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THE DOUBLE OBSTACLE PROBLEM ON METRIC SPACES

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Abstract. We study the double obstacle problem on a metric measure space equipped with a doubling measure and supporting a p-Poincaré inequality. We prove existence and uniqueness. We also prove the continuity of the solution of the double obstacle problem with continuous obstacles and show that the continuous solution is a minimizer in the open set where it does not touch the two obstacles. Moreover we consider the regular boundary points and show that the solution of the double obstacle problem on a regular open set with continuous obstacles is continuous up to the boundary. Regularity of boundary points is further characterized in some other ways using the solution of the double obstacle problem.

1. Introduction

Let $1 < p < \infty$ and $X = (X, d, \mu)$ be a complete metric space endowed with a metric d and a positive complete Borel measure μ which is *doubling*, i.e. there exists a constant $C > 0$ such that for all balls $B = B(x, r) := \{y \in X : d(x, y) < r\}$ in X we have

$$0 < \mu(2B) \leq C\mu(B) < \infty,$$

where $2B = B(x, 2r)$.

In a metric space the gradient has no obvious meaning as in domains in $\mathbf{R}^n$. Therefore the concept of an upper gradient was introduced in Heinonen–Koskela [7] as a substitute for the modulus of the usual gradient. This makes it possible to define and study the Sobolev type spaces $N^{1,p}(X)$ (called Newtonian spaces) in metric spaces which enables us to study variational integrals in metric spaces and to build a nonlinear potential theory for minimizers of the variational integral

$$\int g_u^p \, d\mu, \tag{1}$$

where g_u denotes the minimal p-weak upper gradient of u, see Shanmugalingam [12] and [13]. Indeed, in Kinnunen–Shanmugalingam [10] it was shown that under certain conditions on the space X, the minimizers of (1) satisfy the Harnack inequality and the maximum principle, and are locally Hölder continuous. The Dirichlet problem for p-harmonic functions was studied e.g. in Björn–Björn [2], Björn–Björn–Shanmugalingam [5] and Shanmugalingam [13]. The single obstacle problem in metric spaces has been studied in Kinnunen–Martio [9]. In this note we study the double obstacle problem in metric spaces. Our work extends some results from [9] and [2] in which similar investigations were undertaken for the case of a single obstacle problem.

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Since the single obstacle problem is a special case of the double obstacle problem, one cannot expect better results in the latter case. One significant difference between the single and double obstacle problems is that the solution of the single obstacle problem turns out to be a superminimizer whereas this is no longer true in the double obstacle situation. This does not allow for the use of the weak Harnack inequality for superminimizers, which was a main tool in the analysis of the single obstacle problem. However we are still able to obtain many useful results for the double obstacle problem.

Let Ω be a bounded open subset of X. We study the double obstacle problem of the type

$$\mathscr{K}_{\psi_1,\psi_2,f}(\Omega) = \{v \in N^{1,p}(\Omega) : v - f \in N_0^{1,p}(\Omega) \text{ and } \psi_1 \le v \le \psi_2 \text{ q.e. in } \Omega\},$$

where $f \in N^{1,p}(\Omega)$ and $\psi_j \colon \Omega \to \overline{\mathbf{R}}$, $j = 1, 2$. A function $u \in \mathscr{K}_{\psi_1,\psi_2,f}(\Omega)$ is a *solution of the* $\mathscr{K}_{\psi_1,\psi_2,f}(\Omega)$*-obstacle problem* if

$$\int_\Omega g_u^p \, d\mu \le \int_\Omega g_v^p \, d\mu \quad \text{for all } v \in \mathscr{K}_{\psi_1,\psi_2,f}(\Omega),$$

where g_u is the minimal p-weak upper gradient of u.

In the Euclidean case the double obstacle problem was studied e.g. in Kilpeläinen–Ziemer [8], Dal Maso–Mosco–Vivaldi [6] and Li–Martio [11].

This paper is organized as follows. In Section 2, we define Newtonian spaces, the Sobolev type spaces considered in metric spaces, and give some of their properties. In Section 3, we define the double obstacle problem, and prove that there exists a unique solution (up to sets of capacity zero) of the $\mathscr{K}_{\psi_1,\psi_2,f}(\Omega)$-obstacle problem. We also show that there is a continuous solution of the double obstacle problem provided the two obstacles are continuous, in this case we also prove that the solution is a minimizer in the open set where the continuous solution does not touch the two obstacles.

We end this paper, in Section 4, with boundary regularity for the double obstacle problem, and prove that under certain conditions the solution of the obstacle problem is continuous up to the boundary. Finally we give two new characterizations of regular boundary points.

2. Notation and preliminaries

A nonnegative Borel function g is said to be an *upper gradient* of an extended real-valued function f on X if for all rectifiable curves $\gamma \colon [0, l_\gamma] \to X$ parameterized by arc length ds, we have

$$|f(\gamma(0)) - f(\gamma(l_\gamma))| \le \int_\gamma g \, ds \tag{2}$$

whenever both $f(\gamma(0))$ and $f(\gamma(l_\gamma))$ are finite, and $\int_\gamma g \, ds = \infty$ otherwise. If g is a nonnegative measurable function on X and if (2) holds for p-almost every curve, then g is a *p-weak upper gradient* of f.

By saying that (2) holds for p-almost every curve we mean that it fails only for a curve family with zero p-modulus, see Definition 2.1 in Shanmugalingam [12]. If f has an upper gradient in $L^p(X)$, then it has a *minimal p-weak upper gradient* $g_f \in L^p(X)$ in the sense that for every p-weak upper gradient $g \in L^p(X)$ of f, $g_f \le g$ a.e., see Corollary 3.7 in Shanmugalingam [13].

The operation of taking the upper gradient is not linear. However, we have the following useful property. If $a, b \in \mathbf{R}$ and g_1 and g_2 are upper gradients of u_1 and u_2 respectively, then $|a|g_1 + |b|g_2$ is an upper gradient of $au_1 + bu_2$.

In Shanmugalingam [12], upper gradients have been used to define Sobolev type spaces on metric spaces. We will use the following equivalent definition.

Definition 2.1. Let $u \in L^p(X)$, then we define

$$\|u\|_{N^{1,p}(X)} = \left(\int_X |u|^p \, d\mu + \int_X g_u^p \, d\mu \right)^{1/p},$$

where g_u is the minimal p-weak upper gradient of u. The *Newtonian space* on X is the quotient space

$$N^{1,p}(X) = \left\{ u : \|u\|_{N^{1,p}(X)} < \infty \right\} / \sim,$$

where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(X)} = 0$.

The space $N^{1,p}(X)$ is a Banach space and a lattice, see Theorem 3.7 and p. 249 in Shanmugalingam [12]. We also have the following lemma about minimal p-weak upper gradients, see Björn–Björn [1], Corollary 3.4.

Lemma 2.2. *If $u, v \in N^{1,p}(X)$, then*

$$g_u = g_v \quad \text{a.e. on } \{x \in X : u(x) = v(x)\}.$$

Moreover, if $c \in \mathbf{R}$ is a constant, then $g_u = 0$ a.e. on $\{x \in X : u(x) = c\}$.

Definition 2.3. The *capacity* of a set $E \subset X$ is defined by

$$C_p(E) = \inf_u \|u\|^p_{N^{1,p}(X)},$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u \geq 1$ on E.

We say that a property holds *quasieverywhere* (q.e.) in X, if it holds everywhere except on a set of capacity zero. Newtonian functions are well defined up to sets of capacity zero, i.e. if $u,v \in N^{1,p}(X)$ then $u \sim v$ if and only if $u = v$ q.e. Moreover, Corollary 3.3 in Shanmugalingam [12] shows that if $u, v \in N^{1,p}(X)$ and $u = v$ a.e., then $u = v$ q.e.

From now on we assume that X supports a *p-Poincaré inequality*, i.e. there exist constants $C > 0$ and $\lambda \geq 1$ such that for all balls $B(x,r)$ in X, all integrable functions u on X and all upper gradients g of u we have

$$\fint_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \leq Cr \left(\fint_{B(x,\lambda r)} g^p \, d\mu \right)^{1/p},$$

where $u_{B(x,r)} := \fint_{B(x,r)} u \, d\mu$.

For $\Omega \subset X$ open we define the space $N^{1,p}(\Omega)$ with respect to the restrictions of the metric d and the measure μ to Ω. It is well known in the field that the restriction to Ω of a minimal p-weak upper gradient in X remains minimal with respect to Ω.

A function u is said to belong to the *local Newtonian space* $N^{1,p}_{\mathrm{loc}}(\Omega)$ if $u \in N^{1,p}(A)$ for every open $A \Subset \Omega$, where by $A \Subset \Omega$ we mean that the closure of A is a compact subset of Ω.

To be able to compare the boundary values of Newtonian functions we need to define a Newtonian space with zero boundary values outside of Ω as follows

$$N^{1,p}_0(\Omega) = \left\{ f|_\Omega : f \in N^{1,p}(X) \text{ and } f = 0 \text{ q.e. in } X \setminus \Omega \right\}.$$

The following lemma is useful for proving that a function belongs to $N_0^{1,p}(\Omega)$, see Lemma 5.3 in Björn–Björn [2].

Lemma 2.4. *Let $u \in N^{1,p}(\Omega)$ be such that $v \le u \le w$ q.e. in Ω for some $v, w \in N_0^{1,p}(\Omega)$. Then $u \in N_0^{1,p}(\Omega)$.*

Under our assumptions, Lipschitz functions with compact support are dense in $N_0^{1,p}(\Omega)$, see Shanmugalingam [13]. Moreover the proof of this result in [3] shows that if $0 \le u \in N_0^{1,p}(\Omega)$, then we can choose the Lipschitz approximations to be nonnegative.

We shall need the following Poincaré type inequality. For a proof, see e.g. Kinnunen–Shanmugalingam [10], Lemma 2.1.

Lemma 2.5. *Assume that $\Omega \subset X$ is a nonempty bounded open set with $C_p(X \setminus \Omega) > 0$. Then there exists a constant $C > 0$ such that for all $u \in N_0^{1,p}(\Omega)$ we have*

$$\int_\Omega |u|^p \, d\mu \le C \int_\Omega g_u^p \, d\mu.$$

We shall use the following lemma. For a proof, see Björn–Björn–Parviainen [4].

Lemma 2.6. *Assume that g_j is a p-weak upper gradient of u_j, $j = 1, 2, \dots$, and that both sequences $\{u_j\}_{j=1}^\infty$ and $\{g_j\}_{j=1}^\infty$ are bounded in $L^p(X)$. Then there are $u, g \in L^p(X)$, convex combinations $v_j = \sum_{i=j}^{N_j} a_{j,i} u_i$ with p-weak upper gradients $\bar{g}_j = \sum_{i=j}^{N_j} a_{j,i} g_i$ and a strictly increasing sequence of indices $\{j_k\}_{k=1}^\infty$, such that*

(a) *both $u_{j_k} \to u$ and $g_{j_k} \to g$ weakly in $L^p(X)$;*
(b) *both $v_j \to u$ and $\bar{g}_j \to g$ in $L^p(X)$;*
(c) *$v_j \to u$ q.e.;*
(d) *g is a p-weak upper gradient of u.*

3. The double obstacle problem

Recall that we assume in this paper that X is a complete metric measure space supporting a p-Poincaré inequality and that μ is doubling.

Throughout the rest of this paper we make the additional assumptions that $\Omega \subset X$ is a nonempty bounded open set such that $C_p(X \setminus \Omega) > 0$. Also the letter C represents various constants and can change even within the same line of a calculation.

Let $V \subset X$ be a nonempty bounded open set with $C_p(X \setminus V) > 0$, let $\psi \colon V \to \overline{\mathbf{R}}$ and $f \in N^{1,p}(V)$. In Kinnunen–Martio [9] the single obstacle problem (denoted by $\widetilde{\mathscr{K}}_{\psi,f}(V)$) is defined as follows

$$\widetilde{\mathscr{K}}_{\psi,f}(V) = \bigl\{v \in N^{1,p}(V) : v - f \in N_0^{1,p}(V) \text{ and } v \ge \psi \text{ a.e. in } V\bigr\}$$

and $u \in \widetilde{\mathscr{K}}_{\psi,f}(V)$ is a solution of the $\widetilde{\mathscr{K}}_{\psi,f}(V)$-obstacle problem if

$$\int_V g_u^p \, d\mu \le \int_V g_v^p \, d\mu \quad \text{for all } v \in \widetilde{\mathscr{K}}_{\psi,f}(V).$$

As the Newtonian functions are defined up to sets of capacity zero we see that it is natural to consider the obstacle problem up to sets of capacity zero instead of sets of measure zero and therefore define the double obstacle problem with a slightly different notation from Kinnunen–Martio [9] as follows.

Definition 3.1. Let $V \subset X$ be a nonempty bounded open set such that $C_p(X \setminus V) > 0$, let $f \in N^{1,p}(V)$ and $\psi_i \colon V \to \overline{\mathbf{R}}$, $i = 1, 2$. Then we define

$$\mathscr{K}_{\psi_1,\psi_2,f}(V) = \bigl\{ v \in N^{1,p}(V) : v - f \in N_0^{1,p}(V) \text{ and } \psi_1 \le v \le \psi_2 \text{ q.e. in } V \bigr\}.$$

Furthermore, a function $u \in \mathscr{K}_{\psi_1,\psi_2,f}(V)$ is a *solution of the* $\mathscr{K}_{\psi_1,\psi_2,f}(V)$*-obstacle problem* if

$$\int_V g_u^p \, d\mu \le \int_V g_v^p \, d\mu \quad \text{for all } v \in \mathscr{K}_{\psi_1,\psi_2,f}(V).$$

We also let $\mathscr{K}_{\psi_1,\psi_2,f} = \mathscr{K}_{\psi_1,\psi_2,f}(\Omega)$, $\mathscr{K}_{\psi,f}(V) = \mathscr{K}_{\psi,\infty,f}(V)$ and $\mathscr{K}_{\psi,f} = \mathscr{K}_{\psi,f}(\Omega)$.

The distinction between the two definitions becomes important e.g. when solving the single obstacle problem with obstacle χ_K and boundary values zero where $K \subset \Omega$ is a compact set with positive capacity and zero measure. In this case the $\mathscr{K}_{\chi_K,0}$-obstacle problem leads to a p-harmonic function in $\Omega \setminus K$ with boundary values 1 on K and zero on $\partial\Omega$, whereas the $\widetilde{\mathscr{K}}_{\chi_K,0}$-obstacle problem has the trivial solution. In particular this is evident if K is an $(n-1)$-dimensional sphere contained in $\Omega \subset \mathbf{R}^n$.

At the same time our definition is stronger than the definition used in Kinnunen–Martio [9] and it is possible to have no solution of the $\mathscr{K}_{\psi,f}$-obstacle problem whereas there exists a solution of the $\widetilde{\mathscr{K}}_{\psi,f}$-obstacle problem as the following example shows. If $\Omega = B(0,1) \subset \mathbf{R}^n$ (with the Lebesgue measure), $S_n = \partial B(0, 1 - 1/n)$ and $E = \bigcup_{n=2}^\infty S_n$. Then E has measure zero and positive capacity. Therefore, the obstacle problem $\widetilde{\mathscr{K}}_{\chi_E,0}$ has the trivial solution. On the other hand there is no solution for the $\mathscr{K}_{\psi,f}$-obstacle problem, since no Newtonian function with zero boundary values on $\partial B(0,1)$ will be above χ_E. We also remark here that the proofs of all the results which we use from Kinnunen–Martio [9] can be modified to fit our definition.

A function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is a *minimizer* in Ω if it is a solution of the $\mathscr{K}_{-\infty,u}(\Omega')$-obstacle problem for every open $\Omega' \Subset \Omega$. Similarly, a function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is a *superminimizer* in Ω if it is a solution of the $\mathscr{K}_{u,u}(\Omega')$-obstacle problem for every open $\Omega' \Subset \Omega$. A solution of the $\mathscr{K}_{\psi,f}$-obstacle problem is a superminimizer in Ω, but the converse is not true in general. However, if $u \in N^{1,p}(\Omega)$ and u is a superminimizer in Ω, then u is a solution of the $\mathscr{K}_{u,u}(\Omega)$-obstacle problem.

The following theorem is a generalization of Theorem 3.2 from Kinnunen–Martio [9], where existence and uniqueness was proved for the single obstacle problem.

Theorem 3.2. *Let $f \in N^{1,p}(\Omega)$ and $\psi_i \colon \Omega \to \overline{\mathbf{R}}$, $i = 1, 2$. If $\mathscr{K}_{\psi_1,\psi_2,f}$ is nonempty, then there is a unique solution (up to equivalence in $N^{1,p}(\Omega)$) of the $\mathscr{K}_{\psi_1,\psi_2,f}$-obstacle problem.*

Proof. Let

$$I = \inf_{v \in \mathscr{K}_{\psi_1,\psi_2,f}} \int_\Omega g_v^p \, d\mu.$$

Since $\mathscr{K}_{\psi_1,\psi_2,f} \ne \varnothing$, we have $0 \le I < \infty$. Let $\{u_j\}_{j=1}^\infty \subset \mathscr{K}_{\psi_1,\psi_2,f}$ be a minimizing sequence such that

$$\int_\Omega g_{u_j}^p \, d\mu \searrow I.$$

As $\|g_{u_j}\|^p_{L^p(\Omega)} \le \|g_{u_1}\|^p_{L^p(\Omega)}$, the sequence $\{g_{u_j}\}_{j=1}^\infty$ is bounded in $L^p(\Omega)$. Since Ω is bounded, $C_p(X \setminus \Omega) > 0$ and $u_j - f \in N_0^{1,p}(\Omega)$, it follows from Lemma 2.5 that

$$\int_\Omega |u_j - f|^p \, d\mu \le C \int_\Omega g_{u_j - f}^p \, d\mu \le C \int_\Omega g_{u_j}^p \, d\mu + C \int_\Omega g_f^p \, d\mu,$$

and

$$\begin{aligned} \|u_j\|_{N^{1,p}(\Omega)} &\le \|u_j\|_{L^p(\Omega)} + \left\|g_{u_j}\right\|_{L^p(\Omega)} \\ &\le \|u_j - f\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)} + \left\|g_{u_j}\right\|_{L^p(\Omega)} \\ &\le C\,\|f\|_{N^{1,p}(\Omega)} + C\,\|g_{u_1}\|_{L^p(\Omega)}. \end{aligned}$$

Hence $\{u_j\}_{j=1}^\infty$ is bounded in $N^{1,p}(\Omega)$. Using Lemma 2.6 we can find convex combinations $v_j = \sum_{k=j}^{N_j} a_{j,k} u_k$ with p-weak upper gradients $g_j = \sum_{k=j}^{N_j} a_{j,k} g_{u_k}$ and limit functions v, g such that $v_j \to v$ and $g_j \to g$ in $L^p(\Omega)$, $v_j \to v$ q.e. and g is a p-weak upper gradient of v. It follows that $v \in N^{1,p}(\Omega)$. Let

$$\begin{aligned} E_j &= \{x \in \Omega : v_j(x) < \psi_1(x) \text{ or } v_j(x) > \psi_2(x)\}, \quad j = 1, 2, \dots, \\ E &= \bigcup_{j=1}^\infty E_j. \end{aligned}$$

Since $\psi_1 \le v_j \le \psi_2$ q.e. in Ω, we have $C_p(E_j) = 0$ for all j. By the countable subadditivity of C_p, we get $C_p(E) = 0$ and $\psi_1 \le v \le \psi_2$ q.e. on the complement of E. Thus $\psi_1 \le v \le \psi_2$ q.e. in Ω.

Let further $w_j := v_j - f \in N_0^{1,p}(\Omega)$. We can consider w_j to be zero outside of Ω. Let also $w = v - f$, $g'_j = g_j + g_f$ and $g' = g + g_f$, where all three are considered to be identically zero outside of Ω. Then $w_j \to w$, $g'_j \to g'$ in $L^p(X)$ and $w_j \to w$ q.e. in X. By Lemma 2.6, g' is a p-weak upper gradient of w. Hence $w \in N^{1,p}(X)$. As $w = 0$ outside of Ω, we have $v - f \in N_0^{1,p}(\Omega)$, and thus $v \in \mathscr{K}_{\psi_1,\psi_2,f}(\Omega)$. Since

$$\begin{aligned} I &\le \int_\Omega g_v^p \, d\mu \le \int_\Omega g^p \, d\mu = \lim_{j\to\infty} \int_\Omega g_j^p \, d\mu \\ &\le \lim_{j\to\infty} \sum_{k=j}^{N_j} a_{j,k} \int_\Omega g_{u_k}^p \, d\mu \le \lim_{j\to\infty} \int_\Omega g_{u_j}^p \, d\mu = I, \end{aligned}$$

we conclude that v is a solution of the $\mathscr{K}_{\psi_1,\psi_2,f}$-obstacle problem.

For uniqueness assume that u_1 and u_2 are two solutions. Then

$$\int_\Omega g_{u_1}^p \, d\mu = \int_\Omega g_{u_2}^p \, d\mu$$

and $u' = \frac{1}{2}(u_1 + u_2) \in \mathscr{K}_{\psi_1,\psi_2,f}$. Since $g_{u'} \le \frac{1}{2}(g_{u_1} + g_{u_2})$, we have

$$\|g_{u_1}\|_{L^p(\Omega)} \le \|g_{u'}\|_{L^p(\Omega)} \le \tfrac{1}{2}\|g_{u_1}\|_{L^p(\Omega)} + \tfrac{1}{2}\|g_{u_2}\|_{L^p(\Omega)} \le \|g_{u_1}\|_{L^p(\Omega)}.$$

Hence $g_{u_1} = g_{u_2}$ a.e. in Ω by the strict convexity of $L^p(\Omega)$.

Let $c \in \mathbf{R}$, and

$$u = \max\{u_1, \min\{u_2, c\}\}.$$

Then $u \in N^{1,p}(\Omega)$, by the lattice property of $N^{1,p}(\Omega)$. Let

$$\begin{aligned} E &= \{x \in \Omega : u(x) < \psi_1(x) \text{ or } u(x) > \psi_2(x)\}\,, \\ E_i &= \{x \in \Omega : u_i(x) < \psi_1(x) \text{ or } u_i(x) > \psi_2(x)\}\,, \quad i = 1, 2, \\ A_1 &= \{x \in \Omega : u(x) = u_1(x)\}\,, \\ A_2 &= \{x \in \Omega : u(x) > u_1(x)\}\,. \end{aligned}$$

It is clear that $A_1 \cap E \subset E_1$ and hence $C_p(A_1 \cap E) = 0$. If $x \in A_2 \cap E$ then either we have $\psi_1(x) > u(x) > u_1(x)$ or $u_2(x) \geq u(x) > \psi_2(x)$. Thus $A_2 \cap E \subset E_1 \cup E_2$ and

$$C_p(E) \leq C_p(A_1 \cap E) + C_p(A_2 \cap E) = 0.$$

It follows that $\psi_1 \leq u \leq \psi_2$ q.e. in Ω. Also

$$u - f \leq \max\{u_1, u_2\} - f = \max\{u_1 - f, u_2 - f\} \in N_0^{1,p}(\Omega)$$

and $u - f \geq u_1 - f \in N_0^{1,p}(\Omega)$. Lemma 2.4 shows that $u - f \in N_0^{1,p}(\Omega)$ and hence $u \in \mathscr{K}_{\psi_1,\psi_2,f}$.

Let $V_c = \{x \in \Omega : u_1(x) < c < u_2(x)\}$, then $V_c \subset \{x \in \Omega : u(x) = c\}$ and hence $g_u = 0$ a.e. in V_c, by Lemma 2.2. On $\Omega \setminus V_c$ either we have $u_1 \geq c$ or $u_2 \leq c$. Thus, in the first case we get $u = u_1$ and Lemma 2.2 implies that $g_u = g_{u_1}$ a.e. In the second case we have $u = \max\{u_1, u_2\}$ and by Lemma 2.2 we obtain

$$g_u = g_{u_1} \chi_{\{u_1 > u_2\}} + g_{u_2} \chi_{\{u_2 \geq u_1\}} = g_{u_1},$$

since $g_{u_1} = g_{u_2}$. Thus $g_u = g_{u_1} = g_{u_2}$ a.e. in $\Omega \setminus V_c$. The minimizing property of g_{u_1} then implies that

$$\int_\Omega g_{u_1}^p \, d\mu \leq \int_\Omega g_u^p \, d\mu = \int_{\Omega \setminus V_c} g_u^p \, d\mu = \int_{\Omega \setminus V_c} g_{u_1}^p \, d\mu, \tag{3}$$

and we conclude that $g_{u_1} = g_{u_2} = 0$ a.e. in V_c for all $c \in \mathbf{R}$. Now

$$\{x \in \Omega : u_1(x) < u_2(x)\} \subset \bigcup_{c \in \mathbf{Q}} V_c$$

and hence $g_{u_1} = g_{u_2} = 0$ a.e. in $\{x \in \Omega : u_1(x) < u_2(x)\}$. Similarly, if we define $v = \max\{u_2, \min\{u_1, c\}\}$, we get $g_{u_1} = g_{u_2} = 0$ a.e. in the set $\{x \in \Omega : u_2(x) < u_1(x)\}$. It follows that

$$g_{u_1 - u_2} \leq (g_{u_1} + g_{u_2}) \chi_{\{x \in \Omega : u_1(x) \neq u_2(x)\}} = 0 \text{ a.e. in } \Omega.$$

By Lemma 2.5,

$$\|u_1 - u_2\|_{L^p(\Omega)}^p \leq C \int_\Omega g_{u_1 - u_2}^p \, d\mu = 0.$$

It follows that $u_1 = u_2$ a.e. in Ω and hence $u_1 = u_2$ q.e. in Ω. □

Remark 3.3. The solution of the double obstacle problem need not be locally bounded. However, one can easily see that, if the upper obstacle is essentially locally bounded from above and the lower obstacle is essentially locally bounded from below, then the solution of the obstacle problem is essentially locally bounded.

That u is *locally bounded* in Ω is defined by saying that for every $x \in \Omega$ there is r_x such that u is bounded in $B(x, r_x)$. This is however equivalent to saying that u is bounded in Ω' for every $\Omega' \Subset \Omega$. By saying that u is *essentially locally bounded* we allow for an exceptional set of measure zero.

The following lemma is a generalization of Lemma 5.4 in Björn–Björn [2], where they have $\psi_2 = \psi_2' \equiv \infty$.

Lemma 3.4. *Let $f, f' \in N^{1,p}(\Omega)$ and $\psi_j, \psi_j' : \Omega \to \overline{\mathbf{R}}$, $j = 1, 2$. Assume that $\psi_1 \le \psi_1'$ and $\psi_2 \le \psi_2'$ q.e. in Ω and that $(f - f')_+ \in N_0^{1,p}(\Omega)$. Let u be a solution of the $\mathscr{K}_{\psi_1,\psi_2,f}$-obstacle problem and u' be a solution of the $\mathscr{K}_{\psi_1',\psi_2',f'}$-obstacle problem. Then $u \le u'$ q.e. in Ω.*

Proof. Let $v = \min\{u, u'\}$ and $w = \max\{u, u'\}$. Let also

$$\begin{aligned} E_1 &= \{x \in \Omega : v(x) < \psi_1(x) \text{ or } v(x) > \psi_2(x)\}, \\ E_2 &= \{x \in \Omega : w(x) < \psi_1'(x) \text{ or } w(x) > \psi_2'(x)\}, \\ E &= \{x \in \Omega : u(x) < \psi_1(x) \text{ or } u(x) > \psi_2(x)\}, \\ E' &= \{x \in \Omega : u'(x) < \psi_1'(x) \text{ or } u'(x) > \psi_2'(x)\}, \\ A_1 &= \{x \in \Omega : v(x) = u(x)\}, \\ A_2 &= \Omega \setminus A_1 = \{x \in \Omega : v(x) < u(x)\}. \end{aligned}$$

Then it follows that $E_1 \cap A_1 \subset E$ and hence $C_p(E_1 \cap A_1) = 0$. Note also that for q.e. $x \in E_1 \cap A_2$ either $u'(x) = v(x) < \psi_1(x) \le \psi_1'(x)$ or $u(x) > v(x) > \psi_2(x)$, which implies that $E_1 \cap A_2 \subset E \cup E'$, and hence $C_p(E_1 \cap A_2) = 0$. Thus $C_p(E_1) = 0$ and $\psi_1 \le v \le \psi_2$ q.e. in Ω. Similarly we see that $C_p(E_2) = 0$ i.e. $\psi_1' \le w \le \psi_2'$ q.e. in Ω.

Let $h := u - f - (u' - f') \in N_0^{1,p}(\Omega)$. It follows that

$$h \ge \min\{f' - f, h\} \ge -(f' - f)_- - h_- = (f - f')_+ - h_- \in N_0^{1,p}(\Omega).$$

By Lemma 2.4 we have $\min\{f' - f, h\} \in N_0^{1,p}(\Omega)$ and thus

$$\begin{aligned} v - f &= \min\{u' - f, u - f\} = u' - f' + \min\{f' - f, h\} \in N_0^{1,p}(\Omega), \\ w - f' &= \max\{u' - f', u - f'\} = u - f + \max\{-h, f - f'\} \\ &= u - f - \min\{f' - f, h\} \in N_0^{1,p}(\Omega). \end{aligned}$$

Hence $v \in \mathscr{K}_{\psi_1,\psi_2,f}$ and $w \in \mathscr{K}_{\psi_1',\psi_2',f'}$. Since u' is a solution of the $\mathscr{K}_{\psi_1',\psi_2',f'}$-obstacle problem, we have

$$\int_\Omega g_{u'}^p \, d\mu \le \int_\Omega g_w^p \, d\mu = \int_{A_1} g_{u'}^p \, d\mu + \int_{A_2} g_u^p \, d\mu.$$

Thus

$$\int_{A_2} g_{u'}^p \, d\mu \le \int_{A_2} g_u^p \, d\mu,$$

which implies that

$$\int_\Omega g_v^p \, d\mu = \int_{A_1} g_u^p \, d\mu + \int_{A_2} g_{u'}^p \, d\mu \le \int_{A_1} g_u^p \, d\mu + \int_{A_2} g_u^p \, d\mu = \int_\Omega g_u^p \, d\mu.$$

Since u is a solution of the $\mathscr{K}_{\psi_1,\psi_2,f}$-obstacle problem, also v is a solution of the $\mathscr{K}_{\psi_1,\psi_2,f'}$-obstacle problem. By uniqueness, $u = v = \min\{u, u'\}$ q.e. in Ω, and thus $u \le u'$ q.e. in Ω. □

Theorem 3.5. *The solution of the $\mathscr{K}_{\psi_1,\psi_2,f}$-obstacle problem is a superminimizer if and only if it is a solution of the $\mathscr{K}_{\psi_1,f}$-obstacle problem.*

Proof. Let u be a solution of the $\mathscr{K}_{\psi_1,\psi_2,f}$-obstacle problem. If u is a solution of the $\mathscr{K}_{\psi_1,f}$-obstacle problem, then u is a superminimizer and one direction is proved. As for the other direction, assume that u is a superminmizer and let u' be a solution of the $\mathscr{K}_{\psi_1,f}$-obstacle problem, then the comparison Lemma 3.4 implies that $u \leq u'$ q.e. in Ω. Since u is a solution of the $\mathscr{K}_{u,u}$-obstacle problem another application of the comparison Lemma 3.4 shows that $u' \leq u$ q.e. in Ω and hence $u = u'$ q.e. in Ω. Thus u is a solution of the $\mathscr{K}_{\psi_1,f}$-obstacle problem. □

The following localization lemma is sometimes useful.

Lemma 3.6. *Let $\psi_i \colon \Omega \to \overline{\mathbf{R}}$, $i = 1, 2$, and $f \in N^{1,p}(\Omega)$. Let u be a solution of the $\mathscr{K}_{\psi_1,\psi_2,f}$-obstacle problem and let $\Omega' \subset \Omega$ be open. Then u is a solution of the $\mathscr{K}_{\psi_1,\psi_2,u}(\Omega')$-obstacle problem.*

Proof. Let $v \in \mathscr{K}_{\psi_1,\psi_2,u}(\Omega')$, then we have to show that

$$\int_{\Omega'} g_u^p \, d\mu \leq \int_{\Omega'} g_v^p \, d\mu.$$

Since $v - u \in N_0^{1,p}(\Omega') \subset N^{1,p}(\Omega)$ and $v = (v - u) + u \in N^{1,p}(\Omega)$ we can define $v(x) = u(x)$ when $x \in \Omega \setminus \Omega'$. It follows that $\psi_1 \leq v \leq \psi_2$ q.e. in Ω, since $\psi_1 \leq v \leq \psi_2$ q.e. in Ω' and $v = u$ in $\Omega \setminus \Omega'$. Also

$$v - f = (v - u) + (u - f) \in N_0^{1,p}(\Omega).$$

Thus, $v \in \mathscr{K}_{\psi_1,\psi_2,f}$ and using that u is a solution of the $\mathscr{K}_{\psi_1\psi_2,f}$-obstacle problem we get

$$\int_{\Omega} g_u^p \, d\mu \leq \int_{\Omega} g_v^p \, d\mu.$$

Lemma 2.2 implies $g_u = g_v$ a.e. in $\Omega \setminus \Omega'$ and we obtain

$$\int_{\Omega'} g_u^p \, d\mu \leq \int_{\Omega'} g_v^p \, d\mu.$$

Thus, u is a solution of the $\mathscr{K}_{\psi_1,\psi_2,u}(\Omega')$-obstacle problem. □

Proposition 3.7. *Let $\psi_j \colon \Omega \to \overline{\mathbf{R}}$, $j = 1, 2$, and $f \in N^{1,p}(\Omega)$. Let u be a solution of the $\mathscr{K}_{\psi_1,\psi_2,f}$-obstacle problem, $V \subseteq \Omega$ be open and $r \in \mathbf{R}$. Then*

(a) *If $\psi_2 \geq r$ q.e. in V, then $u_r = \min\{u, r\}$ is a superminimizer in V.*
(b) *If $\psi_1 \leq r$ q.e. in V, then $u^r = \max\{u, r\}$ is a subminimizer in V.*

Here a function w is a *subminimizer* if $-w$ is a superminimizer.

Proof. We shall prove (a) and using that $-u$ is a solution of the $\mathscr{K}_{-\psi_2,-\psi_1,-f}$-obstacle problem, we see that (b) will immediately follows.

Let $\Omega' \Subset V$, $v \in N^{1,p}(\Omega')$, $v \geq u_r$ and $v - u_r \in N_0^{1,p}(\Omega')$. To show that

$$\int_{\Omega'} g_{u_r}^p \, d\mu \leq \int_{\Omega'} g_v^p \, d\mu,$$

let $v_r = \min\{v, r\}$ and $\tilde{v} = \max\{v_r, u\}$, then $\tilde{v} \in N^{1,p}(\Omega')$. It follows from Lemma 2.2 that

$$g_{u_r} = \begin{cases} g_u & \text{a.e. on } \{x \in \Omega' : u(x) < r\}, \\ 0 & \text{a.e. on } \{x \in \Omega' : u(x) \geq r\}. \end{cases}$$

Thus, $g_{u_r} \le g_u$ a.e. in Ω' and similarly $g_{v_r} \le g_v$ a.e. in Ω'. Also

$$g_{\tilde{v}} = \begin{cases} g_{v_r} & \text{a.e. on } \{x \in \Omega' : v_r(x) \ge u(x)\} =: A, \\ g_u & \text{a.e. on } \{x \in \Omega' : v_r(x) < u(x)\}. \end{cases}$$

Furthermore,

$$\psi_1 \le u \le \tilde{v} \le \max\{r, u\} \le \psi_2 \quad \text{q.e. in } V$$

and

$$\begin{aligned} 0 \le \tilde{v} - u &\le \max\{v, u\} - u = \max\{v - u, 0\} \\ &\le \max\{v - u_r, 0\} = v - u_r \in N_0^{1,p}(\Omega'). \end{aligned}$$

By Lemma 2.4, $\tilde{v} - u \in N_0^{1,p}(\Omega')$, and hence $\tilde{v} \in \mathscr{K}_{\psi_1,\psi_2,u}(\Omega')$. Thus, using that u is a solution of the $\mathscr{K}_{\psi_1,\psi_2,u}(\Omega')$-obstacle problem, we obtain

$$\int_{\Omega'} g_u^p \, d\mu \le \int_{\Omega'} g_{\tilde{v}}^p \, d\mu = \int_A g_{v_r}^p \, d\mu + \int_{\Omega' \setminus A} g_u^p \, d\mu.$$

It follows that

$$\int_A g_u^p \, d\mu \le \int_A g_{v_r}^p \, d\mu,$$

and hence

$$\int_A g_{u_r}^p \, d\mu \le \int_A g_u^p \, d\mu \le \int_A g_{v_r}^p \, d\mu \le \int_A g_v^p \, d\mu. \tag{4}$$

Note also that for $x \in \Omega' \setminus A$ either we have $u(x) > v_r(x) = r$, which implies that $u_r(x) = r$, or $u(x) > v_r(x) = v(x)$, which also implies that $u_r(x) = r$, since otherwise we would get $u_r(x) = u(x) > v(x)$ a contradiction. Thus we conclude that $\Omega' \setminus A \subset \{x \in \Omega' : u_r(x) = r\}$ and that $g_{u_r} = 0$ a.e. on $\Omega' \setminus A$. Together with (4) this yield

$$\int_{\Omega'} g_{u_r}^p \, d\mu = \int_A g_{u_r}^p \, d\mu \le \int_A g_v^p \, d\mu \le \int_{\Omega'} g_v^p \, d\mu,$$

i.e. u_r is a superminimizer in V. □

From Theorem 3.5 and Proposition 3.7 we obtain the following immediate corollary.

Corollary 3.8. *Let $r \in \mathbf{R}$, $f \in N^{1,p}(\Omega)$ and $\psi\colon \Omega \to \mathbf{R}$. Assume that u is a solution of the $\mathscr{K}_{\psi,r,f}$-obstacle problem, then u is a superminimizer in Ω. Moreover u is a solution of the $\mathscr{K}_{\psi,f}$-obstacle problem.*

Next we prove that the solution of the double obstacle problem is continuous provided both obstacles are continuous. It generalizes Theorem 5.5 in Kinnunen–Martio [9], where a similar result was proved for the single obstacle problem $\mathscr{K}_{\psi,f}$.

Theorem 3.9. *Let $\psi_1\colon \Omega \to \mathbf{R}$ and $\psi_2\colon \Omega \to \mathbf{R}$. Assume that ψ_2 is continuous. Let also $f \in N^{1,p}(\Omega)$ and u be a solution of the $\mathscr{K}_{\psi_1,\psi_2,f}$-obstacle problem. Then the function $u^*\colon \Omega \to \mathbf{R}$ defined by*

$$u^*(x) = \operatorname*{ess\,lim\,inf}_{y \to x} u(y) = \lim_{r \to 0} \operatorname*{ess\,inf}_{B(x,r)} u$$

is lower semicontinuous in Ω, and belongs to the same equivalence class in $N^{1,p}(\Omega)$ as u. Moreover, if ψ_1 is continuous, then u^* is continuous in Ω.

Proof. Note first that u^* does not take the values $-\infty$ and ∞ which follows from Remark 3.3. Let $\alpha \in \mathbf{R}$, $A = \{x \in \Omega : u^*(x) > \alpha\}$ and $x_0 \in A$. Then we have

$$u^*(x_0) = \lim_{r\to 0} \operatorname*{ess\,inf}_{B(x_0,r)} u > \alpha,$$

hence there is $\delta > 0$ such that $\operatorname*{ess\,inf}_{B(x_0,\delta)} u > \alpha$. As for all $y \in B(x_0,\delta)$ there is $\delta_y > 0$ such that $B(y,\delta_y) \subset B(x_0,\delta)$, we have

$$u^*(y) = \operatorname*{ess\,lim\,inf}_{z\to y} u(z) \geq \operatorname*{ess\,inf}_{B(y,\delta_y)} u \geq \operatorname*{ess\,inf}_{B(x_0,\delta)} u > \alpha.$$

This shows that the set A is open and that u^* is lower semicontinuous in Ω.

To show that u^* and u belong to the same equivalence class in $N^{1,p}(\Omega)$, let $\varepsilon > 0$ and for every $x \in \Omega$ find a ball $B_x = B(x, r_x)$ such that

$$\sup_{B_x} \psi_2 \leq \inf_{B_x} \psi_2 + \varepsilon.$$

Clearly we can cover Ω by countably many such balls. Let further v be the lower semicontinuously regularized solution of the $\mathscr{K}_{\psi_1,u}(B_x)$-obstacle problem provided by Theorem 5.1 in Kinnunen–Martio [9]. Since u is a solution of the $\mathscr{K}_{\psi_1,\psi_2,u}(B_x)$-obstacle problem (by Lemma 3.6), the comparison Lemma 3.4 implies that

$$u \leq v \quad \text{q.e. in } B_x. \tag{5}$$

Next, as $\psi_1 \leq u \leq \psi_2 \leq \sup_{B_x}\psi_2 =: r$ q.e. in B_x, we have by the comparison Lemma 3.4 that $v \leq r$ q.e. in B_x. Thus v is a solution of the $\mathscr{K}_{\psi_1,r,u}(B_x)$-obstacle problem, which implies that $v-\varepsilon$ is a solution of the $\mathscr{K}_{\psi_1-\varepsilon,r-\varepsilon,u-\varepsilon}(B_x)$-obstacle problem. As $\psi_1 - \varepsilon \leq \psi_1$, $r - \varepsilon \leq \inf_{B_x}\psi_2 \leq \psi_2$ and $u-\varepsilon \leq u$ in B_x, another application of the comparison Lemma 3.4 implies that $v-\varepsilon \leq u$ q.e. in B_x. Together with (5) we get

$$v - \varepsilon \leq u \leq v \quad \text{q.e. in } B_x \tag{6}$$

and thus $v - \varepsilon = v^* - \varepsilon \leq u^* \leq v^* = v$ everywhere in B_x. This and (6) imply that

$$|u^* - u| \leq \varepsilon \quad \text{q.e. in } B_x. \tag{7}$$

Hence $|u^*-u| \leq \varepsilon$ q.e. in Ω, since for a given $\varepsilon > 0$ we can cover Ω by countably many balls satisfying (7). Letting $\varepsilon \to 0$ we obtain that $u^* = u$ q.e. in Ω.

Next we prove that u^* is continuous if ψ_1 is continuous. We already know that u^* is lower semicontinuous. To show that it is upper semicontinuous let $\varepsilon > 0$, $x \in \Omega$ and choose B_x as above. Let v be the continuous solution of the $\mathscr{K}_{\psi_1,u}(B_x)$-obstacle problem provided by Theorem 5.5 in Kinnunen–Martio [9]. It is shown above that

$$v(z) - \varepsilon \leq u^*(z) \leq v(z) \quad \text{for all } z \in B_x. \tag{8}$$

Thus using that v is continuous we obtain

$$v(z) - \varepsilon = \limsup_{y\to z} v(y) - \varepsilon \leq \limsup_{y\to z} u^*(y) \leq \limsup_{y\to z} v(y) = v(z)$$

for all $z \in B_x$.

This and (8) give $\left|\limsup_{y\to z} u^*(y) - u^*(z)\right| \leq \varepsilon$ for all $z \in B_x$ and hence

$$\left|\limsup_{y\to z} u^*(y) - u^*(z)\right| \leq \varepsilon \quad \text{for all } z \in \Omega.$$

Letting $\varepsilon \to 0$ we get that

$$\limsup_{y \to z} u^*(y) = u^*(z) \quad \text{for all } z \in \Omega.$$

This means that u^* is continuous in Ω. □

The next theorem shows that the continuous solution of the continuous double obstacle problem is a minimizer in the open set where the solution does not touch the two obstacles.

Theorem 3.10. *Let $\psi_i\colon \Omega \to \mathbf{R}$, $i = 1, 2$, be continuous and $f \in N^{1,p}(\Omega)$. Let u be the continuous solution of the $\mathscr{K}_{\psi_1,\psi_2,f}$-obstacle problem. Let also*

$$\Omega' = \{x \in \Omega : u(x) < \psi_2(x)\}.$$

Then u is a solution of the $\mathscr{K}_{\psi_1,u}(\Omega')$-obstacle problem. Moreover, u is a minimizer in the open set $\{x \in \Omega : \psi_1(x) < u(x) < \psi_2(x)\}$ (with boundary values u).

Proof. Let $v \in \mathscr{K}_{\psi_1,u}(\Omega')$ and note that $\min\{u, v\} \in \mathscr{K}_{\psi_1,\psi_2,u}(\Omega')$. Using that u is a solution of the $\mathscr{K}_{\psi_1,\psi_2,u}(\Omega')$-obstacle problem we get that

$$\int_{\Omega'} g_u^p \, d\mu \le \int_{\Omega'} g_{\min\{u,v\}}^p \, d\mu = \int_{\{u \le v\}} g_u^p \, d\mu + \int_{\{u > v\}} g_v^p \, d\mu.$$

It follows that

$$\int_{\{u>v\}} g_u^p \, d\mu \le \int_{\{u>v\}} g_v^p \, d\mu.$$

Note also that $\max\{u, v\} \in \mathscr{K}_{\psi_1,u}(\Omega')$. Lemma 2.2 and the above inequality then imply that

$$\int_{\Omega'} g_{\max\{u,v\}}^p \, d\mu = \int_{\{u>v\}} g_u^p \, d\mu + \int_{\{u \le v\}} g_v^p \, d\mu \le \int_{\Omega'} g_v^p \, d\mu.$$

Thus we conclude that it is enough to show that

$$\int_{\Omega'} g_u^p \, d\mu \le \int_{\Omega'} g_{\max\{u,v\}}^p \, d\mu.$$

As $\max\{u, v\} \ge u$ in Ω', we may assume without loss of generality that $v = \max\{u, v\} \ge u$ in Ω'.

Let $\varepsilon > 0$. Using that Lipschitz functions with compact support are dense in $N_0^{1,p}(\Omega')$ and that $0 \le v - u \in N_0^{1,p}(\Omega')$ we conclude that there is $0 \le \varphi \in \operatorname{Lip_c}(\Omega')$ such that $\|\varphi - (v - u)\|_{N^{1,p}(\Omega)} < \varepsilon$. Let $\tilde{v} = \varphi + u$, then we have

$$\left(\int_{\Omega'} g_{\tilde{v}}^p \, d\mu \right)^{1/p} \le \left(\int_{\Omega'} g_v^p \, d\mu \right)^{1/p} + \varepsilon.$$

As u and ψ_2 are continuous on the compact set $\operatorname{supp} \varphi$ and $u(x) < \psi_2(x)$ for every $x \in \operatorname{supp} \varphi$, we conclude that there is $\sigma > 0$ such that $u + \sigma \le \psi_2$ on $\operatorname{supp} \varphi$. Let $0 < t < 1$ be such that

$$t \max_{\Omega'} \varphi \le \sigma.$$

Then

$$\psi_1(x) \le w(x) := u(x) + t(\tilde{v}(x) - u(x)) = u(x) + t\varphi(x) \le \psi_2(x)$$

for every $x \in \Omega'$. Since $w - u = t\varphi \in N_0^{1,p}(\Omega')$ and $\psi_1 \le w \le \psi_2$ in Ω', we obtain that $w \in \mathscr{K}_{\psi_1,\psi_2,u}(\Omega')$. The convexity of the function $z \mapsto z^p$ and the fact that u is a solution of the $\mathscr{K}_{\psi_1,\psi_2,u}(\Omega')$-obstacle problem imply that

$$\begin{aligned}\int_{\Omega'} g_u^p \, d\mu \le \int_{\Omega'} g_w^p \, d\mu &= \int_{\Omega'} g_{u+t(\tilde{v}-u)}^p \, d\mu \\ &\le \int_{\Omega'} ((1-t)g_u + t g_{\tilde{v}})^p \, d\mu \le (1-t)\int_{\Omega'} g_u^p \, d\mu + t \int_{\Omega'} g_{\tilde{v}}^p \, d\mu.\end{aligned}$$

This implies that

$$t\int_{\Omega'} g_u^p \, d\mu \le t \int_{\Omega'} g_{\tilde{v}}^p \, d\mu,$$

and hence

$$\int_{\Omega'} g_u^p \, d\mu \le \int_{\Omega'} g_{\tilde{v}}^p \, d\mu \le \left[\left(\int_{\Omega'} g_v^p \, d\mu\right)^{1/p} + \varepsilon\right]^p.$$

Since $\varepsilon > 0$ was arbitrary we obtain that

$$\int_{\Omega'} g_u^p \, d\mu \le \int_{\Omega'} g_v^p \, d\mu$$

and hence u is a solution of the $\mathscr{K}_{\psi_1,u}(\Omega')$-obstacle problem.

Next, since $\{x \in \Omega' : u(x) > \psi_1(x)\} = \{x \in \Omega : \psi_1(x) < u(x) < \psi_2(x)\}$, it follows from Theorem 5.5 in Kinnunen–Martio [9] that u is a minimizer in the open set $\{x \in \Omega : \psi_1(x) < u(x) < \psi_2(x)\}$. □

4. Boundary regularity

Definition 4.1. Let V be a bounded open set with $C_p(X \setminus V) > 0$ and $f \in N^{1,p}(V)$. The *p-harmonic extension* $H_V f$ of f to V is the continuous solution of the $\mathscr{K}_{-\infty,f}$-obstacle problem. We write $Hf = H_\Omega f$.

A Lipschitz function f on $\partial\Omega$ can be extended to a function $\tilde{f} \in \operatorname{Lip}(\overline{\Omega})$ such that $\tilde{f} = f$ on $\partial\Omega$. As $H\tilde{f}$ only depends on $\tilde{f}|_{\partial\Omega} = f$ (by the comparison Lemma 3.4), we define $Hf = H\tilde{f}$.

Definition 4.2. A point $x \in \partial\Omega$ is *regular* if

$$\lim_{\Omega \ni y \to x} Hf(y) = f(x) \quad \text{for all } f \in \operatorname{Lip}(\partial\Omega).$$

If $x \in \partial\Omega$ is not regular, it is *irregular*. We also say that Ω is *regular* if every $x \in \partial\Omega$ is regular.

Regularity can be characterized in many different ways, see Björn–Björn [2], Theorem 6.1.

For $A, \Omega \subset X$ we introduce the space of Newtonian functions with zero boundary values in $A \setminus \Omega$ as follows

$$N_0^{1,p}(\Omega; A) = \{f|_{\Omega\cap A} : f \in N^{1,p}(A) \text{ and } f = 0 \text{ q.e. in } A \setminus \Omega\}.$$

One can see that $N_0^{1,p}(\Omega; A) = N_0^{1,p}(\Omega; A \cap \overline{\Omega})$.

Definition 4.3. For $A \subset X$ and $f\colon A \to \overline{\mathbf{R}}$, let

$$C_p\text{-}\sup_A f = \inf\{k \in \mathbf{R} : C_p(\{x \in A : f(x) > k\}) = 0\},$$
$$C_p\text{-}\inf_A f = \sup\{k \in \mathbf{R} : C_p(\{x \in A : f(x) < k\}) = 0\},$$
$$C_p\text{-}\limsup_{y\to x} f(y) = \lim_{r\to 0} C_p\text{-}\sup_{B(x,r)} f,$$
$$C_p\text{-}\liminf_{y\to x} f(y) = \lim_{r\to 0} C_p\text{-}\inf_{B(x,r)} f.$$

The following theorem is a generalization of Theorem 5.6 from Björn–Björn [2] where it was proved for the single obstacle problem, i.e. for $\psi_2 \equiv \infty$ and $m = m'$.

Theorem 4.4. *Let $\psi_i\colon \Omega \to \mathbf{R}$, $i = 1, 2$, and $f \in N^{1,p}(\Omega)$. Let u be a solution of the $\mathcal{K}_{\psi_1,\psi_2,f}$-obstacle problem. Let $x_0 \in \partial\Omega$ be a regular boundary point. Let*

$$m'(f) = \sup\{k \in \mathbf{R} : (f-k)_- \in N_0^{1,p}(\Omega; B(x_0,r)) \text{ for some } r > 0\},$$
$$M'(f) = \inf\{k \in \mathbf{R} : (f-k)_+ \in N_0^{1,p}(\Omega; B(x_0,r)) \text{ for some } r > 0\},$$
$$m = m(f;\psi_2) = \min\Big\{m'(f), C_p\text{-}\liminf_{\Omega\ni y\to x_0} \psi_2(y)\Big\},$$
$$M = M(f;\psi_1) = \max\Big\{M'(f), C_p\text{-}\limsup_{\Omega\ni y\to x_0} \psi_1(y)\Big\}.$$

Then

$$m \le C_p\text{-}\liminf_{\Omega\ni y\to x_0} u(y) \le C_p\text{-}\limsup_{\Omega\ni y\to x_0} u(y) \le M.$$

Proof. Let v be the lower semicontinuous regularized solution of the $\mathcal{K}_{\psi_1,f}$-obstacle problem, then by the comparison Lemma 3.4, $u \le v$ q.e. in Ω and thus

$$C_p\text{-}\limsup_{\Omega\ni y\to x_0} u(y) \le C_p\text{-}\limsup_{\Omega\ni y\to x_0} v(y) \le \limsup_{\Omega\ni y\to x_0} v(y).$$

On the other hand we have $\limsup_{\Omega\ni y\to x_0} v(y) \le M$, by Theorem 5.6 in Björn–Björn [2]. Hence we obtain

$$C_p\text{-}\limsup_{\Omega\ni y\to x_0} u(y) \le M, \tag{9}$$

which shows one inequality of the theorem.

To prove the other inequality, note first that $-u$ is a solution of the $\mathcal{K}_{-\psi_2,-\psi_1,-f}$-obstacle problem and that

$$\begin{aligned} M(-f;-\psi_2) &= \max\Big\{M'(-f), C_p\text{-}\limsup_{\Omega\ni y\to x_0}(-\psi_2(y))\Big\} \\ &= \max\Big\{-m'(f), -C_p\text{-}\liminf_{\Omega\ni y\to x_0} \psi_2(y)\Big\} \\ &= -\min\Big\{m'(f), C_p\text{-}\liminf_{\Omega\ni y\to x_0} \psi_2(y)\Big\} \\ &= -m(f;\psi_2). \end{aligned}$$

This and (9) applied to $-u$ imply that

$$-C_p\text{-}\liminf_{\Omega\ni y\to x_0} u(y) = C_p\text{-}\limsup_{\Omega\ni y\to x_0}(-u(y)) \le M(-f;-\psi_2) = -m.$$

Hence

$$m \le C_p\text{-}\liminf_{\Omega\ni y\to x_0} u(y) \le C_p\text{-}\limsup_{\Omega\ni y\to x_0} u(y) \le M,$$

which finishes the proof. □

Theorem 4.5. *Let $\psi_i\colon \Omega \to \overline{\mathbf{R}}$, $i=1,2$, and $f \in N^{1,p}(\Omega)$. Let u be a solution of the $\mathscr{K}_{\psi_1,\psi_2,f}$-obstacle problem and $x_0 \in \partial\Omega$ be a regular boundary point. Assume further that either*

(a) *$f(x_0) := \lim_{\Omega\ni y\to x_0} f(y)$ exists, or*

(b) *$f \in N^{1,p}(\overline{\Omega}\cap B)$ for some ball B centered at x_0, and that $f|_{\partial\Omega}$ is continuous at x_0.*

Then

$$C_p\text{-}\lim_{\Omega\ni y\to x_0} u(y) = f(x_0)$$

if and only if

$$C_p\text{-}\limsup_{\Omega\ni y\to x_0} \psi_1(y) \le f(x_0) \le C_p\text{-}\liminf_{\Omega\ni y\to x_0} \psi_2(y). \tag{10}$$

Note that it is possible to have a soluble obstacle problem without (10), see Example 5.7 in Björn–Björn [2].

Proof. Assume first that (10) holds, and let m and M be as in Theorem 4.4. Let further $\varepsilon > 0$ and $B' = B(x_0,r) \subset B$ be such that

$$|f(x)-f(x_0)| < \varepsilon \quad \text{for} \begin{cases} x \in B'\cap\Omega & \text{in case (a)}, \\ x\in B'\cap\partial\Omega & \text{in case (b)}. \end{cases}$$

Then $(f-(f(x_0)-\varepsilon))_+ \in N_0^{1,p}(\Omega;B')$ and hence $M' \le f(x_0)+\varepsilon$. By assumption we have $C_p\text{-}\limsup_{\Omega\ni y\to x_0} \psi_1(y) \le f(x_0) \le f(x_0)+\varepsilon$ and thus $M \le f(x_0)+\varepsilon$ and letting $\varepsilon\to 0$ shows that $M \le f(x_0)$. Similarly as $(f-(f(x_0)-\varepsilon))_- \in N_0^{1,p}(\Omega;B')$ we conclude that $m' \ge f(x_0)-\varepsilon$. It follows that $m \ge f(x_0)-\varepsilon$ and by letting $\varepsilon \to 0$ we get $m \ge f(x_0)$. By Theorem 4.4 we obtain

$$m \le C_p\text{-}\liminf_{\Omega\ni y\to x_0} u(y) \le C_p\text{-}\limsup_{\Omega\ni y\to x_0} u(y) \le M \le f(x_0) \le m$$

and hence

$$C_p\text{-}\lim_{\Omega\ni y\to x_0} u(y) = f(x_0).$$

Conversely assume that $f(x_0) < C_p\text{-}\limsup_{\Omega\ni y\to x_0} \psi_1(y)$. As $u \ge \psi_1$ q.e. in Ω we obtain

$$f(x_0) < C_p\text{-}\limsup_{\Omega\ni y\to x_0} \psi_1(y) \le C_p\text{-}\limsup_{\Omega\ni y\to x_0} u(y).$$

Similarly, it follows that $f(x_0) > C_p\text{-}\liminf_{\Omega\ni y\to x_0} u(y)$, when $f(x_0) > C_p\text{-}\liminf_{\Omega\ni y\to x_0} \psi_2(y)$. Hence $f(x_0) \ne C_p\text{-}\lim_{\Omega\ni y\to x_0} u(y)$. □

Corollary 4.6. *Let $\psi_1\colon\Omega\to[-\infty,\infty)$ and $\psi_2\colon\Omega\to(-\infty,\infty]$ be continuous and $f\in N^{1,p}(\overline{\Omega})\cap C(\partial\Omega)$. Let Ω be regular and such that for every $x\in\partial\Omega$ we have*

$$\limsup_{\Omega\ni y\to x} \psi_1(y) \le f(x) \le \liminf_{\Omega\ni y\to x} \psi_2(y).$$

Let u be the continuous solution of the $\mathscr{K}_{\psi_1,\psi_2,f}$-obstacle problem given by Theorem 3.9. If we let $u = f$ on $\partial\Omega$, then $u \in C(\overline{\Omega})$.

In the following theorem (d) and (e) are new characterizations to regularity and add to the characterizations in Björn–Björn [2],Theorems 4.2 and 6.1.

Theorem 4.7. *Let $x_0 \in \partial\Omega$, $\delta > 0$ and $B = B(x_0, \delta)$. Then the following conditions are equivalent:*

(a) *The point x_0 is a regular boundary point.*

(b) *It is true that*
$$\lim_{\Omega \ni y \to x_0} Hf(y) = f(x_0)$$
for all $f \in N^{1,p}(\Omega)$ such that $f(x_0) := \lim_{\Omega \ni y \to x_0} f(y)$ exists.

(c) *It is true that*
$$\lim_{\Omega \ni y \to x_0} Hf(y) = f(x_0)$$
for all $f \in N^{1,p}(\Omega \cup (B \cap \overline{\Omega}))$ such that $f|_{\partial\Omega}$ is continuous at x_0.

(d) *For all $f \in N^{1,p}(\Omega)$ and all $\psi_1, \psi_2 \colon \Omega \to \overline{\mathbf{R}}$ such that $\mathscr{K}_{\psi_1,\psi_2,f} \neq \varnothing$,*
$$C_p\text{-}\limsup_{\Omega \ni y \to x_0} \psi_1(y) \leq f(x_0) \leq C_p\text{-}\liminf_{\Omega \ni y \to x_0} \psi_2(y)$$
and $f(x_0) := \lim_{\Omega \ni y \to x_0} f(y)$, any solution of the $\mathscr{K}_{\psi_1,\psi_2,f}$-obstacle problem satisfies
$$C_p\text{-}\lim_{\Omega \ni y \to x_0} u(y) = f(x_0).$$

(e) *For all $f \in N^{1,p}(\Omega \cup (B \cap \overline{\Omega}))$ such that $f|_{\partial\Omega}$ is continuous at x_0 and all $\psi_1, \psi_2 \colon \Omega \to \overline{\mathbf{R}}$ such that $\mathscr{K}_{\psi_1,\psi_2,f} \neq \varnothing$ and*
$$C_p\text{-}\limsup_{\Omega \ni y \to x_0} \psi_1(y) \leq f(x_0) \leq C_p\text{-}\liminf_{\Omega \ni y \to x_0} \psi_2(y),$$
any solution u of the $\mathscr{K}_{\psi_1,\psi_2,f}$-obstacle problem satisfies
$$C_p\text{-}\lim_{\Omega \ni y \to x_0} u(y) = f(x_0).$$

Proof. (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) These are Theorems 4.2 and 6.1 in Björn–Björn [2].
(a) $\Rightarrow$ (d) and (a) $\Rightarrow$ (e) This follows from Theorem 4.5.
(d) $\Rightarrow$ (b) and (e) $\Rightarrow$ (c) This is trivial as Hf is the continuous solution of the $\mathscr{K}_{-\infty,\infty,f}$-obstacle problem. □

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