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SOBOLEV CAPACITY AND HAUSDORFF MEASURES IN METRIC MEASURE SPACES

Şerban Costea

McMaster University, Department of Mathematics and Statistics
1280 Main Street West, Hamilton, Ontario L8S 4K1, Canada; secostea@math.mcmaster.ca

Abstract. This paper studies the relative Sobolev p-capacity in proper metric measure spaces when $1 < p < \infty$. We prove that this relative Sobolev p-capacity is Choquet. In addition, if the space X is doubling, unbounded, admits a weak $(1,p)$-Poincaré inequality and has an "upper dimension" Q for some $p \le Q < \infty$, then we obtain lower estimates of the relative Sobolev p-capacities in terms of the Hausdorff content associated with continuous and doubling gauge functions h satisfying the decay condition

$$\int_0^1 \left(\frac{h(t)}{t^{Q-p}}\right)^{1/p} \frac{dt}{t} < \infty. \tag{1}$$

This condition generalizes a well-known condition in $\mathbf{R}^n$.

1. Introduction

In this paper (X,d,μ) is a proper metric space. (That is, closed bounded subsets of X are compact.) We assume that μ is a nontrivial regular Borel measure which is finite on bounded sets and positive on nonempty open sets. We shall impose further restrictions on the space X and the measure μ later.

The Sobolev p-capacity was studied by Maz'ya and Heinonen–Kilpeläinen–Martio in $\mathbf{R}^n$ and by Kinnunen–Martio in metric spaces. The relative Sobolev p-capacity in metric spaces was introduced by Björn in [3] when studying the boundary continuity properties of quasiminimizers.

We develop a theory of the relative Sobolev p-capacity in a proper metric measure space (X,d,μ) when $1 < p < \infty$. We prove that this capacity is a Choquet set function. In $\mathbf{R}^n$ it is known that sets of p-capacity zero have Hausdorff h-measure zero provided that $h\colon [0,\infty) \to [0,\infty)$ is a homeomorphism satisfying the integrability condition $\int_0^1 \left(\frac{h(t)}{t^{n-p}}\right)^{1/p} \frac{dt}{t} < \infty$. (See Theorem 4.1 in Reshetnyak [23], Theorem 3.1 in Martio [20] and Maz'ya [22].) A similar result was proved in $\mathbf{R}^n$ by Havin–Maz'ya and Adams–Hedberg. Theorem 7.1 in Havin–Maz'ya [12] or Theorem 5.1.13 in Adams–Hedberg [1] states that every set in $\mathbf{R}^n$ with zero p-capacity, $1 < p \le n$, has Hausdorff h-measure zero provided that $\int_0^1 \left(\frac{h(t)}{t^{n-p}}\right)^{1/(p-1)} \frac{dt}{t} < \infty$.

In this paper, under the assumption that the metric measure space X is proper, doubling, unbounded, and admits a weak $(1,p)$-Poincaré inequality, we extend Theorem 4.1 from Reshetnyak [23] to metric spaces that in addition have an "upper dimension" Q with $1 < p \le Q < \infty$, provided that the homeomorphisms $h\colon [0,\infty) \to [0,\infty)$

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are doubling and satisfy the integrability condition $\int_0^1 \left(\frac{h(t)}{t^{Q-p}}\right)^{1/p} \frac{dt}{t} < \infty$. Thus we generalize the results obtained by Martio, Maz'ya and Reshetnyak in $\mathbf{R}^n$. Some of the ideas used here when proving the Choquet property of the relative Sobolev p-capacity follow Kinnunen–Martio [17] and [18].

2. Preliminaries

In this section we recall the standard notation and definitions to be used throughout this paper. Here and throughout this paper $B(x,r) = \{y \in X : d(x,y) < r\}$ is the open ball with center $x \in X$ and radius $r > 0$, while $\overline{B}(x,r) = \{y \in X : d(x,y) \le r\}$ is the closed ball with center $x \in X$ and radius $r > 0$. For a positive number λ, $\lambda B(a,r) = B(a, \lambda r)$ and $\lambda \overline{B}(a,r) = \overline{B}(a, \lambda r)$.

Throughout this paper, C will denote a positive constant whose value is not necessarily the same at each occurrence; it may vary even within a line. $C(a, b, \ldots)$ is a constant that depends only on the parameters $a, b, \ldots$. Here Ω will denote a nonempty open subset of X. For $E \subset X$, the boundary, the closure, and the complement of E with respect to X will be denoted by ∂E, $\overline{E}$, and $X \setminus E$, respectively; $\operatorname{diam} E$ is the diameter of E with respect to the metric d and $E \subset\subset F$ means that $\overline{E}$ is a compact subset of F.

Let $\Omega \subset X$ be open. For a measurable function $u \colon \Omega \to \mathbf{R}$, supp u is the smallest closed set such that u vanishes on the complement of supp u. We also define

$$\begin{aligned} Lip(\Omega) &= \{\varphi \colon \Omega \to \mathbf{R} : \varphi \text{ is Lipschitz}\}, \\ Lip_0(\Omega) &= \{\varphi \colon \Omega \to \mathbf{R} : \varphi \text{ is Lipschitz and with compact support in } \Omega\}. \end{aligned}$$

A measure μ is said to be *doubling* if there exists a constant $C \ge 1$ such that

$$\mu(2B) \le C\mu(B)$$

for every ball $B = B(x,r)$ in X. A metric measure space (X, d, μ) is called *doubling* if the measure μ is doubling. Under the assumption that the measure μ is doubling, it is known that (X, d, μ) is proper if and only if it is complete.

A *path* in X is a continuous map γ from an interval I of $\mathbf{R}$ to X. Whenever γ is rectifiable, we use the arc length parametrization $\gamma \colon [0, l_\gamma] \to X$, where l_γ is the length of the curve γ.

A nonnegative Borel function ρ on X is an *upper gradient* of a real-valued function u on X if for all rectifiable paths $\gamma \colon [0, l_\gamma] \to X$,

$$|u(\gamma(0)) - u(\gamma(l_\gamma))| \le \int_\gamma \rho \, ds.$$

Let $1 < p < \infty$ be fixed from now on throughout the paper. If the above inequality fails only for a curve family with zero p-modulus, (see for example Section 2.3 in Heinonen–Koskela [15]), then ρ is called a *p-weak upper gradient* of u. It was proved by Koskela–MacManus in [19] that the L^p-closure of the set of all upper gradients of u that are in L^p is precisely the set of all p-weak upper gradients of u that are in L^p.

Definition 2.1. We say that X supports a *weak* $(1,p)$*-Poincaré inequality* if there exists $C > 0$ and $\lambda \ge 1$ such that for all balls B with radius r, all measurable functions u on X and all upper gradients g of u we have

$$\frac{1}{\mu(B)}\int_B |u-u_B|\,d\mu \le Cr\left(\frac{1}{\mu(\lambda B)}\int_{\lambda B} g^p\,d\mu\right)^{1/p}. \tag{2}$$

In the above definition of the Poincaré inequality, we can equivalently assume that g is a p-weak upper gradient of u. (See the discussion before Definition 2.1.) If (2) holds with $\lambda = 1$, then we say that X satisfies a $(1,p)$-*Poincaré inequality.*

In this paper we use a version of Sobolev-type spaces on a metric measure space X defined by Shanmugalingam in [25]. There are several other definitions of Sobolev-type spaces in the metric setting; see Hajłasz [10], Heinonen–Koskela [15], Cheeger [6], and Franchi–Hajłasz–Koskela [9]. It has been shown that under reasonable hypotheses, the majority of these definitions yields the same space; see Franchi–Hajłasz–Koskela [9] and Shanmugalingam [25].

We define the space $\widetilde{N}^{1,p}(X)$ to be the collection of all p-integrable functions u on X that have a p-integrable p-weak upper gradient g on X. This space is equipped with the norm

$$||u||_{\widetilde{N}^{1,p}(X)} = \left(\int_X |u|^p d\mu + \inf\int_X g^p d\mu\right)^{1/p},$$

where the infimum is taken over all p-weak upper gradients of u. The *Newtonian space* on X is the quotient space

$$N^{1,p}(X) = \widetilde{N}^{1,p}(X)/\sim$$

with the norm $||u||_{N^{1,p}(X)} = ||u||_{\widetilde{N}^{1,p}(X)}$, where $u \sim v$ if and only if $||u-v||_{\widetilde{N}^{1,p}(X)} = 0$.

The space $N^{1,p}(X)$ equipped with the norm $||\cdot||_{N^{1,p}}$ is a lattice; see Shanmugalingam [25]. Corollary 3.7 in Shanmugalingam [26] shows that every $u \in N^{1,p}$ has a minimal p-weak upper gradient g_u in the sense that $g_u \le g$ holds μ-a.e. for all p-weak upper gradients of u. Theorem 1.1 in Björn–Björn–Shanmugalingam [2] shows that if (X,d,μ) is a proper and doubling metric measure space that admits a weak $(1,p)$-Poincaré inequality, then all the functions in $N^{1,p}(X)$ are quasicontinuous in X. We also note that if $u \in N^{1,p}(X)$ and v is a bounded Lipschitz function, then $uv \in N^{1,p}(X)$ and the function $|u|g_v + |v|g_u$ is a p-weak upper gradient of uv.

3. Relative Sobolev capacity

In this section, we establish a general theory of the relative Sobolev p-capacity in proper metric measure spaces. We recall that if (X,d,μ) is a proper metric measure space, then the Sobolev p-capacity of a set $E \subset X$ is (see Björn–Björn–Shanmugalingam [2])

$$\mathrm{Cap}_p(E) = \inf\{||u||^p_{N^{1,p}(X)} : u \in \mathscr{A}(E)\},$$

where

$$\mathscr{A}(E) = \{u \in N^{1,p}(X) : u = 1 \text{ on E}\}.$$

A property is said to hold *p-quasieverywhere* (or p-q.e.) if it holds everywhere except on a set of Sobolev p-capacity zero. We recall (see Björn–Björn–Shanmugalingam [2]) that if the space (X,d,μ) is proper, doubling and admits a weak $(1,p)$-Poincaré inequality, then Cap_p is an outer capacity, that is

$$\mathrm{Cap}_p(E) = \inf\{\mathrm{Cap}_p(U) : E \subset U,\ U \text{ open}\}.$$

In order to introduce the relative Sobolev p-capacity, we need Newtonian spaces with zero boundary values.

Definition 3.1. Suppose $\Omega \subset X$ is an open set. We let (see Björn [3] and Shanmugalingam [26])

$$N_0^{1,p}(\Omega) = \{u \in N^{1,p}(X) : u = 0 \text{ } p\text{-q.e. on } X \setminus \Omega\}.$$

We note that if $\mathrm{Cap}_p(X \setminus \Omega) = 0$, then $N_0^{1,p}(\Omega) = N^{1,p}(X)$. It is also known that if (X, d, μ) is a proper and doubling metric measure space that satisfies a weak $(1, p)$-Poincaré inequality, then $Lip_0(\Omega)$ is dense in $N_0^{1,p}(\Omega)$ with respect to the $N^{1,p}$ norm whenever Ω is an open subset of X.

For $E \subset \Omega$ we define

$$A(E, \Omega) = \{u \in N_0^{1,p}(\Omega) : u \geq 1 \text{ on a neighborhood of } E\}.$$

We call $A(E, \Omega)$ the set of *admissible functions for the condenser* (E, Ω). The *relative Sobolev p-capacity* of the pair (E, Ω) is denoted by

$$\mathrm{cap}_p(E, \Omega) = \inf \left\{ \int_\Omega g_u^p \, d\mu : u \in A(E, \Omega) \right\}.$$

(See Björn [3].) If $A(E, \Omega) = \emptyset$, we set $\mathrm{cap}_{B_p}(E, \Omega) = \infty$. Since $A(E, \Omega)$ is closed under truncations from below by 0 and from above by 1 and since the minimal p-weak upper gradients do not increase under these truncations, we may restrict ourselves to those admissible functions u for which $0 \leq u \leq 1$.

3.1. Basic properties of the relative Sobolev capacity. A capacity is a monotone, subadditive set function. The following theorem expresses, among other things, that this is true for the relative Sobolev p-capacity.

Theorem 3.2. *Suppose that (X, d, μ) is a proper metric measure space. Let $\Omega \subset X$ be a bounded open set. The set function $E \mapsto \mathrm{cap}_p(E, \Omega)$, $E \subset \Omega$, enjoys the following properties:*

(i) *If $E_1 \subset E_2$, then $\mathrm{cap}_p(E_1, \Omega) \leq \mathrm{cap}_p(E_2, \Omega)$.*

(ii) *If $\Omega_1 \subset \Omega_2$ are open, bounded and $E \subset \Omega_1$, then*

$$\mathrm{cap}_p(E, \Omega_2) \leq \mathrm{cap}_p(E, \Omega_1).$$

(iii) $\mathrm{cap}_p(E, \Omega) = \inf\{\mathrm{cap}_p(U, \Omega) : E \subset U \subset \Omega, \ U \text{ open}\}$.

(iv) *If K_i is a decreasing sequence of compact subsets of Ω with $K = \bigcap_{i=1}^\infty K_i$, then*

$$\mathrm{cap}_p(K, \Omega) = \lim_{i \to \infty} \mathrm{cap}_p(K_i, \Omega).$$

(v) *If $E_1 \subset E_2 \subset \ldots \subset E = \bigcup_{i=1}^\infty E_i \subset \Omega$, then*

$$\mathrm{cap}_p(E, \Omega) = \lim_{i \to \infty} \mathrm{cap}_p(E_i, \Omega).$$

(vi) *If $E = \bigcup_{i=1}^\infty E_i \subset \Omega$, then*

$$\mathrm{cap}_p(E, \Omega) \leq \sum_{i=1}^\infty \mathrm{cap}_p(E_i, \Omega).$$

Proof. We follow Kinnunen–Martio [17] and [18]. Properties (i)–(iv) are immediate consequences of the definition.

(v) Monotonicity yields

$$\lim_{i\to\infty} \operatorname{cap}_p(E_i,\Omega) \le \operatorname{cap}_p(E,\Omega).$$

To prove the opposite inequality, we may assume without loss of generality that $\lim_{i\to\infty} \operatorname{cap}_p(E_i,\Omega) < \infty$. Let $\varepsilon > 0$ be fixed. For every $i = 1,2,\dots$ we choose $u_i \in A(E_i,\Omega)$, $0 \le u_i \le 1$ and a corresponding minimal upper gradient g_{u_i} such that

$$||g_{u_i}||^p_{L^p(\Omega)} < \operatorname{cap}_p(E_i,\Omega) + \varepsilon \le \lim_{j\to\infty} \operatorname{cap}_p(E_j,\Omega) + \varepsilon. \tag{3}$$

Since Ω is bounded and $0 \le u_i \le 1$ for every $i = 1,2,\dots,$ it follows via (3) that u_i is a bounded sequence in $N^{1,p}_0(\Omega)$. Hence there exists a subsequence, which we denote again by u_i and functions $u, g \in L^p(\Omega)$ such that $u_i \to u$ weakly in $L^p(\Omega)$ and $g_{u_i} \to g$ weakly in $L^p(\Omega)$ as $i \to \infty$. Using Mazur's lemma simultaneously for u_i and g_{u_i}, we obtain sequences v_i and g_i such that $v_i \in A(E_i,\Omega)$, $v_i \to u$ in $L^p(\Omega)$ and μ-a.e. and $g_i \to g$ in $L^p(\Omega)$, where g is a p-weak upper gradient of u and g_i is a p-weak upper gradient of v_i, $i = 1,2,\dots$. (See Lemma 3.1 in Kallunki–Shanmugalingam [16].) These sequences can be found in the following way. Let i_0 be fixed. Since every subsequence of u_i converges to u weakly in $L^p(\Omega)$, we may use the Mazur lemma for the subsequence $u_i, i \ge i_0$. Similarly and simultaneously, we use the Mazur lemma for the subsequence g_{u_i}, $i \ge i_0$. We obtain finite convex combinations v_{i_0} and g_{i_0} of the functions u_i and g_{u_i}, $i \ge i_0$ as close as we want in $L^p(\Omega)$ to u and g respectively. For every $i = i_0, i_0+1,\dots$ there is an open neighborhood O_i of E_{i_0} such that $u_i = 1$ in O_i. The intersection of finitely many open neighborhoods of E_{i_0} is an open neighborhood of E_{i_0}. Therefore, v_{i_0} equals 1 in an open neighborhood U_{i_0} of E_{i_0}. It is easy to see that g_{i_0} is a p-weak upper gradient for v_{i_0}. Moreover, since for every $i = 1,2,\dots$ we have

$$||g_{u_i}||^p_{L^p(\Omega)} < \operatorname{cap}_p(E_i,\Omega) + \varepsilon \le \lim_{j\to\infty} \operatorname{cap}_p(E_j,\Omega) + \varepsilon,$$

we obtain from the convexity of the p-seminorm and (3) that

$$||g_{v_i}||^p_{L^p(\Omega)} \le ||g_i||^p_{L^p(\Omega)} \le \lim_{j\to\infty} \operatorname{cap}_p(E_j,\Omega) + \varepsilon \tag{4}$$

for every $i = 1,2,\dots$. Passing to subsequences if necessary, we may assume that for every $i = 1,2,\dots$ we have

$$||v_{i+1} - v_i||_{L^p(\Omega)} + ||g_{i+1} - g_i||_{L^p(\Omega)} \le 2^{-i}. \tag{5}$$

For $j = 1,2,\dots$ we set

$$w_j = \sup_{i\ge j} v_i.$$

It is easy to see that $w_j = \lim_{k\to\infty} w_{j,k}$ pointwise in X, where $w_{j,k}$ is defined for every $k > j$ by

$$w_{j,k} = \sup_{k\ge i\ge j} v_i.$$

We notice that $w_{j,k} \in A(E_j,\Omega)$ with p-weak upper gradient $g_{j,k}$ whenever $j < k < \infty$, where $g_{j,k}$ is defined by

$$g_{j,k} = \sup_{k\ge i\ge j} g_i.$$

Moreover,

$$w_{j,k} \leq v_j + \sum_{i=j}^{k-1} |v_{i+1} - v_i| \quad \text{and} \quad g_{j,k} \leq g_j + \sum_{i=j}^{k-1} |g_{i+1} - g_i| \tag{6}$$

whenever $j < k < \infty$. We define $\widetilde{g}_j$ by

$$\widetilde{g}_j = g_j + \sum_{i=j}^{\infty} |g_{i+1} - g_i|.$$

Then, since $w_j = \lim_{k\to\infty} w_{j,k}$ pointwise in X, it follows easily from (6) that $\widetilde{g}_j$ is a p-weak upper gradient of w_j. We obviously have $g_{w_j} \leq \widetilde{g}_j$ μ-a.e. in X.

The convexity and reflexivity of $L^p(\Omega)\times L^p(\Omega)$ together with Mazur's lemma and formula (6) imply that $w_j \in N_0^{1,p}(\Omega)$ with

$$w_j \leq v_j + \sum_{i=j}^{\infty} |v_{i+1} - v_i|$$

pointwise in X. It is easy to see that $w_j = 1$ in a neighborhood of E and this shows, since $w_j \in N_0^{1,p}(\Omega)$, that in fact $w_j \in A(E,\Omega)$ and hence $\text{cap}_p(E,\Omega) \leq ||g_{w_j}||^p_{L^p(\Omega)}$. We notice that

$$||\widetilde{g}_j||_{L^p(\Omega)} \leq ||g_j||_{L^p(\Omega)} + \sum_{i=j}^{\infty} ||g_{i+1} - g_i||_{L^p(\Omega)} \leq ||g_j||_{L^p(\Omega)} + 2^{-j+1} \tag{7}$$

for every $j \geq 1$. Therefore, for all sufficiently large j we have via (4) and (7) that

$$\text{cap}_p(E,\Omega) \leq ||g_{w_j}||^p_{L^p(\Omega)} \leq ||\widetilde{g}_j||^p_{L^p(\Omega)} \leq \lim_{i\to\infty} \text{cap}_p(E_i,\Omega) + 2\varepsilon.$$

By letting $\varepsilon \to 0$, we get the converse inequality so (v) is proved.

(vi) To prove the countable subadditivity, we need to prove the finite subadditivity first. It is enough to prove this for two sets because then the general finite case follows by induction. So let E_1 and E_2 be two subsets of Ω. We can assume without loss of generality that $\text{cap}_p(E_1,\Omega) + \text{cap}_p(E_2,\Omega) < \infty$. Let $u_i \in A(E_i,\Omega)$ with minimal upper gradients g_{u_i} such that

$$0 \leq u_i \leq 1 \quad \text{and} \quad ||g_{u_i}||^p_{L^p(\Omega)} < \text{cap}_p(E_i,\Omega) + \varepsilon \text{ for } i = 1,2.$$

Then $u = \max(u_1,u_2)$ belongs to $A(E_1 \cup E_2,\Omega)$ and $g = \max(g_{u_1}, g_{u_2})$ is a p-weak upper gradient of u. Therefore

$$\begin{aligned}\text{cap}_p(E_1 \cup E_2,\Omega) &\leq ||g_u||^p_{L^p(\Omega)} \leq ||g||^p_{L^p(\Omega)} \leq ||g_{u_1}||^p_{L^p(\Omega)} + ||g_{u_2}||^p_{L^p(\Omega)} \\ &\leq \text{cap}_p(E_1,\Omega) + \text{cap}_p(E_2,\Omega) + 2\varepsilon.\end{aligned}$$

Letting $\varepsilon \to 0$ we complete the proof in the case of two sets, and hence the general finite case.

The general case follows from the finite case together with (v). The theorem is proved. □

A set function that satisfies properties (i), (iv), (v) and (vi) is called a *Choquet capacity* (relative to Ω). We may thus invoke an important capacitability theorem of Choquet and state the following result. See Appendix II in Doob [7].

Theorem 3.3. *Suppose that (X, d, μ) is a proper metric measure space. Suppose that Ω is a bounded open set in X. The set function $E \mapsto \mathrm{cap}_p(E, \Omega)$, $E \subset \Omega$, is a Choquet capacity. In particular, all Borel subsets (in fact, all analytic) subsets E of Ω are capacitable, i.e.,*

$$\mathrm{cap}_p(E, \Omega) = \sup\{\mathrm{cap}_p(K, \Omega) : K \subset E \text{ compact}\}$$

whenever $E \subset \Omega$ is analytic.

It is easy to see that

$$\mathrm{cap}_p(K, \Omega) = \mathrm{cap}_p(\partial K, \Omega)$$

whenever K is a compact subset of Ω.

Remark 3.4. Suppose that (X, d, μ) is a proper and doubling metric measure space that satisfies a weak $(1, p)$-Poincaré inequality. If K is a compact subset of the open set $\Omega \subset X$, we get the same p-capacity for (K, Ω) if we restrict ourselves to a smaller set of admissible functions, namely

$$W(K, \Omega) = \{u \in Lip_0(\Omega) : u = 1 \text{ in a neighborhood of } K\}.$$

Indeed, let $u \in A(K, \Omega)$; we may clearly assume that $u = 1$ in a neighborhood $U \subset\subset \Omega$ of K. Then we choose a cut-off Lipschitz function η, $0 \leq \eta \leq 1$ such that $\eta = 1$ in $X \setminus U$ and $\eta = 0$ in a neighborhood $\widetilde{U}$ of K, $\widetilde{U} \subset\subset U$. Now, if $\varphi_j \in Lip_0(\Omega)$ is a sequence converging to u in $N_0^{1,p}(\Omega)$, then $\psi_j = 1 - \eta(1 - \varphi_j)$ is a sequence belonging to $W(K, \Omega)$ which converges to $1 - \eta(1 - u)$ in $N_0^{1,p}(\Omega)$. But $1 - \eta(1 - u) = u$. This establishes the assertion, since $W(K, \Omega) \subset A(K, \Omega)$. In fact, it is easy to see that if $K \subset \Omega$ is compact we get the same p-capacity if we consider

$$\widetilde{W}(K, \Omega) = \{u \in Lip_0(\Omega) : u = 1 \text{ on } K\}.$$

It is also useful to observe that if $\psi \in N_0^{1,p}(\Omega)$ is such that $\varphi - \psi \in N_0^{1,p}(\Omega \setminus K)$ for some $\varphi \in \widetilde{W}(K, \Omega)$, then

$$\mathrm{cap}_p(K, \Omega) \leq \int_\Omega g_\psi^p \, d\mu.$$

Following an argument very similar to the one from Theorem 3.2, one can conclude:

Theorem 3.5. *Suppose that (X, d, μ) is a proper and doubling metric measure space satisfying a weak $(1, p)$-Poincaré inequality. The set function $E \mapsto \mathrm{Cap}_p(E), E \subset X$, is a Choquet capacity. In particular:*

(i) *If $E_1 \subset E_2$, then $\mathrm{Cap}_p(E_1) \leq \mathrm{Cap}_p(E_2)$.*

(ii) *If $E = \bigcup_i E_i$, then*

$$\mathrm{Cap}_p(E) \leq \sum_i \mathrm{Cap}_p(E_i).$$

Since $Lip_0(X)$ is dense in $N^{1,p}(X)$ with respect to the $N^{1,p}$ norm whenever (X, d, μ) is a proper and doubling metric measure space satisfying a weak $(1, p)$-Poincaré inequality, one can prove (by using an argument similar to the one from Remark 3.4) the following lemma:

Lemma 3.6. *Suppose (X, d, μ) is a proper and doubling metric measure space satisfying a weak $(1, p)$-Poincaré inequality. If $K \subset X$ is compact, then*

$$\mathrm{Cap}_p(K) = \inf\{||u||^p_{N^{1,p}(X)} : u \in \mathscr{A}(K) \cap Lip_0(X)\}.$$

We recall the following relation between the relative Sobolev capacity and the global Sobolev capacity. (See e.g. Lemma 2.6 in Björn–MacManus–Shanmugalingam [4] and Lemma 3.3 in Björn [3].)

Lemma 3.7. *Suppose (X, d, μ) is a proper and unbounded doubling metric measure space that satisfies a weak $(1, p)$-Poincaré inequality. Then for every $\lambda > 1$ there exists a constant $C_\lambda > 0$ such that*

$$\frac{\mathrm{Cap}_p(E \cap B(x, r))}{C_\lambda(1 + r^p)} \le \mathrm{cap}_p(E \cap B(x, r), B(x, \lambda r)) \le C_\lambda(1 + r^{-p})\mathrm{Cap}_p(E \cap B(x, r))$$

for every $E \subset X$, $x \in X$ and $r > 0$.

Definition 3.8. We say that $\mathrm{cap}_p(E) = 0$ if $\mathrm{cap}_p(E \cap \Omega, \Omega) = 0$ for every bounded and open $\Omega \subset X$.

Remark 3.9. If (X, d, μ) is a proper and unbounded doubling metric measure space that satisfies a weak $(1, p)$-Poincaré inequality, one can prove (by using Lemma 3.7 together with the Choquet property of the relative Sobolev capacity) that $\mathrm{cap}_p(E) = 0$ if and only if $\mathrm{Cap}_p(E) = 0$. It is also easy to see by using the aforementioned Lemma that if E is a bounded subset of X, then $\mathrm{cap}_p(E) = 0$ if and only if there exists Ω a bounded and open neighborhood of E such that $\mathrm{cap}_p(E, \Omega) = 0$ provided that (X, d, μ) is proper, doubling, unbounded and satisfies a weak $(1, p)$-Poincaré inequality.

4. Hausdorff measures and relative Sobolev capacity

In this section we examine the relationship between Hausdorff measures and the relative Sobolev p-capacity under some extra assumptions satisfied by the space X.

4.1. Generalized Hausdorff measure. Let h be a real-valued, strictly increasing function on $[0, \infty)$ such that $\lim_{t\to 0} h(t) = h(0) = 0$ and $\lim_{t\to\infty} h(t) = \infty$. Such a function h is called a *measure function.* A measure function h is called *doubling* if there exists a constant $C > 0$ such that

$$h(10t) \le C\,h(t) \text{ for all } t > 0. \tag{8}$$

The smallest constant C such that (8) holds is denoted by $C(h)$ and is called the *doubling constant* of h.

Let $0 < \delta \le \infty$. Throughout this subsection Ω is a closed subset of X. For $E \subset \Omega$ we define

$$\Lambda^\delta_{h,\Omega}(E) = \inf \sum_i h(r_i),$$

where the infimum is taken over all coverings of E by open sets G_i in Ω with diameter r_i not exceeding δ. The set function $\Lambda^\infty_{h,\Omega}$ is called the *h-Hausdorff content relative to* Ω. Clearly $\Lambda^\delta_{h,\Omega}$ is an outer measure for every $\delta \in (0, \infty]$ and every closed set $\Omega \subset X$. We write $\Lambda^\delta_h(E)$ for $\Lambda^\delta_{h,X}(E)$.

Moreover, for every $E \subset \Omega$, there exists a Borel set $\widetilde{E}$ such that $E \subset \widetilde{E} \subset \Omega$ and $\Lambda^\delta_{h,\Omega}(E) = \Lambda^\delta_{h,\Omega}(\widetilde{E})$. Clearly $\Lambda^\delta_{h,\Omega}(E)$ is a decreasing function of δ. It is easy to see that $\Lambda^{\delta}_{h,\Omega_2}(E) \le \Lambda^{\delta}_{h,\Omega_1}(E)$ for every $\delta \in (0,\infty]$ whenever Ω_1 and Ω_2 are two closed sets in X such that $E \subset \Omega_1 \subset \Omega_2$. This allows us to define the *h-Hausdorff measure relative to* Ω of $E \subset \Omega$ by

$$\Lambda_{h,\Omega}(E) = \sup_{\delta>0} \Lambda^{\delta}_{h,\Omega}(E) = \lim_{\delta\to 0} \Lambda^{\delta}_{h,\Omega}(E).$$

The measure $\Lambda_{h,\Omega}$ is Borel regular; that is, it is an additive measure on Borel sets of Ω and for each $E \subset \Omega$ there is a Borel set G such that $E \subset G \subset \Omega$ and $\Lambda_{h,\Omega}(E) = \Lambda_{h,\Omega}(G)$. (See [8, p. 170] and [21, Chapter 4].) We denote $\Lambda_h(E) := \Lambda_{h,X}(E)$. If $h(t) = t^s$, we write Λ_s for $\Lambda_{t^s,X}$. It is immediate from the definition that $\Lambda_s(E) < \infty$ implies $\Lambda_\alpha(E) = 0$ for all $\alpha > s$. The smallest $s \ge 0$ that satisfies $\Lambda_\alpha(E) = 0$ for all $\alpha > s$ is called the *Hausdorff dimension of E*.

For $\Omega \subset X$ closed and $\delta > 0$, the set function $\Lambda^{\delta}_{h,\Omega}$ has the following property:

(i) If K_i is a decreasing sequence of compact sets in Ω, then

$$\Lambda^{\delta}_{h,\Omega}\Big(\bigcap_{i=1}^{\infty} K_i\Big) = \lim_{i\to\infty} \Lambda^{\delta}_{h,\Omega}(K_i).$$

Moreover, if Ω is a compact subset of X and h is a continuous measure function, then $\Lambda^{\delta}_{h,\Omega}$ satisfies the following additional properties:

(ii) If E_i is an increasing sequence of arbitrary sets in Ω, then

$$\Lambda^{\delta}_{h,\Omega}\Big(\bigcup_{i=1}^{\infty} E_i\Big) = \lim_{i\to\infty} \Lambda^{\delta}_{h,\Omega}(E_i).$$

(iii) $\Lambda^{\delta}_{h,\Omega}(E) = \sup\{\Lambda^{\delta}_{h,\Omega}(K) : K \subset E \text{ compact}\}$ whenever $E \subset \Omega$ is a Borel set. (See Chapter 2:6 in Rogers [24].) In other words, $\Lambda^{\delta}_{h,\Omega}$ is a Choquet capacity whenever Ω is a compact subset of X and h is a continuous measure function.

If $h\colon [0,\infty) \to [0,\infty)$ is a measure function that is a homeomorphism, we know that $\Lambda_h(E) = 0$ if and only if $\Lambda^{\infty}_h(E) = 0$. (See Proposition 5.1.5 in Adams–Hedberg [1].) If $h(t) = t^s$, $0 < s < \infty$, we write Λ^{∞}_s for $\Lambda^{\infty}_{t^s,X}$.

We prove now the following version of Cartan's lemma in doubling metric measure spaces.

Lemma 4.1. (Cartan's lemma) *Suppose (X,d,μ) is a doubling metric measure space. Let σ be a finite compactly supported positive measure on X. Let $h\colon [0,\infty) \to [0,\infty)$ be a doubling measure function that is also a homeomorphism. If $\lambda > 0$ and*

$$A_\lambda = \{x \in X : \sigma(B(x,r)) \le \frac{h(r)}{\lambda} \text{ for all } r > 0\},$$

then $\Lambda^{\infty}_h(X \setminus A_\lambda) \le C\lambda\sigma(X)$, where $C > 0$ is the doubling constant of h.

Proof. We can assume without loss of generality that $\sigma(X) > 0$. Let $M > 0$ be such that $h(M) = \lambda\sigma(X)$. For each $x \in X \setminus A_\lambda$ there exists a radius $r_x > 0$ such that

$$h(r_x) < \lambda\sigma(B(x,r_x)) \le \lambda\sigma(X) = h(M).$$

Since h is strictly increasing, the choice of M implies that the supremum of all such radii is less than M. Moreover, $X \setminus A_\lambda$ is bounded. Indeed, this is obvious if X is bounded. If X is unbounded, then $X \setminus A_\lambda \subset B(a, r+M)$, where $B = B(a,r)$ is a ball containing the support of σ. This allows us to apply Theorem 1.16 in Heinonen [13] and select a countable sequence of points (x_i) in $X \setminus A_\lambda$ such that the corresponding balls $B(x_i, r_{x_i})$ are pairwise disjoint and such that

$$X \setminus A_\lambda \subset \bigcup_i B(x_i, 5r_{x_i}).$$

Therefore

$$\Lambda_h^\infty(X \setminus A_\lambda) \le \sum_i h(10r_{x_i}) \le C \sum_i h(r_{x_i}) \le C\lambda \sum_i \sigma(B(x_i, r_{x_i})) \le C\lambda\sigma(X).$$

The lemma follows. □

Remark 4.2. It is easy to see that when X is unbounded the preceding theorem in fact yields

$$\Lambda^{10M}_{h,\overline{B}(a,r+11M)}(X \setminus A_\lambda) \le C\lambda\sigma(X).$$

Now we prove the following relation between a Lipschitz function with compact support and its p-weak upper gradients.

Lemma 4.3. *Let (X, d, μ) be a doubling metric measure space that satisfies a weak $(1, p)$-Poincaré inequality and let $B = B(x_0, r)$ be a ball in X with $0 < r < (1/8)\operatorname{diam} X$. Let u be a function in $Lip_0(B(x_0, r))$ and let g be a p-weak upper gradient of u. There exists a constant $C_0 > 0$ depending only on p and on data of X such that*

$$|u(x)| \le C_0 \int_0^{3r} \left(\frac{1}{\mu(B(x,t))} \int_{B(x,t)} g^p \, d\mu \right)^{1/p} dt$$

for every $x \in X$.

Proof. There exists a similar result when $p = 1$, obtained by Björn–Onninen when proving Theorem 3 in [5]. The proof for the case $p > 1$ is similar but we present it for the convenience of the reader. We can assume without loss of generality that u is nonnegative and that $x \in B(x_0, r)$. We can also assume that $g = 0$ μ-a.e. outside $B(x_0, r)$. We let $r_j = 2r\,(2\lambda)^{-j}$ and $B_j = B(x, r_j)$, $j = 0, 1, 2, \ldots$, where $\lambda \ge 1$ is the constant from the $(1, p)$-Poincaré inequality. Since u is a Lipschitz function, every point in X is a Lebesgue point for u. Therefore

$$\begin{aligned} u(x) &= \lim_{j\to\infty} \frac{1}{\mu(B_j)} \int_{B_j} u \, d\mu \\ &= \frac{1}{\mu(B_0)} \int_{B_0} u \, d\mu + \sum_{j=0}^{\infty} \frac{1}{\mu(B_{j+1})} \int_{B_{j+1}} (u - u_{B_j}) \, d\mu, \end{aligned}$$

where $u_{B_j} = \mu(B_j)^{-1} \int_{B_j} u \, d\mu$. It is easy to see that $B(x_0, r) \subset B_0 = B(x, 2r)$ since $x \in B(x_0, r)$. The first term on the right-hand side can be estimated using the Sobolev inequality (see e.g. Proposition 3.1 in Björn [3] or the proof of Theorem

13.1 in Hajłasz–Koskela [11]), while the second term is estimated by the weak $(1,p)$-Poincaré inequality as follows

$$\begin{aligned} u(x) &\le C r_0 \left(\frac{1}{\mu(B_0)} \int_{B_0} g^p \, d\mu \right)^{1/p} + C \sum_{j=0}^{\infty} r_j \left(\frac{1}{\mu(\lambda B_j)} \int_{\lambda B_j} g^p \, d\mu \right)^{1/p} \\ &\le C_0 \int_0^{3r} \left(\frac{1}{\mu(B(x,t))} \int_{B(x,t)} g^p \, d\mu \right)^{1/p} dt. \end{aligned}$$

This finishes the proof. □

4.2. The Main result and special cases. We now state and prove our main result.

Theorem 4.4. *Suppose $1 < p \le Q < \infty$. Let (X,d,μ) be a proper and unbounded doubling metric measure space that supports a weak $(1,p)$-Poincaré inequality. Suppose $h\colon [0,\infty) \to [0,\infty)$ is a doubling homeomorphism. We also suppose that there exists a constant $C_\mu > 0$ such that*

$$\mu(B(x,t)) \ge C_\mu^{-1} t^Q \tag{9}$$

for all $t > 0$ and $x \in X$. Then there exists a positive constant C_1 depending only on the doubling constant of h, on p, and on data of X such that

$$\frac{\Lambda_h^\infty(E \cap \overline{B}(x_0,r))}{\left(\int_0^{6r} (\frac{h(t)}{t^{Q-p}})^{1/p} \frac{dt}{t}\right)^p} \le C_1 \operatorname{cap}_p(E \cap \overline{B}(x_0,r), B(x_0,2r)) \tag{10}$$

for every $E \subset X$, every $x_0 \in X$ and every $r > 0$.

Proof. We assume first that E is compact. There is nothing to prove if we have

$$\Lambda_h^\infty(E \cap \overline{B}(x_0,r)) = 0 \quad \text{or} \quad \int_0^{6r} \left(\frac{h(t)}{t^{Q-p}} \right)^{1/p} \frac{dt}{t} = \infty.$$

So we can assume without loss of generality that

$$\Lambda_h^\infty(E \cap \overline{B}(x_0,r)) > 0 \quad \text{and} \quad I(r) = \int_0^{6r} \left(\frac{h(t)}{t^{Q-p}} \right)^{1/p} \frac{dt}{t} < \infty.$$

Let $\varepsilon \in (0,1/2)$ be fixed. We choose $u_\varepsilon \in W(E \cap \overline{B}(x_0,r), B(x_0,2r))$ with its corresponding minimal p-weak upper gradient g_{u_ε} such that

$$\int_{B(x_0,2r)} g_{u_\varepsilon}^p \, d\mu < \operatorname{cap}_p(E \cap \overline{B}(x_0,r), B(x_0,2r)) + 2\varepsilon.$$

We can assume without loss of generality that $u_\varepsilon \ge 0$. We define

$$\sigma_\varepsilon(A) = \int_A g_{u_\varepsilon}^p \, d\mu$$

if $A \subset X$ is a Borel set. Suppose that $\alpha > 0$ and let

$$B_{\alpha,\varepsilon} = \{x \in X : \sigma_\varepsilon(B(x,t)) \le \frac{h(t)(1-\varepsilon)^p}{\alpha^p} \text{ for all } t > 0\}.$$

For $x \in B_{\alpha,\varepsilon}$ we have via (9) and Lemma 4.3

$$u_\varepsilon(x) \le \frac{C_\mu^{1/p} C_0 (1-\varepsilon)}{\alpha} \int_0^{6r} \left(\frac{h(t)}{t^Q} \right)^{1/p} dt,$$

where C_0 is the constant from Lemma 4.3. If we let

$$\alpha = C_\mu^{1/p} C_0 \int_0^{6r} \left(\frac{h(t)}{t^Q} \right)^{1/p} dt = C_\mu^{1/p} C_0 I(r),$$

then we notice that $E \cap \overline{B}(x_0, r) \subset \{x \in X : u_\varepsilon(x) > 1 - \varepsilon\} \subset X \setminus B_{\alpha,\varepsilon}$. Therefore from Lemma 4.1 and the choice of α it follows that

$$\Lambda_h^\infty(E \cap \overline{B}(x_0, r)) \le C(h)(1-\varepsilon)^{-p} C_\mu C_0^p I(r)^p \int_X g_{u_\varepsilon}^p \, d\mu.$$

Letting $\varepsilon \to 0$ we obtain the desired conclusion when E is compact.

We assume now that E is an arbitrary set. Since there exists a Borel set $\widetilde{E}$ containing E such that

$$\Lambda_h^\infty(E) = \Lambda_h^\infty(\widetilde{E}) \quad \text{and} \quad \operatorname{cap}_p(E \cap \overline{B}(x_0, r), B(x_0, 2r)) = \operatorname{cap}_p(\widetilde{E} \cap \overline{B}(x_0, r), B(x_0, 2r)),$$

we can assume that E is Borel. We choose $M > 0$ such that

$$h(M) = 2^p \operatorname{cap}_p(\overline{B}(x_0, r), B(x_0, 2r)) + 2^p.$$

From the fact that $\operatorname{cap}_p(\,\cdot\,, B(x_0, 2r))$ is a Choquet capacity, the discussion before Lemma 4.1, the choice of M and Remark 4.2, it follows that the claim holds also when E is Borel. This finishes the proof. □

Remark 4.5. It follows easily that if X is a proper and unbounded doubling metric measure space as in Theorem 4.4 with $Q - s < p \le Q$, then there exists a constant $C = C(Q, p, s, C_\mu)$ such that

$$\frac{\Lambda_s^\infty(E \cap B(a, R))}{R^{s-Q+p}} \le C \operatorname{cap}_p(E \cap B(a, R), B(a, 2R)) \tag{11}$$

whenever $E \subset X$, $R > 0$, and $a \in X$. (We use $h(t) = t^s$.) Inequality (11) was also obtained by Heinonen–Koskela in [15, Theorem 5.9] for compact sets.

Theorem 4.4 has the following corollary.

Corollary 4.6. *Suppose $1 < p \le Q < \infty$. Let (X, d, μ) be a proper and unbounded doubling metric measure space as in Theorem 4.4. Let $E \subset X$ be such that $\operatorname{cap}_p(E) = 0$. Then*

(i) *$\Lambda_h(E) = 0$ for every doubling homeomorphism $h\colon [0, \infty) \to [0, \infty)$ satisfying (1).*

(ii) *The Hausdorff dimension of E is at most $Q - p$.*

(iii) *The set $X \setminus E$ is connected.*

Note that for every $\varepsilon > 0$ we can take $h = h_\varepsilon\colon [0, \infty) \to [0, \infty)$ in Corollary 4.6, where $h_\varepsilon(t) = |\ln t|^{-p-\varepsilon}$ for every $t \in (0, 1/2)$ when $p = Q$. When $1 < p < Q$ we can take $h_\varepsilon(t) = t^{Q-p+\varepsilon}$ for every $t \ge 0$.

Proof. It is enough to prove the first claim for E bounded because Λ_h^∞ is a countably subadditive set function and $\Lambda_h(E) = 0$ if and only if $\Lambda_h^\infty(E) = 0$ whenever h is a continuous measure function. So we assume that E is bounded. Let $B = B(x_0, r)$ be a ball containing E. We have $\operatorname{cap}_p(E, B(x_0, 2r)) = 0$. An appeal to Theorem 4.4 yields the first claim. The second claim is a consequence of (i) because for every $s > Q - p$, the function $h_s\colon [0, \infty) \to [0, \infty)$ defined by $h_s(t) = t^s$ satisfies (1). The third claim is a consequence of the Poincaré inequality. □

Remark 4.7. Since we have

$$\int_0^1 \left(\frac{h(t)}{t^{Q-p}}\right)^{1/(p-1)} \frac{dt}{t} < \infty$$

whenever h is a doubling homeomorphism satisfying (1), Corollary 4.6 follows also via Theorem 3 and Example 2 in Björn–Onninen [5].

4.3. Upper bounds for relative capacity in terms of Hausdorff measures. We recall the following upper bounds for the relative capacity (see Lemma 7.18 in Heinonen [13] and Lemma 3.3 in Björn [3]).

Theorem 4.8. *Let $1 < p \le Q < \infty$ be fixed. Suppose (X, d, μ) is a proper and unbounded metric measure space. We also suppose that there exists a constant $C_\mu > 0$ such that*

$$\mu(B(x,t)) \le C_\mu t^Q \tag{12}$$

for all $t > 0$ and $x \in X$.

(i) *Suppose $1 < p < Q$. There exists a constant C depending only on data of X such that*

$$\operatorname{cap}_p(B(x_0,r), B(x_0,R)) \le Cr^{Q-p},$$

for every $x_0 \in X$ and every pair of positive numbers r, R such that $2r \le R$.

(ii) *Suppose $p = Q$. There exists a constant C depending only on data of X such that*

$$\operatorname{cap}_Q(B(x_0,r), B(x_0,R)) \le C\left(\ln\frac{R}{r}\right)^{1-Q},$$

for every $x_0 \in X$ and every pair of positive numbers r, R such that $2r \le R$.

We also get upper bounds of the relative capacity in terms of some Hausdorff measures. Similar estimates were obtained by Heinonen–Kilpeläinen–Martio [14] in $\mathbf{R}^n$ and by Kinnunen–Martio (see [17, Theorem 4.13]) in metric spaces.

Theorem 4.9. *Let $1 < p \le Q < \infty$ be fixed. Suppose (X, d, μ) is a proper and unbounded metric measure space such that the upper mass bound (12) holds for the measure μ. Let $h\colon [0,\infty) \to [0,\infty)$ be a homeomorphism such that for $0 < t \le \frac{1}{2}$ we have*

$$h(t) = \begin{cases} t^{Q-p} & \text{if } p < Q, \\ \left(\ln\frac{1}{t}\right)^{1-p} & \text{if } p = Q. \end{cases}$$

Then there exists a constant depending only on the data of X such that

$$\operatorname{cap}_p(E,\Omega) \le C\Lambda_h(E)$$

whenever $E \subset \Omega \subset\subset X$ with E compact and Ω open.

Proof. The proof is similar to the proof of Theorem 2.27 in Heinonen–Kilpeläinen–Martio [14]. We present it for the convenience of the reader. Let δ be the distance from E to the complement of Ω. We can assume without loss of generality that $\delta \le 1$. We fix $\varepsilon < \frac{\delta^2}{4}$. We cover the compact set E by finitely many open balls $B(x_i, r_i)$ such that $r_i < \frac{\varepsilon}{2}$. Since we can assume that the balls $B(x_i, r_i)$ intersect E, we have $B(x_i, \frac{\delta}{2}) \subset \Omega$. By using Theorem 4.8 together with our choice of ε, we have

$$\operatorname{cap}_p(B(x_i, r_i), B(x_i, \frac{\delta}{2})) \le \begin{cases} C\, r_i^{Q-p} & \text{if } p < Q, \\ C\, 2^{Q-1} \left(\ln \frac{1}{r_i}\right)^{1-p} & \text{if } p = Q. \end{cases}$$

Here C is the constant from Theorem 4.8. Using Theorem 3.2 (ii) and (vi) we get

$$\operatorname{cap}_p(E, \Omega) \le \sum_i \operatorname{cap}_p(B(x_i, r_i), \Omega) \le \sum_i \operatorname{cap}_p(B(x_i, r_i), B(x_i, \frac{\delta}{2})) \le c \sum_i h(r_i).$$

Taking the infimum over all such coverings and letting $\varepsilon \to 0$, we conclude

$$\operatorname{cap}_p(E, \Omega) \le C \Lambda_h(E). \qquad \square$$

We close this section with sufficient conditions to get sets of relative Sobolev p-capacity zero.

Lemma 4.10. *Suppose (X, d, μ) is a proper and unbounded doubling metric measure space that admits a weak $(1, p)$-Poincaré inequality. Let E be a compact set in X. If there exists a constant $M > 0$ such that*

$$\operatorname{cap}_p(E, \Omega) \le M < \infty$$

for all open sets Ω containing E, then $\operatorname{cap}_p(E) = 0$.

Proof. It is enough to prove (see Remark 3.9) that $\operatorname{cap}_p(E, \Omega) = 0$ for every bounded and open $\Omega \supset E$. We let Ω be a bounded fixed open neighborhood of E. We choose a descending sequence of open sets

$$\Omega = \Omega_1 \supset\supset \Omega_2 \supset\supset \cdots \supset\supset \bigcap_i \Omega_i = E$$

and we choose $\varphi_i \in A(E, \Omega_i)$, $0 \le \varphi_i \le 1$ with $\varphi_i = 1$ on E and

$$\int_{\Omega_i} g_{\varphi_i}^p \, d\mu < \operatorname{cap}_p(E, \Omega_i) + 1 \le M + 1 \text{ for } i = 1, 2, \ldots. \tag{13}$$

Since Ω is bounded and $0 \le \varphi_i \le 1$ for every $i = 1, 2, \ldots$, it follows via (13) that φ_i is a bounded sequence in $N_0^{1,p}(\Omega)$. We notice that φ_i converges pointwise μ-a.e. to a function ψ which is 1 on E and 0 on $X \setminus E$. We also notice that g_{φ_i} converges pointwise μ-a.e. to 0. Hence, from Mazur's lemma ([27, p. 120]) and the reflexivity of $L^p(\Omega) \times L^p(\Omega)$ it follows that there exists a subsequence denoted again by φ_i such that $(\varphi_i, g_{\varphi_i})$ converges weakly to $(\psi, 0)$ in $L^p(\Omega) \times L^p(\Omega)$ and a sequence $\widetilde{\varphi}_i$ of convex combinations of φ_i,

$$\widetilde{\varphi}_i = \sum_{j=i}^{j_i} \lambda_{i,j} \varphi_j, \quad \lambda_{i,j} \ge 0, \quad \sum_{j=i}^{j_i} \lambda_{i,j} = 1,$$

such that $(\widetilde{\varphi}_i, g_{\widetilde{\varphi}_i})$ converges to $(\psi, 0)$ in $L^p(\Omega) \times L^p(\Omega)$. The closedness of $A(E, \Omega_i)$ under finite convex combinations implies that $\widetilde{\varphi}_i \in A(E, \Omega_i)$ for every integer $i \ge 1$. Therefore

$$0 \le \operatorname{cap}_p(E, \Omega) \le \limsup_{i \to \infty} \int_\Omega g_{\widetilde{\varphi}_i}^p d\mu = 0. \qquad \square$$

Theorem 4.11. *Let $1 < p \leq Q < \infty$ be fixed. Suppose (X, d, μ) is a proper and unbounded metric measure space such that the upper mass bound (12) holds for the measure μ. Let E be a subset of X.*

(i) *Suppose $1 < p < Q$. Then $\Lambda_{Q-p}(E) < \infty$ implies $\operatorname{cap}_p(E) = 0$.*

(ii) *Suppose $p = Q$. Let $h\colon [0, \infty) \to [0, \infty)$ be an increasing homeomorphism such that $h(t) = (\ln \frac{1}{t})^{1-p}$ for all $t \in (0, \frac{1}{2})$. Then $\Lambda_h(E) < \infty$ implies $\operatorname{cap}_p(E) = 0$.*

Proof. Since Λ_h is a regular outer measure, it is enough to assume that E is Borel. Let Ω be a bounded and open subset of X. We want to show that $\operatorname{cap}_p(E\cap\Omega, \Omega) = 0$. Since $\operatorname{cap}_p(\,\cdot\,, \Omega)$ is a Choquet capacity, we can assume without loss of generality that $E \cap \Omega$ is compact. We choose a descending sequence of open sets

$$\Omega = \Omega_1 \supset\supset \Omega_2 \supset\supset \cdots \supset\supset \bigcap_i \Omega_i = E \cap \Omega.$$

Then from Theorem 4.9 we have

$$\operatorname{cap}_p(E \cap \Omega, \Omega_i) \leq C\Lambda_h(E \cap \Omega) \leq C\Lambda_h(E) \quad \text{for every } i = 1, 2, \ldots.$$

By using the argument from Lemma 4.10 for the set $E \cap \Omega$, we obtain the desired conclusion. This finishes the proof. □

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