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DEFICIENCIES OF CERTAIN CLASSES OF MEROMORPHIC FUNCTIONS

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Abstract. Denote by B the class of transcendental meromorphic functions for which the set of finite critical and asymptotic values is bounded, and by S that class of functions for which this set is finite. We give some conditions on transcendental deficient functions of members of the class B, including an improvement of a result of Langley and Zheng [10]. This leads to corresponding results for the class S. It is also proved that no derivative of a finite lower order periodic function has non-zero finite deficient values. We show by example that the finite lower order condition is necessary here.

1. Introduction

The asymptotic and critical values of a meromorphic function f (where here and henceforth meromorphic should be taken to be mean meromorphic in the plane) together constitute the singular values of the inverse function f^{-1}. These singular values play a significant role in complex dynamics [1, 3, 13]. The class B consists of all transcendental meromorphic functions for which the inverse has a bounded set of finite singular values. The subclass S contains those functions for which this set of singular values is finite. We consider slowly growing Nevanlinna deficient functions of members of the classes B and S. All definitions and terminology are as in [6].

Theorem 1. *Let f be a member of the class B of finite lower order, and let h be a zero order transcendental meromorphic function with deficient poles; that is, $\delta(\infty,h)>0$. Then $\delta(0,f-h)=0$.*

We shall obtain the following related theorem for functions h of small positive order.

Theorem 2. *Let $0<\delta,\nu<1$ and let f be a member of the class B of finite lower order λ. Then there exists $\rho>0$ such that if h is a transcendental meromorphic function of order less than ρ satisfying $\delta(\infty,h)>26\rho(h)^{1-\nu}$ then $\delta(0,f-h)<\delta$.*

Moreover, for $\varepsilon>0$ we may take $\rho=\delta^{(1+\varepsilon)/\nu}$ provided $\delta\le\delta_0(\varepsilon,\lambda)$ where δ_0 is positive and depends only on ε and λ.

The next result partially extends Theorem 1 to functions $f\in B$ of arbitrary order.

Theorem 3. *Let f belong to the class B and let h be transcendental and meromorphic with deficient poles, and such that*

$$T(r,h)=O(\log r)^P \quad \text{as } r\to\infty$$

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for some P. Then $\delta(0, f-h) = 0$.

Theorems 1, 2 and 3 together substantially improve a result from [10], in which it was shown that if f is the class B and h is transcendental meromorphic with finitely many poles such that $T(r,h) = o(\log r)^2$ as $r \to \infty$, then $\delta(0, f-h) = 0$.

For h to be called a deficient function of f it is normally required that $T(r,h) = o(T(r,f))$ as $r \to \infty$, but this is not necessary for Theorems 1, 2 or 3. Thus in each case we are also considering whether $f \in B$ can be a deficient function of h. Note, however, that $f-h$ is non-constant as we shall see that the deficiency of the poles of h ensures that $h \notin B$.

We can modify the hypotheses of the above three results by using the following observation on deficient functions, the proof of which is given later.

Lemma 1. *If f and h are meromorphic functions such that either*

$$T(r,h) = o(T(r,f)) \quad \text{or} \quad T(r,f) = o(T(r,h)) \quad \text{as } r \to \infty \tag{1.1}$$

then

$$\delta\left(0, \frac{1}{f-a} - \frac{1}{h-a}\right) = \delta(0, f-h) \quad \text{for all } a \in \mathbf{C}.$$

By applying this, Theorems 1, 2 and 3 immediately give the following corollary.

Corollary. *Let $a \in \mathbf{C}$ and let f be a transcendental meromorphic function such that the set of singular values of the inverse function f^{-1} does not accumulate at a.*

(i) *If f has finite lower order and h is a zero order transcendental meromorphic function satisfying* (1.1) *and with deficient value a, then $\delta(0, f-h) = 0$.*
(ii) *Suppose that $0 < \delta, \nu < 1$ and that f has finite lower order. Then there exists $\rho > 0$ such that, for all transcendental meromorphic functions h satisfying* (1.1) *with order less than ρ and $\delta(a,h) > 26\rho(h)^{1-\nu}$, we have $\delta(0, f-h) < \delta$.*
(iii) *If h is a transcendental meromorphic function satisfying* (1.1), *with deficient value a, and such that $T(r,h) = O(\log r)^P$ as $r \to \infty$ for some P, then $\delta(0, f-h) = 0$.*

If f is in the class S then it satisfies the condition in the above corollary for any value of a. Note also that it was shown in [10] that a non-constant rational function cannot be a deficient function of a member of the class S.

Turning our attention to periodic functions, we see from the example $e^z + a$ that omitted values are possible. However, the derivative of this example has no non-zero finite deficient values. In fact, this holds in general for all derivatives of meromorphic periodic functions of finite lower order. Some counterexamples of infinite lower order are constructed in section 7.

Theorem 4. *Let f be a periodic meromorphic function of finite lower order. Then f' has no non-zero finite deficient values.*

2. Preliminaries

The following is included for completeness.

Proof of Lemma 1. We may assume that $a = 0$ and that $T(r,h) = o(T(r,f))$. We need two simple estimates; firstly $T(r, 1/f - 1/h) \geq T(r, f-h)(1+o(1))$, and secondly

$$n\left(r, \frac{hf}{f-h}\right) \leq n(r, 1/(f-h)) + 2n(r, h),$$

as the function $hf/(f-h)$ has poles only where f and h both have poles or where $f-h=0$. It then follows that $\delta(0, 1/f - 1/h) \geq \delta(0, f-h)$, and since $1/(1/f) = f$ we get equality. □

The only property of mappings in the class B on which this paper relies is that described in the next lemma.

Lemma 2. ([3, 13]) *Let f belong to the class B. Then there exist $L > 0$ and $M > 0$ such that if $|z| > L$ and $|f(z)| > M$ then*

$$\left|\frac{zf'(z)}{f(z)}\right| \geq \frac{\log|f(z)/M|}{C}$$

where C is a positive absolute constant.

Lemma 3. ([7]) *Let $S(r)$ be an unbounded positive non-decreasing function on $[r_0, \infty)$, continuous from the right, of order ρ and lower order λ. Let $A > 1$ and $B > 1$. Then*

$$S(Ar) < BS(r)$$

outside an exceptional set G satisfying

$$\overline{\text{logdens}}\, G \leq \rho\left(\frac{\log A}{\log B}\right), \quad \underline{\text{logdens}}\, G \leq \lambda\left(\frac{\log A}{\log B}\right).$$

Lemma 4. ([5]) *Let h be a meromorphic function of order ρ. If $\rho < \sigma < 1/2$ then*

$$\underline{\text{logdens}}\{r > 0 : \log L(r,h) > \cos\pi\sigma m(r,h) - \pi\sigma\sin\pi\sigma T(r,h)\} \geq 1 - \rho/\sigma$$

where $L(r,h) = \min\{|h(z)| : |z| = r\}$.

In particular, it follows from Lemma 4 that if h has deficient poles and order zero then there exists a positive constant d such that

$$\log L(r,h) > dT(r,h)$$

on a set of logarithmic density 1. A standard argument [12, p. 287] shows that such functions h cannot belong to the class B.

The following lemma is a routine consequence of Fuchs' small arcs lemma [4], the version stated here is derived from [8, p. 721].

Lemma 5. ([11]) *Let g be a non-constant meromorphic function and let $0 < \eta < 1$.*

(i) *There exist a constant $K(\eta) \geq 1$ depending only on η and a subset $I_\eta \subseteq [0, \infty)$ of lower logarithmic density at least $1 - \eta$ such that if $r \in I_\eta$ is large and F_r is a subinterval of $[0, 2\pi]$ of length m then*

$$\int_{F_r} \left|\frac{rg'(re^{i\theta})}{g(re^{i\theta})}\right| d\theta \leq K(\eta)T(er, g)m\log\left(\frac{2\pi e}{m}\right).$$

(ii) *If g has finite lower order then there exist a positive constant L and a subset $J_\eta \subseteq [0,\infty)$ of upper logarithmic density at least $1-\eta$ such that if $r \in J_\eta$ is large and F_r is a subinterval of $[0,2\pi]$ of length m then*

$$\int_{F_r} \left| \frac{rg'(re^{i\theta})}{g(re^{i\theta})} \right| d\theta \le LT(r,g) m \log\left(\frac{2\pi e}{m}\right).$$

The following Fuchs type result is key to the proof of Theorems 1 and 2.

Lemma 6. *Let h be a meromorphic function.*

(i) *Suppose that h has order zero (respectively lower order zero) and let $\delta_1, \delta_2 \in (0,1)$. Then*

$$\int_0^{2\pi} \left| \frac{rh'(re^{i\theta})}{h(re^{i\theta})} \right| d\theta < \delta_1 T(r,h)$$

for all r outside an exceptional set E of upper (respectively lower) logarithmic density at most δ_2.

(ii) *There exists a positive absolute constant K_0 such that if the order of h satisfies $0 < \rho(h) < \frac{1}{32}$ then*

$$\int_0^{2\pi} \left| \frac{rh'(re^{i\theta})}{h(re^{i\theta})} \right| d\theta < K_0 \rho(h) T(r,h) \tag{2.1}$$

outside an exceptional set of upper logarithmic density at most $\frac{1}{4}$.

Remark. It is standard (using for example [11, Lemma 6]) that Lemma 6(i) implies that the integral is $o(T(r,h))$ as $r \to \infty$ outside a set of zero logarithmic density (respectively zero lower logarithmic density).

Proof of Lemma 6. For $0 < r < R$, integrating the differentiated Poisson–Jensen formula [9, p. 65] leads to

$$\int_0^{2\pi} \left| \frac{rh'(re^{i\theta})}{h(re^{i\theta})} \right| d\theta \le \frac{8\pi R r}{(R-r)^2}(T(R,h)+O(1)) + 2\sum_{|c_k|<R} H_k, \tag{2.2}$$

where the c_k are the zeroes and poles of h repeated according to multiplicity and

$$H_k = r\int_0^{2\pi} \frac{d\theta}{|re^{i\theta}-c_k|} = 2r\int_0^{\pi} \frac{d\theta}{|re^{i\theta}-|c_k||}. \tag{2.3}$$

We proceed to estimate the H_k. Defining $\gamma_k = |r-|c_k||/r$, for a given r, and following Fuchs [4] we divide the c_k into two classes:

(I) those c_k for which $\gamma_k < \pi/2$, i.e., $|r-|c_k|| < \pi r/2$,
(II) those c_k for which $\gamma_k \ge \pi/2$, i.e., $|r-|c_k|| \ge \pi r/2$.

For $c_k \in$ (II)

$$H_k \le \frac{2\pi r}{|r-|c_k||} \le 4 \quad \text{and so} \quad \sum_{\substack{|c_k|<R \\ c_k \in \text{(II)}}} H_k \le 4n(R), \tag{2.4}$$

where $n(R) = n(R,h) + n(R,1/h)$ is the number of c_k lying in $|z| \le R$.

For $c_k \in$ (I), using (2.3) shows that

$$H_k \le 2r\int_0^{\gamma_k} \frac{d\theta}{|r-|c_k||} + 2r\int_{\gamma_k}^{\pi/2} \frac{d\theta}{r\sin\theta} + 2r\int_{\pi/2}^{\pi} \frac{d\theta}{r} \le 2+\pi+\pi\log\frac{\pi r}{2|r-|c_k||}. \tag{2.5}$$

To count the number of $|c_k|$ near r we define

$$\mu(r,t) = \#\{|c_k| < R : |r-|c_k|| < t\},$$

counting with multiplicities. Set $R = \alpha 2^n$ for $\alpha > 2$. An application of Cartan's Lemma (formula (6.5.17) of [8, p. 367]) with $h_n = 2^{n-3}\delta_2/3$ gives that

$$\mu(r,t) < \frac{n(R)t}{eh_n} = \frac{48n(R)t}{2^{n+1}e\delta_2}, \quad 0 < t < \infty, \tag{2.6}$$

for $r \in [2^n, 2^{n+1}]$ outside an exceptional set E_n of linear measure at most $12h_n = 2^{n-1}\delta_2$. Combining (2.5) and (2.6) yields

$$\sum_{\substack{|c_k|<R\\ c_k\in(\mathrm{I})}} H_k \le \int_{t=0}^{\pi r/2}\left(2+\pi+\pi\log\frac{\pi r}{2t}\right)d\mu(r,t) \le \frac{48\pi(1+\pi)}{e\delta_2}n(R), \tag{2.7}$$

for $r \in [2^n, 2^{n+1}] \setminus E_n$. Observe that

$$n(R) \le n(\alpha r, h) + n(\alpha r, 1/h) \le \frac{2}{\log\alpha}(T(\alpha^2 r, h) + O(1)).$$

Using this, (2.4) and (2.7), the estimate (2.2) becomes

$$\begin{aligned} I &= \int_0^{2\pi}\left|\frac{rh'(re^{i\theta})}{h(re^{i\theta})}\right|d\theta \\ &\le \frac{8\pi Rr}{(R-r)^2}(T(R,h)+O(1)) + \left(8+\frac{96\pi(1+\pi)}{e\delta_2}\right)n(R) \\ &\le 16\left(\frac{\pi\alpha}{(\alpha-2)^2} + \frac{1}{\log\alpha}\left(1+\frac{12\pi(1+\pi)}{e\delta_2}\right)\right)(T(\alpha^2 r,h)+A), \end{aligned} \tag{2.8}$$

for $r \in [2^n, 2^{n+1}]\setminus E_n$ and some constant A. Hence for $2^m < s \le 2^{m+1}$ inequality (2.8) holds for all $r \in [1, s]$ outside a set of linear measure at most $\delta_2(2^{-1}+2+\ldots+2^{m-1}) < \delta_2 s$. Therefore (2.8) holds for all $r > 0$ outside an exceptional set E' with upper linear density at most δ_2. We now prove the two parts of the lemma separately.

(i) Assume that h has order zero (respectively lower order zero). Then Lemma 3 gives that $T(\alpha^2 r, h) + A \le 2T(r,h)$ outside a set E'' of upper (respectively lower) logarithmic density zero. Hence $E = E' \cup E''$ has upper (respectively lower) logarithmic density at most δ_2 and for $r \notin E$,

$$\int_0^{2\pi}\left|\frac{rh'(re^{i\theta})}{h(re^{i\theta})}\right|d\theta \le 32\left(\frac{\pi\alpha}{(\alpha-2)^2} + \frac{1}{\log\alpha}\left(1+\frac{12\pi(1+\pi)}{e\delta_2}\right)\right)T(r,h).$$

The proof of part (i) is thus completed by choosing α sufficiently large.

(ii) Assume now that the order of h satisfies $0 < \rho(h) < \frac{1}{32}$. Applying Lemma 3 gives that

$$T(\alpha^2 r, h) + A \le eT(r,h) + A \le 3T(r,h)$$

outside a set E'' of upper logarithmic density at most $2\rho(h)\log\alpha$. Thus taking $\delta_2 = \frac{1}{8}$ and $\log\alpha = \frac{1}{16\rho(h)} > 2$, the upper logarithmic density of $E' \cup E''$ does not exceed $\frac{1}{4}$ and by (2.8)

$$\int_0^{2\pi} \left| \frac{rh'(re^{i\theta})}{h(re^{i\theta})} \right| d\theta \leq 48(16\rho(h)) \left(\frac{\pi\alpha\log\alpha}{(\alpha-2)^2} + 1 + \frac{96\pi(1+\pi)}{e} \right) T(r,h)$$

for $r \notin E' \cup E''$. Since the term $\pi\alpha\log\alpha/(\alpha-2)^2$ is bounded for $\alpha > e^2$, we can find an absolute constant K_0 to complete the proof. □

3. Proof of Theorem 1

Let f and h be as in the hypothesis, but suppose that $\delta(0, f-h) > 0$.

Lemma 7. *There exist positive constants m, c and a set J of positive upper logarithmic density such that, for $r \in J$,*

$$\log|f(z) - h(z)| < -cT(r, f-h) \tag{3.1}$$

on a subset Σ_r of $S(0,r)$ of angular measure at least m. Furthermore, for $z \in \Sigma_r$,

$$zf'(z) = zh'(z) + o(1) \quad \text{as } r \to \infty \text{ in } J. \tag{3.2}$$

Proof. Since $\delta(0, f-h) > 0$ we can pick z_0 with $|z_0| = r$, for all large r, such that

$$\log|f(z_0) - h(z_0)| < -\frac{1}{2}\delta(0, f-h)T(r, f-h).$$

Let Ω_r be that arc of $S(0,r)$ with midpoint z_0 and angular measure $2m$. Choosing m sufficiently small and applying Lemma 5(ii) to $f-h$ gives a set J, of positive upper logarithmic density, such that for $r \in J$ the estimate (3.1) holds on Ω_r. Furthermore, by applying Lemma 5(ii) with $F_r = \Omega_r$, we see that the subset of Ω_r on which

$$\left| \frac{z(f'(z) - h'(z))}{f(z) - h(z)} \right| \leq 2LT(r, f-h)\log(\pi e/m)$$

must have measure at least m. Let Σ_r be this subset. Now (3.2) follows from (3.1) for $z \in \Sigma_r$. □

The remark following Lemma 4 shows that we can find a positive constant d such that

$$\log L(r,h) > dT(r,h) \tag{3.3}$$

on a set of logarithmic density 1. Let J' be that subset of J on which (3.3) holds and note that J' has positive upper logarithmic density. Using Lemma 7 and (3.3) gives that, for $z \in \Sigma_r$,

$$f(z) \to \infty \quad \text{and} \quad \frac{zf'(z)}{f(z)} = \frac{zh'(z)}{h(z)}(1+o(1)) + o(1), \quad \text{as } r \to \infty \text{ in } J'. \tag{3.4}$$

Hence the hypothesis of Lemma 2 is satisfied by f and $z \in \Sigma_r$ for all sufficiently large $r \in J'$, and using Lemma 2, (3.1), (3.3) and (3.4) yields

$$dT(r,h) \leq \log|f(z) + o(1)| \leq O\left(\left| \frac{zf'(z)}{f(z)} \right| \right) \leq O\left(\left| \frac{zh'(z)}{h(z)} \right| \right)$$

for $z \in \Sigma_r$ as $r \to \infty$ in J'. Since the angular measure of Σ_r is at least m, this leads to a contradiction with Lemma 6(i) thus proving the theorem.

4. Proof of Theorem 2

Let f, δ and ν be as in the hypothesis. Assume that the transcendental meromorphic function h satisfies $\delta(\infty, h) > 26\rho(h)^{1-\nu}$ and $\delta(0, f-h) \geq \delta$.

Let K_0 be as in Lemma 6(ii) and let K_1 be the constant $K(\frac{1}{8})$ of Lemma 5(i); then $K_1 \geq 1$. Define the constants $C_1 = 16CK_0K_1$ and $C_2 = \pi/2CK_0$ where C is as in Lemma 2. We may assume that $C_2 < \frac{1}{16}$ since Lemma 6(ii) continues to hold if we demand that $K_0 > 8\pi/C$. The function

$$\phi(x) = C_1 e^{4\lambda+1} x \log(C_2 e/x)$$

is strictly increasing for $0 < x < C_2$ and $\phi(C_2/2) \geq 4\pi K_1 > \delta$ so that we may define $\rho^\nu < C_2/2$ by $\phi(\rho^\nu) = \delta$.

We aim to show that $\rho(h) \geq \rho$. We will then be done, because for $\varepsilon > 0$ and δ less than some positive $\delta_0(\varepsilon, \lambda)$ we see that $\phi(\delta^{1+\varepsilon}) < \delta$. Since we have $\rho < C_2/2$ we may assume that $\rho(h) < \frac{1}{32}$. It follows that the lower order $\lambda(f-h)$ is less than $\lambda + \frac{1}{32}$.

By Theorem 1 and the above we have that $0 < \rho(h) < \frac{1}{32}$. Applying Lemma 4 to h and taking $\sigma = 8\rho(h)$ in the notation there now leads to

$$\underline{\text{logdens}} \left\{ r > 0 : \log L(r,h) > \frac{\sqrt{2}}{2} \left(\frac{m(r,h)}{T(r,h)} - 8\pi\rho(h) \right) T(r,h) \right\} \geq \frac{7}{8}.$$

Therefore, recalling that $\delta(\infty, h) > 26\rho(h)^{1-\nu}$, we get that

$$\log L(r,h) > \frac{\sqrt{2}}{2}(26 - 8\pi\rho(h)^\nu)\rho(h)^{1-\nu}T(r,h) > \frac{\rho(h)^{1-\nu}T(r,h)}{2} \tag{4.1}$$

on a set of lower logarithmic density at least $\frac{7}{8}$. Hence $h \notin B$ and $f-h$ is non-constant [12, p. 287].

Applying Lemma 5(i) to $f-h$ with $\eta = \frac{1}{8}$, followed by Lemma 3 with $S(r) = T(r, f-h)$, $A = e$ and $B = e^{4\lambda(f-h)}$, we obtain a set H of upper logarithmic density at least $\frac{5}{8}$ such that, for $r \in H$,

$$\int_{F_r} \left| \frac{r(f'(re^{i\theta}) - h'(re^{i\theta}))}{f(re^{i\theta}) - h(re^{i\theta})} \right| d\theta \leq K_1 e^{4\lambda+1} T(r, f-h) m \log\left(\frac{2\pi e}{m}\right), \tag{4.2}$$

where F_r is any interval of length m. Let H' be that subset of H on which (4.1) holds; then H' has upper logarithmic density at least $\frac{1}{2}$.

Choose $m = C_1\rho^\nu/4K_1 = 2\pi\rho^\nu/C_2 < \pi$. Then

$$K_1 e^{4\lambda+1} m \log\left(\frac{2\pi e}{m}\right) = \frac{\phi(\rho^\nu)}{4} = \frac{\delta}{4}. \tag{4.3}$$

Lemma 8. *There exist $c > 0$ and, for each $r \in H$, a subset Σ_r of $S(0,r)$ of angular measure at least m on which*

$$\log|f(z) - h(z)| < -cT(r, f-h) \tag{4.4}$$

and

$$zf'(z) = zh'(z) + o(1) \quad \text{as } r \to \infty \text{ in } H. \tag{4.5}$$

Proof. Since $\delta(0, f-h) \geq \delta$ we can pick z_0 with $|z_0| = r$, for all large r, such that

$$\log|f(z_0) - h(z_0)| < -\frac{1}{2}\delta T(r, f-h).$$

Let Ω_r be that arc of $S(0,r)$ with midpoint z_0 and angular measure $2m$. Using (4.2) and (4.3) we see that for $r \in H$ the estimate (4.4) holds on Ω_r with $c = \delta/4$. By considering $F_r = \Omega_r$ in (4.2) with m replaced by $2m$, we see that the subset of Ω_r on which

$$\left|\frac{z(f'(z) - h'(z))}{f(z) - h(z)}\right| \leq 2K_1 e^{4\lambda+1} T(r, f-h)\log(\pi e/m)$$

must have measure at least m. Let Σ_r be this subset. Now (4.5) follows from (4.4) for $z \in \Sigma_r$. □

It follows from Lemma 8 and (4.1) that, for $z \in \Sigma_r$,

$$f(z) \to \infty \quad \text{and} \quad \frac{zf'(z)}{f(z)} = \frac{zh'(z)}{h(z)}(1 + o(1)) + o(1), \quad \text{as } r \to \infty \text{ in } H'. \tag{4.6}$$

The hypothesis of Lemma 2 is now satisfied by f and $z \in \Sigma_r$ for all sufficiently large $r \in H'$. Therefore Lemma 2, (4.1), (4.4) and (4.6) now yield

$$\frac{\rho(h)^{1-\nu}T(r,h)}{2} \leq \log|f(z) + o(1)| \leq 2C\left|\frac{zh'(z)}{h(z)}\right| \tag{4.7}$$

for $z \in \Sigma_r$ as $r \to \infty$ in H'.

By Lemma 6(ii) there exist large $r \in H'$ satisfying (2.1). Since Σ_r has angular measure at least m, upon integrating (4.7) and comparing with (2.1) we see that we must have $\rho(h)^\nu > m/4CK_0 = \rho^\nu$.

5. Proof of Theorem 3

Most of the proof of Theorem 3 will be contained in the next three lemmas.

Lemma 9. *Suppose that for $j = 1, \ldots, N$ the functions $\psi_j(r)$ are positive and non-decreasing on $[e, \infty)$, continuous from the right and such that $\psi_j(r) = O(\log r)^P$ as $r \to \infty$ for some P. Let $\alpha > 1$ and $\delta > 0$. Then there exist a constant B and a set E of lower logarithmic density at most δ such that, for $r \notin E$,*

$$\psi_j(r^\alpha) \leq B\psi_j(r)$$

for each $j = 1, \ldots, N$.

Proof. Define $\phi_j(s) = \psi_j(e^s) = O(s^P)$ for $s \geq 1$. Then Lemma 3 applies to ϕ_j (we may assume that ψ_j is unbounded) to give $\phi_j(\alpha s) < B\phi_j(s)$ for s outside an exceptional set G_j. The constant B is chosen so large that $\overline{\text{logdens}}\, G_j \leq \delta/N$, so that the lower logarithmic density of $G = \bigcup G_j$ does not exceed δ. Taking $E = \{r \geq e : \log r \in G\}$ and $r = e^s \notin E$, we now have that, for each j,

$$\psi_j(r^\alpha) = \phi_j(\alpha s) < B\phi_j(s) = B\psi_j(r).$$

Suppose now that $\underline{\text{logdens}}\, E > l > \delta$. Let χ_E be the characteristic function of E. Then

$$L(r) = \int_e^r \chi_E(t)\frac{dt}{t} > l\log r - c$$

for some constant c and all $r \geq e$, and we calculate

$$\int_{[1,s]\cap G} \frac{d\tau}{\tau} = \int_e^r \frac{dL(t)}{\log t} = \frac{L(r)}{\log r} + \int_e^r \frac{L(t)}{t(\log t)^2}\, dt > l \log s + l - c,$$

so that $\underline{\text{logdens}}\, G \geq l > \delta$, in contradiction to our choice of B. □

We apply the previous lemma to obtain the following new pointwise estimate for the logarithmic derivative of a slowly growing meromorphic function.

Lemma 10. *Let h be meromorphic such that $T(r,h) = O(\log r)^P$ for some P and let $0 < \delta \leq 1$. Then*

$$M\left(r, \frac{zh'}{h}\right) = o(T(r,h))$$

for r outside a set of lower logarithmic density δ.

We remark that by applying, for example, [11, Lemma 6] we can in fact take $\delta = 0$ in the above statement.

Proof of Lemma 10. We may assume that h is transcendental. Define $n(r) = n(r,h) + n(r,1/h)$. Then

$$n(r) \leq \frac{2T(r^2,h)}{\log r} + o(1) = O(\log r)^{P-1}. \tag{5.1}$$

Using (5.1) and applying Lemma 9 to $n(r)$ and $T(r,h)$ with $\alpha = 2$ in the notation there, we obtain a set E of lower logarithmic density at most $\delta/2$ such that

$$n(r^2) = O\left(\frac{T(r,h)}{\log r}\right) \quad \text{for } r \notin E. \tag{5.2}$$

Since h has zero order, we see from the Weierstrass product representation [6] that

$$\left|\frac{h'(z)}{h(z)}\right| \leq \sum \frac{1}{|r - |a_k||}, \tag{5.3}$$

where $r = |z|$ and the a_k are the zeroes and poles of h repeated according to multiplicity. Suppose that $r \in [2^{n-1}, 2^n)$ and let $s = 2^n$ and

$$\mu(r,t) = \#\{|a_k| < s(\log s)^P : |r - |a_k|| < t\}.$$

Cartan's Lemma [8, p. 367, (6.5.17)] gives, with $h_n = \delta s/96$,

$$\frac{\mu(r,t)}{t} < \frac{96n(s(\log s)^P)}{e\delta s}$$

for $0 < t < \infty$ and $r \in [2^{n-1}, 2^n) \setminus F_n$, where the exceptional set F_n has measure at most $\delta s/8$. Since μ is integer-valued we have

$$\mu(r,t) = 0 \quad \text{for} \quad t \leq t_0 = \frac{e\delta s}{96n(s(\log s)^P)}.$$

Therefore, for $r \in [2^{n-1}, 2^n) \setminus F_n$,

$$\begin{aligned}\sum_{|a_k| \le r(\log r)^P} |r - |a_k||^{-1} &\le \int_{t_0}^{s(\log s)^P} \frac{d\mu(r,t)}{t} \\ &\le \frac{96 n(s(\log s)^P)}{e\delta s}\left(1 + \int_{t_0}^{s(\log s)^P} \frac{dt}{t}\right) \qquad (5.4) \\ &\le \frac{96 n(r^2)}{e\delta r}\left(1 + \log \frac{96 n(r^2)(\log 2r)^P}{e\delta}\right)\end{aligned}$$

since $s(\log s)^P \le r^2$ for r sufficiently large. Hence if $2^{m-1} < R < 2^m$ then (5.4) holds for $r \in [1, R]$ outside a set of measure $\delta(\frac{1}{4} + \frac{1}{2} + \ldots + 2^{m-3}) < \frac{1}{2}\delta R$. Therefore (5.4) holds for all r outside a set F of upper linear density $\delta/2$. Using (5.1) and (5.2) this gives

$$(5.5) \qquad \sum_{|a_k| \le r(\log r)^P} |r - |a_k||^{-1} = O\left(\frac{T(r,h)\log\log r}{r \log r}\right) = o\left(\frac{T(r,h)}{r}\right)$$

for $r \notin E \cup F$, and furthermore $\underline{\text{logdens}}(E \cup F) \le \delta$.

We now consider those a_k for which $|a_k| > r(\log r)^P$. For such a_k we have $|r - |a_k|| > |a_k|/2$ for r large enough. Using this,

$$(5.6) \qquad \sum_{|a_k| > r(\log r)^P} |r - |a_k||^{-1} \le \int_{r(\log r)^P}^{\infty} \frac{2}{t}\, dn(t) \le C \int_{r(\log r)^P}^{\infty} \frac{(\log t)^{P-1}}{t^2}\, dt$$

for some constant C by (5.1). We observe that

$$I_q = \int_R^\infty \frac{(\log t)^q}{t^2}\, dt = O\left(\frac{(\log R)^q}{R}\right), \quad \text{for } q \in \mathbf{R},$$

as for $q \le 0$ this is trivial, and since $I_q = qI_{q-1} + (\log R)^q/R$ the above holds for all q by induction. Using this and (5.6) now gives that

$$(5.7) \qquad \sum_{|a_k| > r(\log r)^P} |r - |a_k||^{-1} = O\left(\frac{(\log(r(\log r)^P))^{P-1}}{r(\log r)^P}\right) = o\left(\frac{1}{r}\right).$$

The result now follows from (5.3), (5.5) and (5.7). □

The proof of the next lemma is due to James Langley.

Lemma 11. *Let G be a transcendental meromorphic function and suppose that 0 is a deficient value of G. Then for all r outside a set of finite logarithmic measure there exists some z with $|z| = r$ such that $|G(z)| = o(1)$ and $|zG'(z)| = o(1)$ as $r \to \infty$.*

Proof. Write $T(r) = T(r, G)$ and let $p(s) = T(e^s)^{\frac{1}{2}}$. Applying Borel's Lemma [6, Lemma 2.4] to $p(s)$ and taking $r = e^s$ and $R = r\exp(T(r)^{-\frac{1}{2}})$ gives

$$(5.8) \qquad T(R) < 4T(r)$$

for r outside a set of finite logarithmic measure. Let

$$H_r = \left\{t \in [0, 2\pi] : \log|G(re^{it})| < -\frac{1}{2}\delta(0, G)T(r)\right\}.$$

Then

$$\frac{1}{2\pi}\int_{H_r}\log^+\frac{1}{|G(re^{it})|}\,dt \geq \frac{1}{2}\delta(0,G)T(r)(1-o(1)). \tag{5.9}$$

Let $m(r)$ be the measure of H_r. Lemma III of [2] gives that

$$\frac{1}{2\pi}\int_{H_r}\log^+\frac{1}{|G(re^{it})|}\,dt \leq \frac{11R}{R-r}m(r)\left(1+\log^+\frac{1}{m(r)}\right)T(R,1/G). \tag{5.10}$$

Observing that $R/(R-r) = T(r)^{\frac{1}{2}}(1+o(1))$, and using (5.8) and (5.9), the inequality (5.10) becomes, for small $m(r)$,

$$\delta(0,G)(1-o(1)) \leq 88m(r)^{\frac{3}{4}}T(r)^{\frac{1}{2}}(1+o(1)),$$

and it follows that $m(r) > T(r)^{-\frac{3}{4}}$ for all r outside a set of finite logarithmic measure. Now consider

$$H_r' = \left\{t \in H_r : \log\left|\frac{G'(re^{it})}{G(re^{it})}\right| > T(r)^{\frac{7}{8}}\right\}.$$

If $H_r' = H_r$ then

$$m(r,G'/G) \geq \frac{m(r)}{2\pi}T(r)^{\frac{7}{8}} > \frac{T(r)^{\frac{1}{8}}}{2\pi},$$

but this contradicts the standard estimate that $m(r,G'/G) = O(\log T(r) + \log r)$ outside a set of finite logarithmic measure. Therefore we can pick $z = re^{it}$ with $t \in H_r \setminus H_r'$ and this z satisfies the statement of the lemma. □

We now proceed to prove Theorem 3. Let f and h be as in the hypothesis, but assume that $\delta(0,f-h) > 0$. By the remark following Lemma 4 there exists a positive constant d such that

$$\log L(r,h) > dT(r,h)$$

on a set of logarithmic density 1. Using this, and applying Lemma 11 to $f-h$ gives, for each r in a set of logarithmic density 1, a point $z = z_r$ with $|z| = r$ such that

$$f(z) = h(z) + o(1) \quad \text{and} \quad \frac{zf'(z)}{f(z)} = \frac{zh'(z)}{h(z)}(1+o(1)) + o(1)$$

as $r \to \infty$. Lemmas 2 and 10 combined with the above now give, for $z = z_r$,

$$dT(r,h) < \log|f(z)+o(1)| = O\left(\left|\frac{zf'(z)}{f(z)}\right|\right) = O\left(\left|\frac{zh'(z)}{h(z)}\right|\right) = o(T(r,h))$$

as $r \to \infty$ outside a set of small lower logarithmic density. This contradiction completes the proof of the theorem.

6. Proof of Theorem 4

We begin with the following elementary lemma.

Lemma 12. *For $r > 0$ and small positive m, let $L(\phi)$ be the length of the interval*

$$\left\{\operatorname{Re}\left(re^{i\theta}\right) : \theta \in [\phi, \phi+m]\right\}.$$

Then $L(\phi) \geq r\left(1-\cos\frac{m}{2}\right)$.

Proof.

$$L(\phi)=\begin{cases} r(1-\cos(\phi+m)), & \phi\in\left[-\frac{m}{2},0\right], \\ r(\cos\phi-\cos(\phi+m)), & \phi\in\left[0,\frac{\pi}{2}-\frac{m}{2}\right]. \end{cases}$$

L is clearly increasing over $\left[-\frac{m}{2},0\right]$. For $\phi\in\left(0,\frac{\pi}{2}-\frac{m}{2}\right)$, we find that $L'(\phi)\ge 0$ and so L is in fact increasing on $\left[-\frac{m}{2},\frac{\pi}{2}-\frac{m}{2}\right]$. By symmetry considerations we see that this implies that $L(\phi)\ge L\left(-\frac{m}{2}\right)=r\left(1-\cos\frac{m}{2}\right)$ for all ϕ. □

Now let f be a periodic meromorphic function of finite lower order and suppose that f' has a non-zero finite deficient value. Without loss of generality we may take both the period and the deficient value to be 1. Let δ be such that $\delta(1,f')>3\delta>0$.

Using Lemma 5 we find a small positive m and a set $J\subseteq[0,\infty)$ of upper logarithmic density at least $\frac{1}{2}$ such that, if $r\in J$ is large and F_r is a subinterval of $[0,2\pi]$ of length m, then

$$\int_{F_r}\left|\frac{rf''(re^{i\theta})}{f'(re^{i\theta})-1}\right|d\theta<\delta T(r,f'). \tag{6.1}$$

For large $r\in J$ there exists z_0 with $|z_0|=r$ and $\log|f'(z_0)-1|\le -3\delta T(r,f')$. Let Ω be an arc of $S(0,r)$ with endpoint z_0 and angular measure m. Then using (6.1) we see that

$$\log|f'(z)-1|\le -2\delta T(r,f'),\quad z\in\Omega. \tag{6.2}$$

For $n\in A=\mathbf{Z}\cap[-2r,2r]\setminus\{0\}$ the circle $S(0,r)$ intersects $S(n,r)$ at one or two points with real part $\frac{n}{2}$. For large r, Lemma 12 shows that the interval $\{\mathrm{Re}(z):z\in\Omega\}$ has length greater than 2, and so it must contain $\frac{N-1}{2},\frac{N}{2}$ for some $N-1,N\in A$. Hence Ω meets both $S(N-1,r)$ and $S(N,r)$, and we pick points of intersection α and β respectively. Note that $\alpha+1\in S(N,r)$ and that reflection of Ω in the line $\mathrm{Re}(z)=\frac{N}{2}$ gives an arc Ω' of $S(N,r)$ that contains $\alpha+1$ and β. Using (6.1) and the periodicity of f' and f'' we have,

$$\int_{\Omega'}\left|\frac{f''(z)}{f'(z)-1}\right||dz|<\delta T(r,f').$$

Since $\beta\in\Omega\cap\Omega'$, the above and (6.2) yield

$$\log|f'(z)-1|<-\delta T(r,f'),\quad z\in\Omega\cap\Omega'.$$

Let γ be the path joining α to β along Ω followed by the path from β to $\alpha+1$ along Ω'. Then the length of γ is at most $2mr$ and so

$$1=\left|\int_\gamma(f'(z)-1)dz\right|<2mr\exp(-\delta T(r,f')),$$

a contradiction for r sufficiently large.

7. Infinite order counterexamples

The entire periodic function

$$\int_0^{e^z}\frac{1-e^t}{t}\,dt$$

has derivative $1-e^{e^z}$ which omits the value 1. In fact, there exist derivatives of entire periodic functions having arbitrarily many deficient values. The rest of this section is devoted to constructing such an example.

For an integer $q \geq 2$ let

$$F(z) = \int_0^{e^z} \frac{1}{w} \left(\int_0^w e^{-t^q} dt \right) dw, \quad F'(z) = \int_0^{e^z} e^{-t^q} dt, \tag{7.1}$$

then F is entire and periodic. It shall be useful to define the function $G(z) = e^{-e^{qz}}$ and the set S as the union of the sectors

$$S_k = \left\{ z : \left| \arg z - \frac{2\pi k}{q} \right| \leq \frac{\pi}{2q} \right\}.$$

Lemma 13. *Taking $G(z)$ and S_k as above, the contribution to $m(r, 1/G)$ of the set where $e^z \in S_k$ is*

$$J_k = \frac{T(r,G)}{q}(1+o(1)), \quad \text{as } r \to \infty.$$

To exhibit the deficiencies of F' let $\omega = e^{2\pi i/q}$, and for integer k let

$$I_k = \int_0^{\omega^k \infty} e^{-t^q} dt = \omega^k I_0$$

where the path of integration is given by $t = \omega^k s$ for $s \in [0, \infty)$. Note that $I_k \neq 0, \infty$ and $I_j \neq I_k$ for $0 \leq j < k < q$. By Cauchy's Theorem

$$F'(z) = I_k - \int_{\gamma_k} e^{-t^q} dt$$

where γ_k follows the circular arc from e^z to $\omega^k |e^z|$ and then the ray $\omega^k s$ for $s \in [|e^z|, \infty)$. If $e^z \in S_k$, then for t lying on γ_k we have that $\left| e^{-t^q} \right| \leq |G(z)|$, so that writing $e^{-t^q} = qt^{q-1} e^{-t^q} / qt^{q-1}$ and integrating by parts yields

$$|F'(z) - I_k| \leq |G(z)| \left(\frac{e^{|(q-1)z|}}{q} + \frac{q-1}{q} \int_{\gamma_k} \frac{|dt|}{|t|^q} \right) = O(e^{qr} |G(z)|)$$

as $|z| = r \to \infty$. Using this together with Lemma 13 now leads to

$$\frac{T(r,G)}{q}(1+o(1)) \leq m\left(r, \frac{1}{F' - I_k}\right) + O(r), \quad \text{as } r \to \infty. \tag{7.2}$$

If $e^z \in S$ and t lies on the straight line joining the origin to e^z then $|e^{-t^q}| \leq 1$ so that $|F'(z)| \leq |e^z|$ by (7.1). If instead $e^z \notin S$ and t lies on the straight line joining the origin to e^z we see that $|e^{-t^q}| \leq |G(z)|$ so that by (7.1) we have $|F'(z)| \leq |e^z G(z)|$. Therefore

$$T(r,G) \geq T(r,F') - r.$$

Comparison with (7.2) now reveals that $\delta(I_k, F') \geq 1/q$ for $k = 0, \ldots, q-1$. Since F' is entire, the sum of the deficiencies cannot exceed 1 and so we must have equality here.

Proof of Lemma 13. We first observe that if $e^z \notin S$ then $|G(z)| \geq 1$. Hence these points contribute nothing to $m(r, 1/G)$ and so $T(r,G) = J_0 + \ldots + J_{q-1} + O(1)$. Thus it will suffice to prove that $J_k = J_l + o(T(r,G))$.

We remark that

$$e^z \in S_k \quad \Leftrightarrow \quad \left|\mathrm{Im}(z) - 2\pi\left(n + \frac{k}{q}\right)\right| \le \frac{\pi}{2q} \quad \text{for some integer } n.$$

From [6, p. 7] we have that

$$T(r,G) \sim \frac{e^{qr}}{\sqrt{2\pi^3 qr}}. \tag{7.3}$$

Calculate, for $z = re^{i\theta} \in S_k$,

$$\log^+ \frac{1}{|G(re^{i\theta})|} = e^{qr\cos\theta}\cos(qr\sin\theta) \tag{7.4}$$

and fix a small angle $\alpha > 0$. Then for $\theta \in [\alpha, 2\pi - \alpha]$, we have $\log^+ |G(re^{i\theta})|^{-1} = o(T(r,G))$ by (7.3). Note also that the angular measure of $\{z : |\mathrm{Im}\ z| \le 4\pi\}$ with respect to $S(0,r)$ is $O(1/r)$, so that the contribution to J_k from this region is $O(e^{qr}/r) = o(T(r,G))$.

Let J_k^+ and J_k^- denote the contributions to J_k from the upper and lower half planes respectively. It now follows from the above that, for $k = 0,\dots,q$,

$$J_k^+ = \sum_{n=1}^{N} H_{k,n} + o(T(r,G))$$

where $H_{k,n}$ is the contribution to J_k from

$$E_{k,n} = S(0,r) \cap \left\{z : \mathrm{Re}(z) > 0, \left|\mathrm{Im}(z) - 2\pi\left(n + \frac{k}{q}\right)\right| \le \frac{\pi}{2q}\right\}$$

and N is the least integer exceeding $1 + (r/2\pi)\sin\alpha$. In particular, $2\pi N \approx r\sin\alpha$ and N is independent of k.

Using (7.4) and changing from the angular variable θ to the scaled imaginary part $t = qr\sin\theta$ shows that

$$H_{k,n} = \int_{2\pi(nq+k)-\pi/2}^{2\pi(nq+k)+\pi/2} \cos t \frac{e^{\sqrt{q^2r^2-t^2}}}{\sqrt{q^2r^2-t^2}}\, dt. \tag{7.5}$$

For $0 < \theta < 2\alpha$ the variable t is positive but small compared to qr. Using this, and the fact that the function e^x/x is increasing for $x > 1$, it follows from (7.5) that $H_{k+1,n} \le H_{k,n}$. Therefore $J_{k+1}^+ \le J_k^+ + o(T(r,G))$ for $k = 0,\dots,q$. However $J_0^+ = J_q^+$ because $S_0 = S_q$, and so we must have that $J_k^+ = J_l^+ + o(T(r,G))$ for all k, l.

This argument can be repeated to show that $J_{k+1}^- \ge J_k^- + o(T(r,G))$ and hence equality (in this case t is negative so the inequality is reversed). □

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