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A POWER OF A MEROMORPHIC FUNCTION SHARING A SMALL FUNCTION WITH ITS DERIVATIVE

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Abstract. In this paper, we investigate uniqueness problems of meromorphic functions that share a small function with one of their derivatives, and give some results which are related to a conjecture of Brück, and also improve several previous results.

1. Introduction and results

In this paper, a meromorphic function will mean meromorphic in the whole complex plane. We shall use the standard notations in Nevanlinna value distribution theory of meromorphic functions such as $T(r,f)$, $N(r,f)$, $m(r,f)$ (see e.g., [5, 8]). For any nonconstant meromorphic function f, we denote by $S(r,f)$ any quantity satisfying

$$\lim_{r\to\infty}\frac{S(r,f)}{T(r,f)}=0,$$

possibly outside of a set of finite linear measure in R_+. A meromorphic function $a(z)$ is said to be a small function of f, provided $T(r,a)=S(r,f)$.

We say that two meromorphic functions f and g share a small function a IM (ignoring multiplicities) when $f-a$ and $g-a$ have the same zeros. If $f-a$ and $g-a$ have the same zeros with the same multiplicities, then we say that f and g share a CM (counting multiplicities).

The uniqueness theory of entire and meromorphic functions has grown up to an extensive subfield of the value distribution theory, see e.g. the monograph [8] by Yang and Yi. A widely studied subtopic of the uniqueness theory has been to considering shared value problems relative to a meromorphic function f and its derivative $f^{(k)}$. Some of the basic papers in this direction are due to Rubel and Yang [7], Gundersen [3], Mues and Steinmetz [6] and Yang [9]. A much investigated problem in this direction is the following conjecture proposed by Brück [2]:

Conjecture. Let f be a non-constant entire function. Suppose that

$$\rho_1(f):=\limsup_{r\to\infty}\frac{\log\log T(r,f)}{\log r}$$

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is not a positive integer or infinite. If f and f' share one finite value a CM, then

$$\frac{f'-a}{f-a}=c$$

for some non-zero constant c.

The conjecture has been verified in special cases only by now: (1) f is of finite order, see [4], (2) $a=0$, see [2] and (3) $N(r,1/f')=S(r,f)$, see [2]. However, the corresponding conjecture for meromorphic functions fails in general, as shown by Gundersen and Yang [4], while it remains true in the case of $N(r,1/f')=S(r,f)$, see Al-Khaladi [1].

Recently, Yang and the present author [10] considered the case that $F=f^n$, where f is a nonconstant meromorphic function, assuming value sharing with F and F':

Theorem A. *Let f be a nonconstant entire function, $n\geq 7$ be an integer. Denote $F=f^n$. If F and F' share 1 CM, then $F=F'$, and f assumes the form*

$$f(z)=ce^{\frac{1}{n}z},$$

where c is a nonzero constant.

Theorem B. *Let f be a nonconstant meromorphic function and $n\geq 12$ be an integer. Denote $F=f^n$. If F and F' share 1 CM, then $F=F'$, and f assumes the form*

$$f(z)=ce^{\frac{1}{n}z},$$

where c is a nonzero constant.

Subsequently, the present author [13] improved Theorem A and B and gave the following theorems.

Theorem C. *Let f be a nonconstant entire function and $n\geq 6$ be an integer. Denote $F=f^n$. If F and F' share 1 CM, then $F=F'$, and f assumes the form*

$$f(z)=ce^{\frac{1}{n}z},$$

where c is a nonzero constant.

Theorem D. *Let f be a nonconstant meromorphic function and $n\geq 7$ be an integer. Denote $F=f^n$. If F and F' share 1 CM, then $F=F'$, and f assumes the form*

$$f(z)=ce^{\frac{1}{n}z},$$

where c is a nonzero constant.

In this paper, we improve Theorem C and D by obtaining the following results:

Theorem 1.1. *Let f be a nonconstant entire function, n, k be positive integers and $a(z)$ be a small meromorphic function of f such that $a(z)\not\equiv 0,\infty$. If f^n-a and $(f^n)^{(k)}-a$ share the value 0 CM and $n>k+1$, then $f^n=(f^n)^{(k)}$, and f assumes the form*

$$f(z)=ce^{\frac{\lambda}{n}z},$$

where c is a nonzero constant and $\lambda^k=1$.

Theorem 1.2. *Let f be a nonconstant meromorphic function, n, k be positive integers and $a(z)$ be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. If $f^n - a$ and $(f^n)^{(k)} - a$ share the value 0 CM and*

$$n > k + 1 + \sqrt{k+1}, \tag{1.1}$$

then $f^n = (f^n)^{(k)}$, and f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a nonzero constant and $\lambda^k = 1$.

From Theorem 1.1 and 1.2, we can deduce the following two corollaries.

Corollary 1.3. *Let f be a nonconstant entire function and $n \geq 3$ be an integer. Denote $F = f^n$. If F and F' share 1 CM, then $F = F'$, and f assumes the form*

$$f(z) = ce^{\frac{1}{n}z},$$

where c is a nonzero constant.

Corollary 1.4. *Let f be a nonconstant meromorphic function and $n \geq 4$ be an integer. Denote $F = f^n$. If F and F' share 1 CM, then $F = F'$, and f assumes the form*

$$f(z) = ce^{\frac{1}{n}z},$$

where c is a nonzero constant.

Remark. Obviously, Corollary 1.3 and Corollary 1.4 improve Theorem C and Theorem D respectively.

For the case sharing the small function IM, we have the following results.

Theorem 1.5. *Let f be a nonconstant entire function, n, k be positive integers and $a(z)$ be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. If $f^n - a$ and $(f^n)^{(k)} - a$ share the value 0 IM and $n > 2k + 3$, then $f^n = (f^n)^{(k)}$, and f assumes the form*

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a nonzero constant and $\lambda^k = 1$.

Theorem 1.6. *Let f be a nonconstant meromorphic function, n, k be positive integers and $a(z)$ be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. If $f^n - a$ and $(f^n)^{(k)} - a$ share the value 0 IM and*

$$n > 2k + 3 + \sqrt{(2k+3)(k+3)}, \tag{1.2}$$

then $f^n = (f^n)^{(k)}$, and f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a nonzero constant and $\lambda^k = 1$.

2. Some lemmas

Let F and G be two non-constant meromorphic functions such that F and G share the value 1 IM. Let z_0 be a 1-point of F of order p, a 1-point of G of order q. We denote by $N_L(r, \frac{1}{F-1})$ the counting function of those 1-points of F where $p > q$; by $N_E^{1)}(r, \frac{1}{F-1})$ the counting function of those 1-points of F where $p = q = 1$; by

$N_E^{(2}(r,\frac{1}{F-1})$ the counting function of those 1-points of F where $p=q\geq 2$; each point in these counting functions is counted only once. In the same way, we can define $N_L(r,\frac{1}{G-1})$, $N_E^{1)}(r,\frac{1}{G-1})$, and $N_E^{(2}(r,\frac{1}{G-1})$(see [12]). Particularly, if F and G share 1 CM, then

$$N_L\left(r,\frac{1}{F-1}\right)=N_L\left(r,\frac{1}{G-1}\right)=0. \tag{2.1}$$

With these notations, if F and G share 1 IM, it is easy to see that

$$\begin{aligned}\overline{N}\left(r,\frac{1}{F-1}\right) &= N_E^{1)}\left(r,\frac{1}{F-1}\right)+N_L\left(r,\frac{1}{F-1}\right)\\ &+ N_L\left(r,\frac{1}{G-1}\right)+N_E^{(2}\left(r,\frac{1}{G-1}\right)=\overline{N}\left(r,\frac{1}{G-1}\right).\end{aligned} \tag{2.2}$$

Lemma 2.1. [11, Lemma 3] *Let*

$$H=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right), \tag{2.3}$$

where F and G are two nonconstant meromorphic functions. If $H\not\equiv 0$, then

$$N_E^{1)}\left(r,\frac{1}{F-1}\right)\leq N(r,H)+S(r,F)+S(r,G). \tag{2.4}$$

Let p be a positive integer and $a\in \mathbf{C}\bigcup\{\infty\}$. We denote by $N_{p)}\left(r,\frac{1}{f-a}\right)$ the counting function of the zeros of $f-a$ with the multiplicities less than or equal to p, and by $N_{(p+1}\left(r,\frac{1}{f-a}\right)$ the counting function of the zeros of $f-a$ with the multiplicities larger than p. And we use $\overline{N}_{p)}\left(r,\frac{1}{f-a}\right)$ and $\overline{N}_{(p+1}\left(r,\frac{1}{f-a}\right)$ to denote the corresponding reduced counting functions (ignoring multiplicities). However, $N_p\left(r,\frac{1}{f-a}\right)$ denotes the counting function of the zeros of $f-a$ where m-fold zeros are counted m times if $m\leq p$ and p times if $m>p$. Obviously, $\overline{N}\left(r,\frac{1}{f-a}\right)=N_1\left(r,\frac{1}{f-a}\right)$.

Lemma 2.2. [14, Lemma 3] *Suppose that f is a nonconstant meromorphic function and k, p are positive integers. Then*

$$N_p\left(r,1/f^{(k)}\right)\leq T(r,f^{(k)})-T(r,f)+N_{p+k}\left(r,1/f\right)+S(r,f), \tag{2.5}$$

$$N_p\left(r,1/f^{(k)}\right)\leq k\overline{N}(r,f)+N_{p+k}\left(r,1/f\right)+S(r,f). \tag{2.6}$$

Lemma 2.3. *Let f be a nonconstant meromorphic function, n, k be positive integers and $a(z)$ be a small meromorphic function of f such that $a(z)\not\equiv 0,\infty$. Suppose that f^n-a and $(f^n)^{(k)}-a$ share the value 0 IM and H is given by* (2.3), *where $F=\frac{f^n}{a}$, $G=\frac{(f^n)^{(k)}}{a}$. If $H\not\equiv 0$ and $n>k+1$, then*

$$T(r,f)=O\left(\overline{N}(r,f)+\overline{N}(r,1/f)\right). \tag{2.7}$$

Proof. Since $f^n - a$ and $(f^n)^{(k)} - a$ share the value 0 IM, then F and G share 1 IM possibly except at the zeros and poles of $a(z)$. By the definition of H, we have

$$\begin{aligned} N(r,H) \le{}& \overline{N}_{(2}\left(r,\frac{1}{F}\right) + \overline{N}_{(2}\left(r,\frac{1}{G}\right) + \overline{N}(r,F) + N_L\left(r,\frac{1}{F-1}\right) \\ &+ N_L\left(r,\frac{1}{G-1}\right) + N_0\left(r,\frac{1}{F'}\right) + N_0\left(r,\frac{1}{G'}\right) + S(r,f), \end{aligned} \tag{2.8}$$

where $N_0(r,\frac{1}{F'})$ denotes the counting function corresponding to the zeros of F' which are not the zeros of F and $F-1$, and correspondingly for G'.

From (2.2), (2.4) and (2.8), we obtain

$$\begin{aligned} &\overline{N}\left(r,\frac{1}{F-1}\right) + \overline{N}\left(r,\frac{1}{G-1}\right) \\ &= 2N_E^{1)}\left(r,\frac{1}{F-1}\right) + 2N_L\left(r,\frac{1}{F-1}\right) + 2N_L\left(r,\frac{1}{G-1}\right) + 2N_E^{(2}\left(r,\frac{1}{G-1}\right) \\ &\le \overline{N}_{(2}\left(r,\frac{1}{F}\right) + \overline{N}_{(2}\left(r,\frac{1}{G}\right) + \overline{N}(r,F) + N_E^{1)}\left(r,\frac{1}{F-1}\right) + 3N_L\left(r,\frac{1}{F-1}\right) \\ &\quad + 3N_L\left(r,\frac{1}{G-1}\right) + 2N_E^{(2}\left(r,\frac{1}{G-1}\right) + N_0\left(r,\frac{1}{F'}\right) + N_0\left(r,\frac{1}{G'}\right). \end{aligned} \tag{2.9}$$

Noting that

$$\begin{aligned} &N_E^{1)}\left(r,\frac{1}{F-1}\right) + 2N_L\left(r,\frac{1}{F-1}\right) + N_L\left(r,\frac{1}{G-1}\right) + 2N_E^{(2}\left(r,\frac{1}{G-1}\right) \\ &\le N\left(r,\frac{1}{F-1}\right) \le T(r,F) + O(1), \end{aligned}$$

(2.9) yields

$$\begin{aligned} &\overline{N}\left(r,\frac{1}{F-1}\right) + \overline{N}\left(r,\frac{1}{G-1}\right) \\ &\le \overline{N}_{(2}\left(r,\frac{1}{F}\right) + \overline{N}_{(2}\left(r,\frac{1}{G}\right) + \overline{N}(r,F) + N_L\left(r,\frac{1}{F-1}\right) \\ &\quad + 2N_L\left(r,\frac{1}{G-1}\right) + T(r,F) + N_0\left(r,\frac{1}{F'}\right) + N_0\left(r,\frac{1}{G'}\right) + O(1). \end{aligned} \tag{2.10}$$

From the second fundamental theorem, we have

$$T(r,F) \le \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}(r,F) + \overline{N}\left(r,\frac{1}{F-1}\right) - N_0\left(r,\frac{1}{F'}\right) + S(r,F), \tag{2.11}$$

$$T(r,G) \le \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}(r,G) + \overline{N}\left(r,\frac{1}{G-1}\right) - N_0\left(r,\frac{1}{G'}\right) + S(r,G). \tag{2.12}$$

Combining with (2.10), (2.11) and (2.12), we get

$$\begin{aligned} T(r,F) + T(r,G) \le{}& N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + 3\overline{N}(r,f) + N_L\left(r,\frac{1}{F-1}\right) \\ &+ 2N_L\left(r,\frac{1}{G-1}\right) + T(r,F) + S(r,f), \end{aligned}$$

which means

$$T(r,(f^n)^{(k)}) \le N_2\left(r,\frac{1}{f^n}\right) + N_2\left(r,\frac{1}{(f^n)^{(k)}}\right) + 3\overline{N}(r,f) + N_L\left(r,\frac{1}{F-1}\right) + 2N_L\left(r,\frac{1}{G-1}\right) + S(r,f).$$

From above inequality and (2.5), we have

$$T(r,f^n) \le N_2\left(r,\frac{1}{f^n}\right) + N_{2+k}\left(r,\frac{1}{f^n}\right) + 3\overline{N}(r,f) + N_L\left(r,\frac{1}{F-1}\right) + 2N_L\left(r,\frac{1}{G-1}\right) + S(r,f). \tag{2.13}$$

By the definition, we have

$$\overline{N}\left(r,\frac{1}{G}\right) = N_1\left(r,\frac{1}{G}\right) \le N_1\left(r,\frac{1}{(f^n)^{(k)}}\right) + S(r,f).$$

From this and (2.6), we get

$$\begin{aligned}\overline{N}\left(r,\frac{1}{G}\right) &\le N_{1+k}\left(r,\frac{1}{f^n}\right) + k\overline{N}\left(r,f\right) + S(r,f)\\ &\le (k+1)\overline{N}\left(r,\frac{1}{f}\right) + k\overline{N}\left(r,f\right) + S(r,f).\end{aligned} \tag{2.14}$$

Noting that $n > k+1$, we get from (2.14)

$$\begin{aligned}N_L\left(r,\frac{1}{F-1}\right) &\le N\left(r,\frac{F}{F'}\right) \le N\left(r,\frac{F'}{F}\right) + S(r,f)\\ &\le \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,F\right) + S(r,f)\\ &\le \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,f\right) + S(r,f),\end{aligned} \tag{2.15}$$

$$\begin{aligned}N_L\left(r,\frac{1}{G-1}\right) &\le N\left(r,\frac{G}{G'}\right) \le N\left(r,\frac{G'}{G}\right) + S(r,f)\\ &\le \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}\left(r,G\right) + S(r,f)\\ &\le (k+1)\overline{N}\left(r,\frac{1}{f}\right) + (k+1)\overline{N}\left(r,f\right) + S(r,f).\end{aligned} \tag{2.16}$$

Substituting (2.15) and (2.16) into (2.13), we obtain the conclusion of Lemma 2.3. □

Lemma 2.4. *Let*

$$V = \left(\frac{F'}{F-1} - \frac{F'}{F}\right) - \left(\frac{G'}{G-1} - \frac{G'}{G}\right), \tag{2.17}$$

where F and G are given by Lemma 2.3. If $V = 0$ and $n \ge 2$, then $F = G$.

Proof. From $V = 0$, we get

$$1 - \frac{1}{F} = B - \frac{B}{G}, \tag{2.18}$$

where B is a non-zero constant. We discuss the following two cases.

Case 1. Suppose that $N(r,f) \neq S(r,f)$. Then there exists a z_0 which is not a zero or pole of a such that $\frac{1}{f(z_0)} = 0$, thus $\frac{1}{F(z_0)} = \frac{1}{G(z_0)} = 0$. We get $B = 1$ from (2.18).

Case 2. Suppose that $N(r,f) = S(r,f)$. If $B \neq 1$, then $\overline{N}\left(r, \frac{1}{F-\frac{1}{1-B}}\right) = \overline{N}(r,G) = S(r,f)$. From the second fundamental theorem, we have

$$\begin{aligned} T(r,F) &\leq \overline{N}(r,F) + \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{F-\frac{1}{1-B}}\right) + S(r,F) \\ &\leq \overline{N}\left(r,\frac{1}{f}\right) + S(r,f), \end{aligned}$$

which is a contradiction since $n \geq 2$. Therefore $B = 1$. Thus $F = G$, completing the proof of Lemma 2.4. □

Lemma 2.5. *Let V be given by* (2.17) *and suppose that $V \not\equiv 0$. Then*

$$(n-1)\overline{N}(r,f) \leq N(r,V) + S(r,f). \tag{2.19}$$

Proof. We get from (2.17) that

$$V = \frac{F'}{F(F-1)} - \frac{G'}{G(G-1)}.$$

Suppose that z_0 is a pole of f with the multiplicity p such that $a(z_0) \neq 0$ and $a(z_0) \neq \infty$. Then z_0 is a zero of $\frac{F'}{F(F-1)}$ with the multiplicity $np-1$ and a zero of $\frac{G'}{G(G-1)}$ with the multiplicity $np+k-1$. So z_0 is zero of V with the multiplicity at least $n-1$. Noting that $m(r,V) = S(r,f)$, we have

$$(n-1)\overline{N}(r,f) \leq N\left(r,\frac{1}{V}\right) + S(r,f) \leq T(r,V) + S(r,f) \leq N(r,V) + S(r,f). \quad \square$$

Lemma 2.6. *Assume that the conditions of Lemma* 2.5 *are satisfied and $n > k+1$.*

(1) *If F and G share 1 CM, then*

$$(n-k-1)\overline{N}(r,f) \leq (k+1)\overline{N}\left(r,\frac{1}{f}\right) + S(r,f). \tag{2.20}$$

(2) *If F and G share 1 IM, then*

$$(n-2k-3)\overline{N}(r,f) \leq (2k+3)\overline{N}\left(r,\frac{1}{f}\right) + S(r,f). \tag{2.21}$$

Proof. (1) From (2.17), we have

$$N(r,V) \leq \overline{N}\left(r,\frac{1}{G}\right) + S(r,f).$$

From this, (2.19) and (2.14), we obtain (2.20).

(2) From (2.17), we have

$$N(r,V)\le \overline{N}\left(r,\frac{1}{G}\right)+N_L\left(r,\frac{1}{F-1}\right)+N_L\left(r,\frac{1}{G-1}\right)+S(r,f).$$

Substituting (2.14), (2.15) and (2.16) into above inequality and using (2.19), we obtain (2.21). □

Lemma 2.7. *Let*

$$U=\frac{F'}{F-1}-\frac{G'}{G-1}, \tag{2.22}$$

where F and G are given by Lemma 2.3. If $U=0$ and $n>k+1$, then $F=G$.

Proof. Suppose to the contrary that $F\ne G$. From $U=0$, we get

$$F=DG+1-D, \tag{2.23}$$

where D is a non-zero constant. We get from (2.23) that $D\ne 1$, $N(r,f)=S(r,f)$. Suppose that there exists a point z_0 such that $f(z_0)=0$ and $a(z_0)\ne 0$. Since $n>k+1$, we have $F(z_0)=G(z_0)=0$. So, $D=1$, a contradiction. Therefore $N(r,1/f)=S(r,f)$. From the second fundamental theorem, we get from (2.23) and (2.14) that

$$\begin{aligned}nT(r,f)&\le T(r,F)+S(r,f)\\&\le \overline{N}(r,F)+\overline{N}(r,1/F)+\overline{N}\left(r,\frac{1}{F+D-1}\right)+S(r,f)\\&\le \overline{N}(r,1/f)+\overline{N}(r,f)+\overline{N}(r,1/G)+S(r,f)\\&\le \overline{N}(r,1/G)+S(r,f)\\&\le (k+1)\overline{N}(r,1/f)+k\overline{N}(r,f)+S(r,f)\\&\le S(r,f),\end{aligned}$$

a contradiction. The proof of Lemma 2.7 is complete. □

Lemma 2.8. *Let U be given by Lemma 2.7. If $U\ne 0$ and $n>k+1$, then*

$$(n-k-1)\overline{N}(r,1/f)\le N(r,U)+S(r,f). \tag{2.24}$$

Proof. Suppose that z_0 is a zero of f with the multiplicity p such that $a(z_0)\ne 0$ and $a(z_0)\ne\infty$. Then z_0 is a zero of $\frac{F'}{F-1}$ with the multiplicity $np-1$ and a zero of $\frac{G'}{G-1}$ with the multiplicity $np-k-1$. So z_0 is zero of U with the multiplicity at least $n-k-1$. Noting that $m(r,U)=S(r,f)$, we have

$$\begin{aligned}(n-k-1)\overline{N}(r,1/f)&\le N\left(r,\frac{1}{U}\right)+S(r,f)\le T(r,U)+S(r,f)\\&\le N(r,U)+S(r,f).\end{aligned}$$

□

Lemma 2.9. *Assume that the conditions of Lemma 2.8 are satisfied.*

(1) *If F and G share 1 CM, then*

$$(n-k-1)\overline{N}(r,1/f)\le \overline{N}(r,f)+S(r,f). \tag{2.25}$$

(2) *If F and G share 1 IM, then*

$$(n-2k-3)\overline{N}(r,1/f) \le (k+3)\overline{N}(r,f)+S(r,f). \tag{2.26}$$

Proof. (1) From (2.22), we have

$$N(r,U) \le \overline{N}(r,f)+S(r,f).$$

From this and (2.24), we obtain (2.25).

(2) From (2.22), we have

$$N(r,U) \le \overline{N}(r,f)+N_L\left(r,\frac{1}{F-1}\right)+N_L\left(r,\frac{1}{G-1}\right)+S(r,f).$$

From this, (2.24), (2.15) and (2.16), we obtain (2.26). □

Lemma 2.10. *Let F and G be given by Lemma 2.3. If $F=G$ and $n>k+1$, then f assumes the form*

$$f(z)=ce^{\frac{\lambda}{n}z},$$

where c is a nonzero constant and $\lambda^k=1$.

Proof. From $F=G$, we have

$$f^n=(f^n)^{(k)}. \tag{2.27}$$

We claim that 0 is a Picard exceptional value of f. In fact, if z_0 is a zero of f with the multiplicity p, then z_0 is a zero of f^n with the multiplicity np and a zero of $(f^n)^{(k)}$ with the multiplicity $np-k$, which is impossible from (2.27). Then from (2.27), we have

$$f(z)=ce^{\frac{\lambda}{n}z},$$

where c is a nonzero constant and $\lambda^k=1$. □

Lemma 2.11. *Let H be given by Lemma 2.3 and*

$$\overline{N}(r,f)=\overline{N}(r,1/f)=S(r,f). \tag{2.28}$$

If $H=0$, then $F=G$.

Proof. By integration, we get from (2.3) that

$$\frac{1}{F-1}=\frac{A}{G-1}+B, \tag{2.29}$$

where $A(\neq 0)$ and B are constants. From (2.29) we have

$$G=\frac{(B-A)F+(A-B-1)}{BF-(B+1)}. \tag{2.30}$$

We discuss the following three cases.

Case 1. Suppose that $B \neq 0, -1$. From (2.30) we have $\overline{N}\left(r, 1/\left(F - \frac{B+1}{B}\right)\right) = \overline{N}(r, G)$. From (2.28) and the second fundamental theorem, we have

$$\begin{aligned} nT(r,f) &\le T(r,F) + S(r,f) \\ &\le \overline{N}(r,F) + \overline{N}(r,1/F) + \overline{N}\left(r, \frac{1}{F - \frac{B+1}{B}}\right) + S(r,f) \\ &\le \overline{N}(r,1/f) + \overline{N}(r,F) + \overline{N}(r,G) + S(r,f) \\ &\le S(r,f), \end{aligned}$$

which is impossible.

Case 2. Suppose that $B = 0$. From (2.30) we have

$$G = AF - (A-1). \tag{2.31}$$

If $A \neq 1$, from (2.31) we can obtain $\overline{N}\left(r, 1/\left(F - \frac{A-1}{A}\right)\right) = \overline{N}(r, 1/G)$. By (2.6), (2.28) and the second fundamental theorem, we have

$$\begin{aligned} nT(r,f) &\le T(r,F) + S(r,f) \\ &\le \overline{N}(r,F) + \overline{N}(r,1/F) + \overline{N}\left(r, \frac{1}{F - \frac{A-1}{A}}\right) + S(r,f) \\ &\le S(r,f), \end{aligned}$$

which is impossible. Thus $A = 1$. From (2.31) we have $F = G$.

Case 3. Suppose that $B = -1$. From (2.30) we have

$$G = \frac{(A+1)F - A}{F}. \tag{2.32}$$

If $A \neq -1$, we obtain from (2.32) that $\overline{N}\left(r, 1/\left(F - \frac{A}{A+1}\right)\right) = \overline{N}(r, 1/G)$. By the same reasoning discussed in Case 2, we obtain a contradiction. Hence $A = -1$. From (2.32), we get $F \cdot G = 1$, that is

$$f^n \cdot (f^n)^{(k)} = a^2.$$

From above equation and (2.28), we have

$$\begin{aligned} 2T\left(r, \frac{f^n}{a}\right) &= T\left(r, \frac{f^{2n}}{a^2}\right) = T\left(r, \frac{a^2}{f^{2n}}\right) + O(1) \\ &= T\left(r, \frac{(f^n)^{(k)}}{f^n}\right) + O(1) = S(r,f). \end{aligned}$$

So $T(r,f) = S(r,f)$, which is impossible. This completes the proof of Lemma 2.11. □

3. Proofs of results

Proof of Theorem 1.6. Let H, U and V be given by (2.3), (2.22) and (2.17) respectively.

If $UV = 0$, we get $F = G$ from Lemma 2.7 or Lemma 2.4. Then Theorem 1.6 follows by Lemma 2.10. Next, we assume $UV \neq 0$. From (2.21) and (2.26), we have

$$\big((n-2k-3)^2 - (2k+3)(k+3)\big)\,\overline{N}(r,f) \le S(r,f),$$
$$\big((n-2k-3)^2 - (2k+3)(k+3)\big)\,\overline{N}(r,1/f) \le S(r,f).$$

Substituting (1.2) into above two inequalities, resulting in

$$\overline{N}(r,1/f) = \overline{N}(r,f) = S(r,f). \tag{3.1}$$

If $H \neq 0$, combining with (3.1) and (2.7) yields

$$T(r,f) = S(r,f),$$

which is a contradiction. Hence $H = 0$. Theorem 1.6 follows from (3.1), Lemma 2.11 and Lemma 2.10. □

Proof of Theorem 1.5. Let H, U and V be given by (2.3), (2.22) and (2.17) respectively.

If $U = 0$, we get $F = G$ from Lemma 2.7. Then Theorem 1.6 follows from Lemma 2.10. Next, we assume $U \neq 0$. Noting that $n > 2k+3$, we get from (2.26) that $\overline{N}(r,1/f) = S(r,f)$.

If $H \neq 0$, then $T(r,f) = S(r,f)$ from (2.7), which is a contradiction. Hence $H = 0$. Theorem 1.5 follows from Lemma 2.11 and Lemma 2.10. □

If we use (2.20) and (2.25) instead of (2.21) and (2.26) in the proofs of Theorem 1.5 and 1.6, we can get the proofs of Theorem 1.1 and 1.2 similarly. We omit the details here.

4. Open problem

Let F be as in Corollary 1.4. Suppose that $n \ge 4 (\ge 3)$ when f is a meromorphic (entire) function. The two corollaries tell us $F = F'$ if F and F' share 1 CM. Examples given by [4] show that the conclusion may fail when $n = 1$. A natural question is:

Question. Can n in Corollary 1.3 and 1.4 be reduced?

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