r^{-1+\gamma}|\bar{u}(y)|^2\,dy \ge c\int\limits_{\Omega\cap\mathscr{U}} r^{-2}|\bar{u}(y)|^2\,dx, \quad c>0. \tag{2.7}$$

For the analysis of the integral I_3 we make use of the rule of differentation of integrals with variable limits and obtain

$$\Big|\nabla_y\int\limits_{\Upsilon(y)} u_\perp(y,z)\,dz - \int\limits_{\Upsilon(y)} \nabla_y u_\perp(y,z)\,dz\Big| \le \sum_{\pm}|\nabla_y H_\pm(y)|\,|u_\perp(y,\pm H_\pm(y))|$$

for almost all $y\in\mathbf{B}_R^{n-1}$. Since the first integral on the left-hand side is null by (2.2), we have

$$|I_3| \le C\int\limits_{\mathbf{B}_R^{n-1}} |\nabla_y\bar{u}(y)|\sum_{\pm}|\nabla_y H_\pm(y)|\,|u_\perp(y,\pm H_\pm(y))|\,dy$$
$$\le C\Big(\int\limits_{\mathbf{B}_R^{n-1}} H(y)|\nabla_y\bar{u}(y)|^2\,dy\Big)^{1/2}\Big(\sum_{\pm}\int\limits_{\Gamma_\pm} H(y)^{-1}|u_\perp(x)|^2\,ds_x\Big)^{1/2} \tag{2.8}$$
$$\le \varepsilon I_1 + c\varepsilon^{-1}I_0,$$

with an arbitrary $\varepsilon>0$. Assume that $\varepsilon = 1/4$; then from (2.5), (2.4) and (2.8) it follows that each of integrals I_1 and I_2 is bounded by the quantity $c\|\nabla_x u; L^2(\Omega\cap\mathscr{U})\|$. The estimates (2.3) and (2.7) for the component in decomposition (2.2) lead to the required relation (2.1). □

3. Solvability of the variational problem

Assume that the right-hand sides of problem (1.3) are smooth for the time being and take the form

$$f(x) = f_0(x) - \nabla_x\cdot\tilde{f}(x), \quad g(x) = g_0(x) + \nu(x)\cdot\tilde{f}(x), \tag{3.1}$$

where $\tilde{f} = (f_1,\ldots,f_n)$ and $\nu = (\nu_1,\ldots,\nu_n)$ is the outer unit normal. In such a case the variational formulation [7] of the Neumann problem (1.3) reads as

$$(\nabla_x u, \nabla_x v)_\Omega = (f_0,v)_\Omega + (\tilde{f},\nabla_x v)_\Omega + (g_0,v)_{\partial\Omega}, \quad v\in C_c^\infty(\bar{\Omega}\setminus\mathscr{O}); \tag{3.2}$$

here $C_c^\infty(\bar\Omega \setminus \mathscr{O})$ is the space of infinitely smooth functions which vanish in a neighbourhood of the point $\mathscr{O}$ and $(\cdot\,,\cdot)_\Xi$ denotes the scalar product in the Lebesgue space $L^2(\Xi)$ as well as its extension to the duality pairing between appropriate weighted spaces. In view of Lemma 2.1, the expression on the right hand side of (3.2) defines a continuous functional on $H^1(\Omega)$, if the inclusions

$$\tilde f \in L_2^2(\Omega)^n, \quad r f_0 \in L_2^2(\Omega), \quad r^{(1-\gamma)/2} g_0 \in L^2(\partial\Omega), \tag{3.3}$$

hold true. In the sequel we understand (3.2) as a generalized (cf. [7]) or variational formulation of the problem (1.3), even though the functions (3.1) cannot always be interpreted properly.

By $H^1(\Omega)_\perp$ we denote the subspace of the Sobolev space, which contains the functions with null mean value

$$\int\limits_\Omega u(x)\,dx = 0. \tag{3.4}$$

The left hand side of (3.2) can be taken for the scalar product in $H^1(\Omega)_\perp$. Indeed, it is sufficient to verify that for each function $u \in H^1(\Omega)_\perp$ the following inequality holds true (we shall give a proof for the convenience of the reader, though the result is included in the theory presented, e.g., in [15]):

$$\|u; L^2(\Omega)\| \le C\|\nabla_x u; L^2(\Omega)\|.$$

We define

$$u(x) = \mathbf{a} + \mathbf{v}(x), \qquad \int\limits_{\Omega(\delta)} \mathbf{v}(x)\,dx = 0,$$

where $\Omega(\delta) = \Omega \setminus \bar{\mathbf{B}}_\delta^n$ and $\delta > 0$ is small enough. According to (3.4) we have

$$|\mathbf{a}| \le (\mathrm{mes}_n\Omega)^{-1} \int\limits_\Omega |\mathbf{v}(x)|\,dx \le c\|\mathbf{v}; L^2(\Omega)\|,$$

i.e., it is sufficient to estimate the latter norm. The domain $\Omega(\delta)$ has Lipschitz boundary, which ensures, owing to the Poincaré inequality, that

$$\|\mathbf{v}; L^2(\Omega(\delta))\| \le c_\delta\|\nabla_x \mathbf{v}; L^2(\Omega(\delta))\| = c_\delta\|\nabla_x u; L^2(\Omega(\delta))\| \le c_\delta\|\nabla_x u; L^2(\Omega)\|.$$

We define a cut-off function $\chi\colon \bar\Omega \to [0,1]$ to be equal to one for $x \in \Omega \cap \mathscr{U}$, $r < R/2$ and to null in the exterior of $\Omega \cap \mathscr{U}$. Multiplying $\mathbf{v}$ by χ we obtain, for $\delta = R/2$,

$$\begin{aligned} \|r^{-1}\chi\mathbf{v}; L^2(\Omega)\| &\le c\|\nabla_x(\chi\mathbf{v}); L^2(\Omega)\| \\ &\le c_\delta(\|\nabla_x\mathbf{v}; L^2(\Omega)\| + \|\mathbf{v}; L^2(\Omega(\delta))\|) \le C_\delta\|\nabla_x u; L^2(\Omega)\|. \end{aligned}$$

Here, Lemma 2.1 is applied. In the proof of the lemma it was established that for a function u with the support in the set $\overline{\Omega} \cap \mathscr{U}$, the norm $\|u; H^1(\Omega)\|$ can be replaced by the quantity $\|\nabla_x u; L^2(\Omega)\|$ in the inequality (2.1). It remains to note that

$$\|\mathbf{v}; L^2(\Omega)\| \le \|\chi\mathbf{v}; L^2(\Omega)\| + \|\mathbf{v}; L^2(\Omega(\delta))\| \le C\|\nabla_x u; L^2(\Omega)\|. \qquad \square$$

By the Riesz representation theorem for continuous functionals on a Hilbert space, the following assertion holds true.

Proposition 3.1. *Assume that the conditions* (3.3) *and*

$$\int_\Omega f_0(x)\,dx + \int_{\partial\Omega} g_0(x)\,ds_x = 0 \tag{3.5}$$

are satisfied. Then the variational problem (3.2) *admits a unique solution* $u \in H^1(\Omega)_\perp$ *and the inequality*

$$\|u; H^1(\Omega)\| \le c\Big(\|\tilde f; L^2_2(\Omega)^n\| + \|rf_0; L^2_2(\Omega)\| + \|r^{(1-\gamma)/2}g_0; L^2(\partial\Omega)\|\Big),$$

is valid, and the constant c *is independent of functions* (3.3).

Remark 3.2. The identity (3.5) is just a substitute for (1.4), since for smooth functions the two identities are equivalent, by the representation (3.1). The inclusion (3.3) implies the convergence of the integrals in (3.5), since the integrals on the right-hand sides of the following relations

$$\int_\Omega |f_0(x)|\,dx \le \|rf_0; L^2(\Omega)\|\Big(\int_\Omega r^{-2}\,dx\Big)^{1/2},$$
$$\int_{\partial\Omega} |g_0(x)|\,ds_x \le \|r^{(1-\gamma)/2}g_0; L^2(\Omega)\|\Big(\int_{\partial\Omega} r^{\gamma-1}\,ds_x\Big)^{1/2}$$

converge on Ω and $\partial\Omega$, respectively.

Remark 3.3. The condition (3.3) does not seem necessary at the first glance, since for smooth functions u in the class $H^1(\Omega)$, the traces of normal derivatives on the smooth surfaces $\Gamma_\pm$ are also smooth. However, they need not belong to the weighted space indicated by (3.3). In fact, we can get better results using the following observation, changing the function $\tilde f$ in (3.1). Indeed, dealing with the traces $g_\pm$ of the function g_0 at $\Gamma_\pm$ and taking into account the formula for the normal vector

$$\nu^\pm(y) = (1 + |\nabla_y H_\pm(y)|^2)^{-1/2}(\nabla_y H_\pm(y), \pm 1), \quad x \in \Gamma_\pm, \tag{3.6}$$

we replace $\tilde f$ by the vector function F with the components

$$F_k(x) = \tilde f_k(x) + \chi(y)(1 + |\nabla_y H(y)|^2)^{-1/2}\frac{\partial H}{\partial y_k}(y)g_-(y), \quad k = 1, \dots, n-1,$$
$$F_n(x) = \tilde f_n(x) - \chi(y)(1 + |\nabla_y H(y)|^2)^{-1/2}g_-(y),$$

and note that now $g''_- = 0$ in a neighbourhood of the point $\mathscr{O} \in \Gamma_-$. In other words, it is sufficient to require that the difference $g_+ - g_-$ belongs to the weighted space (compare with [24], Lemma 4.2.).

Finally, let G be a continuous function such that $G(\mathscr{O}) \neq 0$ and let (1.5) hold true. Then $r^{(1-\gamma)/2}G \in L^2(\mathbf{B}^{n-1}_R)$, if and only if $\gamma < n$. As a consequence of the above considerations we obtain the following result.

Proposition 3.4. *The condition* $\gamma < n$ *is sufficient for the existence of a solution* $u \in H^1(\Omega)$ *with the data satisfying* (1.5) *and* (1.4).

4. Asymptotic structure of solutions

In view of the formulae (1.1), (1.2), the domain Ω becomes thin in the direction of the z–coordinate in the vicinity of the point $\mathscr{O}$. It is to be expected that the solution of the Neumann problem is, in the vicinity of $\mathscr{O}$, a perturbation of a function depending on y only. This can indeed be seen from the representation (4.1) combined with (4.17), (4.18), and (4.19).

We are going to apply the asymptotic ansatz known in the theory of elliptic problems in thin domains (see, e.g. [20], for further details). Such an analysis is required for the description of solutions at the point of tangency for smooth connected components of the boundary (cf. [22, 16, 17] for some problems in mathematical physics). Indeed, we are going to look for the asymptotic solution of problem (1.3) with data (1.5) in the form

$$u(x) = v(y) + H(y)V(y,\zeta), \tag{4.1}$$

where v and V are functions to be determined, and $\zeta = H(y)^{-1}z$ is the *fast* variable. Replacing the solution u in (1.3) by the ansatz (4.1) and taking into account the formula (3.6) for the normal, we obtain

$$\Delta_x u(x) = \Delta_y v(y) + H(y)^{-1}\partial_\zeta^2 V(y,\zeta) + \dots, \tag{4.2}$$

$$\partial_{\nu^\pm} u(y, \pm H_\pm(y) = \pm\partial_\zeta V\Big(y, \frac{1}{2} \pm \frac{1}{2}\Big) - \nabla_y H_\pm(y)\cdot\nabla_y v(y) + \dots. \tag{4.3}$$

We denote by $\dots$ the terms which are not substantial for the present formal asymptotical analysis and can thus be neglected here. Now, we observe that the terms in (4.3) are of the same order (for $y \to 0$) as the function G in (1.5), and conclude that the discrepancies of the solution (4.1) in the problem (1.3) are small provided the following relations are satisfied:

$$\begin{aligned} -\partial_\zeta^2 V(y,\zeta) &= H(y)\Delta_y v(y), \quad \zeta\in(0,1),\\ \partial_\zeta V(y,1) &= \nabla_y H_+(y)\cdot\nabla_y v(y) + G(\mathscr{O}),\\ -\partial_\zeta V(y,0) &= \nabla_y H_-(y)\cdot\nabla_y v(y). \end{aligned} \tag{4.4}$$

These form a Neumann boundary value problem for an ordinary differential equation on the line segment. Since

$$\int_0^1 H(y)\Delta_y v(y)\,d\zeta + \nabla_y H_+\cdot\nabla_y v(y) + G(\mathscr{O}) + \nabla_y H_-\cdot\nabla_y v(y) = \nabla_y\cdot H(y)\nabla_y v(y) + G(\mathscr{O}),$$

the compability condition for the problem is the equality

$$-\nabla_y\cdot H(y)\nabla_y v(y) = G(\mathscr{O}), \tag{4.5}$$

which we consider as a degenerate partial differential equation in the punctured space $\mathbf{R}^{n-1}\setminus\mathscr{O}$. Myltiplying (1.5) by $r^{-1-\gamma}$ and taking into account the relation (1.2), we obtain the differential operator

$$L^0(y,\nabla_y) = -r^{-1-\gamma}\nabla_y\cdot r^{1+\gamma}H_0(\theta)\nabla_y, \tag{4.6}$$

which, in a sense, is the main part of the operator $-\nabla_y\cdot H(y)\nabla_y$ at the point $y=0$. In the spherical coordinates, the operator takes the form $r^{-2}\mathscr{L}(\theta,\nabla_\theta, r\partial_r)$, where ∇_θ stands for the spherical gradient, and can be considered in the framework of elliptic boundary value problems in domains with conical and angular boundary points. We apply the theory to construct the power solutions $r^\lambda \Psi(\theta)$ of the homogeneous and nonhomogeneous equations

$$L^0(y, \nabla_y)U(y) = r^{\mu-1-\gamma}\psi(\theta), \quad y \in \mathbf{R}^{n-1} \setminus \{0\}. \tag{4.7}$$

Lemma 4.1. *The spectral equation on the sphere*

$$-\nabla_\theta \cdot H_0(\theta)\nabla_\theta \phi(\theta) = \Lambda H_0(\theta)\phi(\theta), \quad \theta \in \mathbf{S}^{n-1}, \tag{4.8}$$

admits the infinite sequence of eigenvalues

$$0 = \Lambda_0 < \Lambda_1 \leq \Lambda_2 \leq \Lambda_3 \leq \ldots \leq \Lambda_m \leq \ldots \to +\infty, \tag{4.9}$$

which are listed in (4.9) *taking into account the multiplicities. The corresponding eigenfunctions* $\phi_0, \phi_1, \phi_2, \ldots, \phi_m, \ldots$, *where*

$$(\mathrm{mes}_{n-2}\mathbf{S}^{n-2})^{-1/2} = \phi_0, \tag{4.10}$$

are normalized by the orthogonality conditions

$$(H_0\phi_j, \phi_k)_{\mathbf{S}^{n-2}} = \delta_{j,k} \tag{4.11}$$

where $j, k = 0, 1, 2, \ldots$ *and* $\delta_{j,k}$ *is the Kronecker symbol.*

Proof. Multiplication of the equation (4.8) by $H_0^{-1/2}$ and the changes of variables $\phi \mapsto H_0^{-1/2}\phi$, $\Lambda \mapsto \Lambda+1$ lead to the expression of the positive and selfadjoint operator $-H_0^{-1/2}\nabla_\theta \cdot H_0 \nabla_\theta H^{-1/2} + 1$, for which the general theory applies. From the strong maximum principle it follows that ϕ_0 is constant and that the first eigenvalue Λ_0 is simple. □

Lemma 4.2. *Each power solution*

$$U(y) = r^\lambda \phi(\theta) \tag{4.12}$$

of the homogeneous problem (4.7) *has the exponent*

$$\lambda_m^\pm = \frac{1}{2}\Big(2 - n - \gamma \pm \sqrt{(2-n-\gamma)^2 + 4\Lambda_m}\Big) \tag{4.13}$$

and the eigenfunction ϕ, *corresponding to the eigenvalue* Λ_m *in equation* (4.8). *In addition,*

$$\lambda_0^+ = 0\ ,\ \lambda_0^- = 2 - n - \gamma < 0. \tag{4.14}$$

Proof. The power solutions of the form (4.12) can be obtained by direct computations and separation of variables from the homogenuous equation (4.7), i.e., with $\psi = 0$. □

The following statement is well known, see [4], and e.g., [21], Lemma 3.5.11.

Proposition 4.3. *If the number* $\mu + 1 - \gamma$ *is excluded from the set defined in* (4.13), *then the equation* (4.7) *has a unique power-law solution* (4.12) *with the exponent* $\lambda = \mu + 1 - \gamma$. *In the case* $\mu + 1 - \gamma = \lambda_m^\pm$, *the equation* (4.7) *gains the power-logarithmic solutions*

$$r^{\lambda_m^\pm}(c(\psi)\phi_m(\theta)\ln r + \phi(\theta)), \tag{4.15}$$

which differ by the term $cr^{\lambda_m^{\pm}}\phi_m(\theta)$. *The constant* $c(\psi)$ *in* (4.15) *is null, if and only if the right hand-side of equation* (4.15) *satisfies the condition*

$$\int\limits_{\mathbf{S}^{n-2}} \phi_m(\theta)\psi(\theta)\,ds_\theta = 0. \tag{4.16}$$

In the case $G(\mathscr{O}) \neq 0$ in (4.5) the right hand side of the equation (4.7) must be $r^{-1-\gamma}G(\mathscr{O})$, i.e., $\mu = 0$ and $\psi(\theta) = G(\mathscr{O})$. By Proposition 4.3 and Lemma 4.2, the equation (4.7) has in the case of $\gamma = 1$ (see Figures 1 and 2) the logarithmic solution

$$U(y) = C\ln r + \psi(\theta). \tag{4.17}$$

On the other hand, for $\gamma > 0$ and $\gamma \neq 1$ (see Figure 4), the solution turns out to be a power solution,

$$U(y) = r^{1-\gamma}\psi(\theta) \tag{4.18}$$

Indeed, for $\gamma > 1$ the exponent $1-\gamma$ belongs to the segment $(\lambda_0^-, \lambda_0^+) = (2-n-\gamma, 0)$, free from the numbers (4.13). For $\gamma \in (0,1)$ the exponent $1-\gamma$ may coincide with λ_m^+, but then necessarily $m \geq 1$ and therefore, by (4.7), (4.7), the right-hand side $\psi(\theta) = G(\mathscr{O})$ is orthogonal to the eigenfunction ϕ_m. Hence, the condition (4.16) is satisfied.

We introduce some simplifying assumptions, which will be omitted in the following section. First, $H(y) = r^{1+\gamma}H_0(\theta)$, which implies that in (1.2) the small remainder is absent, and the expression $r^{1+\gamma}L^0(y,\nabla_y)v(y)$ from (4.6) coincides with the right hand side of (4.5). Second, functions $H_\pm$ are assumed twice continuously differentiable, and moreover, in accordance with Section 1, we require

$$|\nabla_y H_\pm(y)| + r|\nabla_y^2 H_\pm(y)| \leq C.$$

Solutions of the Neumann problem are of the form

$$V(y,\zeta) = -\frac{1}{2}\zeta^2 H(y)\Delta_y v(y) - \zeta\nabla_y H_-(y)\cdot\nabla_y v(y), \tag{4.19}$$

and the boundary condition for $\zeta = 1$ is verified for such a solution by the fact that $v = U$ is a solution (of the form (4.17) or (4.18)) to the equation (4.7) for $\mu = 0$ and $\psi(0) = G(\mathscr{O})$. In the case of $G(\mathscr{O}) \neq 0$, we have

$$|\nabla_y v(y)| \geq c_0 r^{-\gamma}, \quad c_0 > 0,$$
$$|\nabla_y (H(y)V(y, H(y)^{-1}\zeta)| \leq c_1.$$

In this way, for $\gamma > n$ the Dirichlet integral of the function (4.1) diverges:

$$\begin{aligned}\int\limits_{\Omega\cap\mathscr{U}} |\nabla_x u(x)|^2\,dx &\geq c\int\limits_{\mathbf{B}_R^{n-1}} H(y)\Big(\frac{1}{2}c_0^2 r^{-2\gamma} - c_1^2\Big)\,dy \\ &\geq c\int\limits_0^R \Big(\frac{1}{2}c_0^2 r^{1-\gamma} - c_1^2 r^{1+\gamma}\Big) r^{n-2}\,dr.\end{aligned} \tag{4.20}$$

On the other hand, for $\gamma \in (0,n)$ the Dirichlet integral converges.

5. Necessity of the condition $\gamma < n$

We assume that the problem (1.3) admits a generalized solution $u \in H^1(\Omega)$. We introduce the following test functions for the integral identity (3.2):

$$v_k(x) = \begin{cases} 0, & x \in \Omega \setminus \mathscr{U}, \\ 2^{k(n-2+\gamma)/2}\sigma(2^k y), & y \in \Omega \cap \mathscr{U}. \end{cases}$$

Here, $k = 1, 2, \ldots$ and σ is a nontrivial function from the space $C^\infty(\mathbf{R}^{n-1})$ with the support in the annulus $\{y : r \in (R/2, R)\}$. Because of the normalization factor $2^{k(n-2+\gamma)/2}$, one readily verifies that

$$\|v_k; H^1(\Omega)\| \leq c, \quad k = 1, 2, \ldots.$$

Since the sets $\operatorname{supp} v_k$ form a sequence diminishing to the point $\mathscr{O}$, the left-hand side of identity (3.2) tends to zero as $k \to \infty$. Under the condition (1.5) the right hand side of (3.2) equals to

$$\begin{aligned} &\int_{\mathbf{B}_R^{n-1}} v_k(y) G(y, H_+(y))(1 + |\nabla_y H_+(y)|^2)^{1/2}\, dy \\ &= 2^{k(n-2+\gamma)/2} 2^{-k(n-1)} \int_{\mathbf{B}_{2^{-k}R}^{n-1}} \sigma(\eta)(G(\mathscr{O}) + o(1))\, d\eta \\ &= 2^{k(\gamma-n)/2}\Big(G(\mathscr{O}) \int_{\mathbf{R}^{n-1}} \sigma(\eta)\, d\eta + o(1)\Big), \quad k \to +\infty. \end{aligned} \tag{5.1}$$

Therefore, for $g(\mathscr{O}) \int \sigma(\eta)\, d\eta \neq 0$ and $\gamma - n \geq 0$ the right hand side of (3.2) admits a non null limit. The obtained contradiction shows the necessity of the condition (1.7).

Remark 5.1. Small changes in the proof of (5.1) lead to the following result. In the case of $\gamma + m \geq n$, with an integer m, problem (1.3) is unsolvable in the class $H^1(\Omega)$, if some derivative of order $k \leq m$ of the function $y \mapsto G(y, H(y))$ does not vanish at $y = 0$.

6. Conclusions

It is clear that the obtained results remain valid for any scalar, formally self adjoint differential operator $-\nabla_x \cdot A \nabla_x$, where A is an $n \times n$-matrix valued function, uniformly positive definite, with measurable, bounded components almost everywhere in Ω.

It seems possible that there are different conditions on the tangency exponent γ (compare to (1.2)) for the existence of bounded energy solutions for elasticity boundary value problems on non smooth domains with tractions prescribed on the boundary $\partial\Omega$ of elastic body Ω, i.e., with the Neumann boundary conditions. This view is supported by some known results in the elasticity theory, including, the weighted Korn inequality and the results on asymptotic structures of the elastic fields in the vicinity of cuspidal singularities of boundaries (see [22], [23]). The precise formulation of such conditions remains an open problem.

Changing to some other types of boundary conditions in general leads to new constraints on the exponent γ. For example, in the case of the Dirichlet boundary conditions (1.6), the bounded energy solutions are obtained only for

$$\gamma < n - 2 \ . \tag{6.1}$$

This is just the reason in two spatial dimensions (see [8], [9], Chapter 14) why the Dirichlet integral of a solution to (1.6) with data (1.5) is infinite for all $\gamma > 0$ for $G(\mathscr{O}) \neq 0$. This can be justified using the methods of [28] and [8]. We only recall, that for a smooth function G the asymptotic anszatz (4.1) is simplified and takes the form $u(x) \sim \zeta G(\mathscr{O})$, and the constraint (6.1) for γ comes out from the reasoning similar to (4.20).

The geometrical form designated in Figure 5 is still a subject for further research. If, for example, $H_- = 0$, and the surface Γ_+ inside of the neighbourhood $\mathscr{U}$ is given by the following equation

$$y_{n-1}^{2q} + P(y_1, \ldots, y_{n-2}) = 1 \ , \tag{6.2}$$

where P is a homogeneous polynomial of order $2(q+p)$, and p, q are positive integers, then our arguments can be used to show that the problem (1.3) along with boundary conditions (1.5) admits a solution in the class $H^1(\Omega)$ for $2(q+p) < n-1$, and that there is no such solution for $2q \geq n$. However, a complete analysis would still require a new approach.

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Received 8 November 2007