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A CONTINUOUS FUNCTION THAT NO MEROMORPHIC FUNCTION CAN AVOID

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Abstract. We give an example of a function G with the property described in the title. We had previously given an example of a continuous function Φ that no holomorphic function can avoid. The function G is a modification of the function Φ, but the proof to show that a meromorphic function cannot avoid G involves a modification of the argument principle.

1. Introduction

Let $D = \{z : |z| < 1\}$, the unit circle in the complex plane. Each function in this note will have D as its domain, except for the function given in Theorem 2, and functions may assume complex values as well as the value ∞. Two functions f and g with common domain Ω will be said to avoid each other if $f(z) \neq g(z)$ for each $z \in \Omega$. When we say that a function f is continuous in a domain Ω, we mean that it is continuous *with respect to the spherical metric* $\chi(-,-)$, where

$$\chi(w,\tau) = \frac{|w-\tau|}{\sqrt{1+|w|^2}\sqrt{1+|\tau|^2}} \quad \text{for } w \neq \infty \neq \tau$$

and

$$\chi(w,\infty) = \frac{1}{\sqrt{1+|w|^2}} \quad \text{for } w \neq \infty.$$

Thus, f is continuous at the point $z \in \Omega$ means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such $\chi(f(z), f(\zeta)) < \varepsilon$ whenever $\zeta \in \Omega$ and $|z - \zeta| < \delta$.

In [3, Theorem 3, p. 140], we gave an example of a continuous function Φ that no holomorphic function can avoid. Rubel and Yang [4, Theorem 3, p. 294] proved that given any two meromorphic functions g_1 and g_2, there exists a meromorphic function f such that f avoids both g_1 and g_2. Note that g_1 and g_2 are not required to avoid each other. The proof of Rubel and Yang is given for meromorphic functions in which the domain is the full complex plane, but the the result is also true for meromorphic functions whose domain is any open subset of the complex plane, as was shown by Hayman and Rubel [1]. It is an easy corollary, by setting $g_1 = g$ and $g_2 \equiv 1$, that if g is a meromorphic function then there is a meromorphic function f that avoids g.

We will prove the following two results.

Theorem 1. *There exists a function G continuous in D such that no meromorphic function avoids G.*

Theorem 2. *There exists a function H continuous in the complex plane such that no function meromorphic in the plane avoids H.*

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The construction of the function G in the proof of Theorem 1 is similar to the construction of Φ in [3, Theorem 3, p. 140], but the proof involves a modification of the argument principle for meromorphic functions.

2. Proofs of the Theorems

To prove Theorem 1, let $A = \{\alpha_n\}$ be a countable dense subset of the complex plane, let $\{\beta_n\}$ be a sequence containing the elements of A such that each element of A occurs infinitely often in the sequence $\{\beta_n\}$, and let $\{r_n\}$ be a strictly increasing sequence of positive real numbers such that $r_n \geq r_0 = 1/2$ and $r_n \to 1$. For each positive integer n, let $C_n = \{z : r_n \leq |z| \leq r_n + (r_{n+1} - r_n)/10\}$, and let D_n be the closed disc with center at $(r_n + r_{n+1})/2$ and radius $(r_{n+1} - r_n)/4$. Then $D_n \cap C_j = \emptyset$ for each pair of positive integers n and j. Now let

$$E = \bigcup_{n=1}^{\infty} (C_n \cup D_n).$$

Define a function G_0 such that $G_0(z) = 1/z$ for $|z| \leq 1/4$, and, for each positive integer n, both $G_0(z) = \beta_n$ for $z \in C_n$, and $G_0(z)$ is a bounded continuous function from D_n onto the disc $\{w : |w| \leq n\}$—for example, we could take

$$G_0(z) = \frac{4n}{r_{n+1} - r_n}\left(z - \frac{r_n + r_{n+1}}{2}\right), \quad z \in D_n.$$

As in the proof of [3, Theorem 3], there exists a function G continuous on D such that

$$G(z) = G_0(z) \quad \text{for } z \in \{z : |z| \leq 1/2\} \cup E\ ,$$

where $G(z) = \infty$ only when $z = 0$. We claim that there is no meromorphic function that avoids G.

Suppose that f is a function meromorphic in D that avoids G. If f is a constant function, say $f(z) \equiv a$, where a is a complex number, there is an integer n such that $n > |a|$, and then $G(z) = a$ for some $z \in D_n$ with $n > |a|$. If $a = \infty$, then $G(0) = \infty$. Therefore, no constant function can avoid G, so we may assume that f is a non-constant meromorphic function. We may further assume that f has no pole at $z = 0$, for, if so, $f(0) = G(0) = \infty$ and f would not avoid G in this case.

Since f is non-constant, it must assume a value in A, so we may assume that f assumes the value α_{n_0}. Thus, there exists a point $z_0 \in D$ with $f(z_0) = \alpha_{n_0}$. Let r_{n_1} be such that both $r_{n_1} > |z_0|$ and $G(z) = \alpha_{n_0}$ for $z \in C_{n_1}$. Let $h(z) = f(z) - \alpha_{n_0}$ and let $p(z) = G(z) - \alpha_{n_0}$. Then $h(z) - p(z) = f(z) - G(z)$, and $p(z) = 0$ for $z \in C_{n_1}$. There exists a number s such that $\{z : |z| = s\} \subset C_{n_1}$ and f (and thus h) has no poles on the circle $\{z : |z| = s\}$. Further, $f(z)$ does not assume the value α_{n_0} on this circle since f is assumed to avoid the function G in D. Since there is a line through the origin that contains no poles of f in D, we may assume, by a rotation of the domain, if necessary, that this line is the real axis. Let $s_1 < s_2 < \cdots < s_k$ be the moduli of the poles of f in $\{z : |z| < s\}$. (Note that $s_1 > 0$, and so we can denote $s_0 = 0$.) Let

$$K_0 = \{z = re^{i\theta} : 0 < r < s_1, 0 < \theta < 2\pi\},$$

$$K_j = \{z = re^{i\theta} : s_j < r < s_{j+1}, 0 < \theta < 2\pi\}, \quad 1 \leq j \leq k-1,$$

and

$$K_k = \{z = re^{i\theta} : s_k < r < s,\ 0 < \theta < 2\pi\}\ .$$

Choose a value for $\arg(f(-1/4) - G(-1/4))$ and denote that value by $T(-1/4)$, and from this value, obtain a function $T(x)$ defined on the interval $[-s, 0)$ of the real axis such that each $T(x)$ is a value of $\arg((f(x) - G(x))$ and $T(x)$ is continuous for $-s \leq x < 0$.

Since we are assuming that the function $f(z) - G(z)$ has no zeros in the set $\{z : |z| \leq s\}$, the function $T(x)$ can be extended continuously from the values on the interval $[-s, 0)$ on the negative real axis to each of the sets $K_j,\ 0 < j \leq k$, such that for each $z \in K_j, T(z)$ is a value of $\arg(f(z) - G(z))$. Also, since $f(x) - G(x)$ is well defined on the positive real axis, we have that for $0 < r < s$, $r \notin S = \{s_j : 1 \leq j \leq k\}$, the value

$$\Delta(r) = \lim_{\theta \to 2\pi} \frac{1}{2\pi} T(re^{i\theta}) - \lim_{\theta \to 0} \frac{1}{2\pi} T(re^{i\theta})$$

is an integer. Further, by the continuity of $T(z)$ on K_j, for a fixed j, $\Delta(r)$ is constant on the interval (s_j, s_{j+1}). This is true for the intervals $(0, s_1)$ and (s_k, s), also.

We can easily compute $\Delta(r)$ for $0 < r < s_1$ and for $s_k < r \leq s$. First, for $r = s$, we have that $p(re^{i\theta}) = 0$ so $f(re^{i\theta}) - G(re^{i\theta}) = f(re^{i\theta}) - \alpha_{n_0}$ and $\Delta(r)$ is simply the winding number of $f(z) - \alpha_{n_0}$ on $|z| = s$. Let $Z(s)$ be the number of zeros of $f(z) - \alpha_{n_0}$ in the set $\{z : |z| \leq s\}$ and let $P(s)$ be the number of poles of $f(z) - \alpha_{n_0}$ in the set $\{z : |z| \leq s\}$. (Note that $Z(s) \geq 1$ by the assumption that $f(z_0) = \alpha_{n_0}$.) Then, by the argument principle, $\Delta(s) = Z(s) - P(s)$. Thus, by continuity, for $s_k < r \leq s, \Delta(r) = Z(s) - P(s)$. Also, for $0 < r < \varepsilon_1 = \min\{s_1, 1/4\}$, $f - G$ is meromorphic and has exactly one pole (at $z = 0$) and no zeros in the set $\{z : |z| < \varepsilon_1\}$, and so $\Delta(r) = -1$ for $0 < r < s_1$.

Next, for $j \geq 0$ if $s_j < r < s_{j+1} < t < s_{j+2}$, we show that $\Delta(r) - \Delta(t)$ is exactly the number of poles of $f(z)$ on the circle $|z| = s_{j+1}$. Let $m = \inf\{|f(z) - G(z)| : |z| \leq s\}$. This value is greater than zero by the assumption that f and G avoid each other, and in all that follows, we may replace m by any smaller number, so we will assume throughout that $m \leq 1/P(s)$. Further, for each $j, 1 \leq j \leq k$, let $\{p_{j,q} : 1 \leq q \leq q_j\}$ be the poles of f on the circle $\{z : |z| = s_j\}$, listed in counterclockwise order, beginning from the positive real ray. Here each pole is listed only once, regardless of its order. Let $M = \sup\{|G(z)| : s_1/2 \leq |z| \leq s\}$, and let $\delta_1 > 0$ be such that $|f(z)| > 100M$ whenever

$$z \in J = \bigcup_{j=1}^{k} \bigcup_{q=1}^{q_j} \{\zeta : |\zeta - p_{j,q}| < \delta_1\}.$$

Further, $f - G$ is uniformly continuous in the usual complex metric on the closure of the set $\{z : \varepsilon \leq |z| \leq s\} \setminus J$ for each positive ε less than s. Thus, there exists a $\delta_2 > 0$ such that $|(f(z) - G(z)) - (f(\zeta) - G(\zeta))| < m/100$ whenever $z, \zeta \notin J, s_1/2 \leq |z|, |\zeta| \leq s$, and $|z - \zeta| < \delta_2$. Let $\delta_3 = (1/2)\min\{\delta_1, \delta_2, s_1/2, s - s_k\}$. Then, we have

$$|(f(z) - G(z)) - (f(\zeta) - G(\zeta))| < m/100$$

whenever $|z - \zeta| < \delta_3$ and z and ζ are in the closure of the set $\{w : s_1/2 \leq |w| \leq s\} \setminus J$.

We will use the following notation:

$$C(r) = \{w : |w| = r\},$$
$$D(j, q) = \{w : |w - p_{j,q}| \leq \delta_1\}$$

and

$$C(j,q) = \partial D(j,q) = \{w : |w - p_{j,q}| = \delta_1\}.$$

We may assume that δ_1 is so small that the discs $D(j,q)$ are all mutually disjoint and that none of them contains the point $z = 0$. Fix j, $1 \le j \le k$. Let λ_1 be the arc of $C(s_j + \delta_3)$ between $D(j,q)$ and $D(j,q+1)$ (or between the positive axis and $D(j,1)$, or between $D(j,q_j)$ and the positive real axis), and let λ_2 be the arc of $C(s_j - \delta_3)$ between $D(j,q)$ and $D(j,q+1)$ (or between the positive real axis and $D(j,1)$, or between $D(j,q_j)$ and the positive real axis). For $z \in \lambda_1$ and $\zeta \in \lambda_2$ with $\arg z = \arg \zeta$, we have that $|z-\zeta| < \delta_2$ and so $|(f(z)-G(z))-(f(\zeta)-G(\zeta))| < m/100$, which means that

$$T(z) - T(\zeta) = 2\pi N(j,q) + o(1),$$

where $N(j,q)$ is an integer that depends only on j and q, and $o(1)$ is of the order of $m/100$. (For $z \in \lambda_1, \zeta \in \lambda_2$, and $\arg z = \arg \zeta$, the difference between $f(z) - G(z)$ and $f(\zeta) - G(\zeta)$ is very small compared to either of these values, so the basic angles of the values of $f - G$ at z and at ζ are very close together, so that the values of $A(z)$ and $A(\zeta)$ are very close to an integer multiple of 2π apart.)

Now fix j and q, where $p_{j,q}$ is a pole of f, $1 \le j \le k$, and let $\Gamma_1 = C(s_j + \delta_3) \cap D(j,q)$ and let $\Gamma_3 = C(s_j - \delta_3) \cap D(j,q)$. Also, let Γ_2 and Γ_4 be the two arcs of $C(j,q)$ bounded by $C(s_j + \delta_3)$ and $C(s_j - \delta_3)$ so that $\Gamma_{j,q} = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ is a Jordan curve containing $p_{j,q}$ in its interior, where we give this curve a counterclockwise orientation. For $z \in \Gamma_{j,q}$, we have that $z \in D(j,q)$, which means that $|f(z) - G(z)| \ge 100M$ and $|(f(z) - G(z)) - f(z)| = |G(z)| \le M$, and so we have

$$|f(z)| \ge |f(z) - G(z)| - |G(z)| \ge 99M$$

and thus

$$|(f(z) - G(z)) - f(z)| \le \frac{1}{99}|f(z)|,$$

which means that the basic angle made by $f(z) - G(z)$ from the real axis is very close to the basic angle made by $f(z)$, and so the change of argument of $f - G$ taken around $\Gamma_{j,q}$ is the same as the change of argument of f around $\Gamma_{j,q}$, which we know to be $-2\pi m_{j,q}$ if we take this in the counterclockwise direction, where $m_{j,q}$ is the order of the pole of f at $p_{j,q}$. But the change of angle along both Γ_2 and Γ_4 is very small by the argument above, since, for example, if z and ζ are both in Γ_2 or both in Γ_4, then $|z - \zeta| < \delta_2$. Thus, the change of argment of $f - G$ along Γ_3, taken in the counter-clockwise direction around $\Gamma_{j,q}$ is of the order of $2\pi m_{j,q}$ more than the change of argument of $f - G$ along Γ_1, taken in the counterclockwise direction, where the error is very small. Thus, if we travel along the circles $C(s_j + \delta_3)$ and $C(s_j - \delta_3)$, as we pass by a pole of f on $|z| = s_j$, the the change of argment on the circle $C(s_j - \delta_3)$ is approximately 2π times the order of the pole more than the change of argment on the circle $C(s_j + \delta_3)$. However, between poles, the change of argument is approximately the same on both circles. As a result, we have

$$\Delta(s_j - \delta_3) - \Delta(s_j + \delta_3) = \sum_{q=1}^{q_j} m(j,q).$$

It follows from this that

$$\Delta(r) - \Delta(s) = P(s) = \text{number of poles of } f \text{ in } \{\zeta : |\zeta| \le s\}$$

for r near zero.

From our original calculations we have that $\Delta(s) = Z(s) - P(s)$ and $\Delta(r) = -1$ for r near to zero, and from this last calculation we have that $\Delta(r) - \Delta(s) = P(s)$ for r near to zero, it follows that $Z(s) = -1$. But we know that $Z(s) \geq 1$, and we have arrived at a contradiction. This contradiction arose because we assumed that $f(z) - G(z)$ was never zero, so we must have that $f(z) - G(z)$ does assume the value zero somewhere in $\{\zeta : |\zeta| \leq s\}$, and there is no meromorphic function f that avoids G. This completes the proof of Theorem 1.

For the proof of Theorem 2, we only need to repeat the details verbatim of the proof of Theorem 1, except that now we let $r_n \to \infty$ (rather than $r_n \to 1$) and denote the function constructed by H rather than G.

3. Some final remarks

Theorems 1 and 2 taken together say that the construction is possible on each simply connected domain of the plane. It is not at all clear if this construction can be modified to deal with a multiply connected domain. It would appear that, if the result is true for multiply connected domains, some new approach is needed, as the basic idea in the proof above will not work for a multiply connected domain.

In the construction of the function G in Theorem 1 above, there was no attempt to make G a particularly "nice" function, except to give it the property that no meromorphic function can avoid G. However, it is possible, by taking a bit more care in the construction, to make G a fairly "nice" function. We say that a continuous function h is a *normal function* if the family $\{h(\phi(z)) : \phi \in \mathscr{M}\}$ has the property that each sequence contains a subsequence that converges spherically uniformly on each compact subset of D to a function continuous on D, where $\mathscr{M}$ denotes the collection of Möbius transformations of D. Most often, the term "normal" function is used when the function is meromorphic, but there are some cases where it is useful to use this term for functions that are continuous in D but not meromorphic (see, for example, [2], [3]). By doing a more careful construction—that is by controlling the sequence $\{r_n\}$ more carefully and by a careful use of the linearization used in the proof of the Tietze Extension Theorem—the function G can be constructed so that G is a normal (continuous) function. The details of this are left to the reader. With respect to normal functions, we note that in a previous work it was proved that there exists a meromorphic function f_1 such that no normal meromorphic function avoids f_1 [2, Theorem 5, p. 228]. However, the funtion f_1 is not itself a normal function. And as noted in the introduction, the Rubel–Yang result in [4] guarantees that there are meromorphic (non-normal) functions that avoid f_1.

References

[1] Hayman, W., and L. A. Rubel: Unavoidable systems of functions. - Math. Proc. Cambridge Philos. Soc. 117, 1995, 345–351.

[2] Lappan, P.: Avoidance criteria for normal families and normal functions. - In: Progress in Analysis: Proceedings of the 3rd International ISAAC Congress 1, World Scientific 2003, 221–228.

[3] Lappan, P.: Special classes of normal families. - Comput. Methods Funct. Theory 8, 2008, 133–142.

[4] Rubel, L. A., and C.-C. Yang: Interpolation and unavoidable families of meromorphic functions. - Michigan Math. J. 20, 1973, 289–296.

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