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HARDY TYPE INEQUALITY IN VARIABLE LEBESGUE SPACES

Humberto Rafeiro and Stefan Samko

Universidade do Algarve, Departamento de Matemática
Campus de Gambelas, 8005-139 Faro, Portugal; hrafeiro@ualg.pt

Universidade do Algarve, Departamento de Matemática
Campus de Gambelas, 8005-139 Faro, Portugal; ssamko@ualg.pt

Abstract. We prove that in variable exponent spaces $L^{p(\cdot)}(\Omega)$, where $p(\cdot)$ satisfies the log-condition and Ω is a bounded domain in $\mathbf{R}^n$ with the property that $\mathbf{R}^n\backslash\overline{\Omega}$ has the cone property, the validity of the Hardy type inequality

$$\left\|\frac{1}{\delta(x)^\alpha}\int_\Omega \frac{\varphi(y)}{|x-y|^{n-\alpha}}\,dy\right\|_{p(\cdot)} \leqq C\|\varphi\|_{p(\cdot)}, \quad 0<\alpha<\min\left(1,\frac{n}{p_+}\right),$$

where $\delta(x)=\operatorname{dist}(x,\partial\Omega)$, is equivalent to a certain property of the domain Ω expressed in terms of α and χ_Ω.

1. Introduction

We consider the Hardy inequality of the form

$$\left\|\frac{1}{\delta(x)^\alpha}\int_\Omega \frac{\varphi(y)}{|x-y|^{n-\alpha}}\,dy\right\|_{p(\cdot)} \leqq C\|\varphi\|_{p(\cdot)}, \quad 0<\alpha<\min\left(1,\frac{n}{p_+}\right), \tag{1}$$

within the frameworks of Lebesgue spaces with variable exponents $p(x)$, $p_+=\sup\limits_{x\in\Omega} p(x)$, where $\delta(x)=\operatorname{dist}(x,\partial\Omega)$. We refer to [8, 9, 18] for Hardy type inequalities. The multidimensional Hardy inequality of the form

$$\int_\Omega |u(x)|^p\delta(x)^{-p+a}\,dx \leqq C\int_\Omega |\nabla u(x)|^p\delta(x)^a\,dx, \quad u\in C_0^1(\Omega), \tag{2}$$

appeared in [23] for bounded domains $\Omega\subset\mathbf{R}^n$ with Lipschitz boundary and $1<p<\infty$ and $a>p-1$. This inequality was generalized by Kufner [17, Theorem 8.4] to domains with Hölder boundary, and after that by Wannebo [40] to domains with generalized Hölder condition. Hajłasz [10] and Kinnunen and Martio [12] obtained a pointwise inequality

$$|u(x)|\leqq \delta(x)\mathscr{M}|\nabla u|(x),$$

where $\mathscr{M}$ is a kind of maximal function depending on the distance of x to the boundary. This pointwise inequality combined with the knowledge of boundedness of Hardy–Littlewood maximal operator implies a "local version near the boundary"

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of Hardy's inequality. This approach was used in the paper of Hajłasz [10] in the case of classical Lebesgue spaces.

Within the frameworks of variable exponent Lebesgue spaces, the Hardy inequality in one variable was first obtained in [15], and later generalized in [7], where the necessary and sufficient conditions for the validity of the Hardy inequality on $(0,\infty)$ were obtained under the assumption that the log-condition on $p(x)$ is satisfied only at the points $x=0$ and $x=\infty$, see also [19, 20].

For the multidimensional versions of Hardy inequality of form (1) with $\delta(x)^\alpha$ replaced by $|x-x_0|^\alpha, x_0\in\overline{\Omega}$, we refer to [32, 33]. Harjulehto, Hästö and Koskenoja in [11] obtained the estimate

$$\left\|\frac{u(x)}{\delta(x)^{1-a}}\right\|_{p(\cdot)} \leqq C\|\nabla u(x)\delta(x)^a\|_{p(\cdot)}, \quad u\in W_0^{1,p(\cdot)}(\Omega),$$

making use of the approach of [10], under the assumption that a is sufficiently small, $0\leqq a<a_0$.

Basing on some ideas and results of fractional calculus, in Theorem 12 we show that the problem of the validity of inequality (1) is equivalent to a certain property of Ω expressed in terms of α and χ_Ω, see Definition 9 and Theorem 12. We did not find mentioning such an equivalence in the literature even in the case of constant p.

Note that the continuing interest to the variable exponent Lebesgue spaces $L^{p(\cdot)}$ observed last years was caused by possible applications (elasticity theory, fluid mechanics, differential equations, see for example [29]). We refer to papers [16, 35] for basics on the Lebesgue spaces with variable exponents and to the surveys [6, 13, 34] on harmonic analysis in such spaces. One of the breakthrough results obtained for variable $p(x)$ was the statement on the boundedness of the Hardy–Littlewood maximal operator in the generalized Lebesgue space $L^{p(\cdot)}$ under certain conditions on $p(x)$, see [3] and the further development in the above survey papers. The importance of the boundedness of the maximal operator is known in particular due to the fact that many convolution operators occurred in applications may be dominated by the maximal operator, which is also used in this paper.

Note also that the study of pointwise multipliers in the spaces of Riesz potentials is in fact an open question in case of variable $p(x)$. Meanwhile, the topic of pointwise multipliers (in particular, in the case of characteristic functions χ_Ω) in spaces of differentiable functions, is of importance in the theory of partial differential equations and other applications, see for instance [28].

The study of pointwise multipliers of spaces of Riesz or Bessel potentials in the case of constant p may be found in [21, 22, 36], see also [28] for the pointwise multipliers in the case of more general spaces. We refer also, in the case of constant p as well, to recent papers [37, 38] on the study of characteristic functions $\chi_\Omega(x)$ as pointwise multipliers.

2. Preliminaries

2.1. On Lebesgue spaces with variable exponent. The basics on variable Lebesgue spaces may be found in [16, 30], but we recall here some necessary definitions. Let $\Omega\subset\mathbf{R}^n$ be an open set. For a measurable function $p\colon\Omega\to[1,\infty)$, we put

$$p_+ = p_+(\Omega) := \operatorname*{ess\,sup}_{x\in\Omega} p(x) \quad \text{and} \quad p_- = p_-(\Omega) := \operatorname*{ess\,inf}_{x\in\Omega} p(x).$$

In the sequel we use the notation

$$\mathscr{P}(\Omega) := \{p \in L^\infty(\Omega) : 1 < p_- \leqq p(x) \leqq p_+ < \infty\}. \tag{3}$$

The generalised Lebesgue space $L^{p(\cdot)}(\Omega)$ with variable exponent is introduced as the set of all functions φ on Ω for which

$$\varrho_{p(\cdot)}(\varphi) := \int_\Omega |\varphi(x)|^{p(x)}\, dx < \infty.$$

Equipped with the norm

$$\|\varphi\|_{L^{p(\cdot)}(\Omega)} := \inf\left\{\lambda > 0 : \varrho_{p(\cdot)}\left(\frac{\varphi}{\lambda}\right) \leqq 1\right\},$$

this is a Banach space. The modular $\varrho_{p(\cdot)}(f)$ and the norm $\|f\|_{p(\cdot)}$ are related to each other by

$$\|f\|^\sigma_{p(\cdot)} \leqq \varrho_{p(\cdot)}(f) \leqq \|f\|^\theta_{p(\cdot)}, \tag{4}$$

where $\sigma = \begin{cases} \operatorname*{ess\,inf}_{x\in\Omega} p(x), & \|f\|_{p(\cdot)} \geqq 1, \\ \operatorname*{ess\,sup}_{x\in\Omega} p(x), & \|f\|_{p(\cdot)} \leqq 1 \end{cases}$ and $\theta = \begin{cases} \operatorname*{ess\,inf}_{x\in\Omega} p(x), & \|f\|_{p(\cdot)} \leqq 1, \\ \operatorname*{ess\,sup}_{x\in\Omega} p(x), & \|f\|_{p(\cdot)} \geqq 1. \end{cases}$

By w-Lip(Ω) we denote the class of all exponents $p \in L^\infty(\Omega)$ satisfying the (local) logarithmic condition

$$|p(x) - p(y)| \leqq \frac{C}{-\ln|x-y|}, \quad |x-y| \leqq \frac{1}{2}, \quad x, y \in \Omega. \tag{5}$$

By $p'(\cdot)$ we denote the conjugate exponent, given by $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} \equiv 1$.

2.2. Hardy–Littlewood maximal operator. As usual, the Hardy–Littlewood maximal operator of a function φ on $\Omega \subseteq \mathbf{R}^n$ is defined as

$$\mathscr{M}\varphi(x) = \sup_{r>0} \frac{1}{|\tilde{B}(x,r)|} \int_{\tilde{B}(x,r)} |\varphi(y)|\, dy, \quad \tilde{B}(x,r) = B(x,r) \cap \Omega. \tag{6}$$

We use the notation

$$\mathbf{P}(\Omega) := \left\{p : 1 < p_- \leqq p_+ \leqq \infty,\ \|\mathscr{M}f\|_{L^{p(\cdot)}(\Omega)} \leqq C\|f\|_{L^{p(\cdot)}(\Omega)}\right\}. \tag{7}$$

Proposition 1. [3, Theorem 3.5] *If Ω is bounded, $p \in \mathscr{P}(\Omega) \cap w$-Lip$(\Omega)$, then $p \in \mathbf{P}(\Omega)$.*

2.3. Potential and hypersingular integral operators.

Definition 2. For a function φ on $\mathbf{R}^n$, the Riesz potential operator I^α is defined by

$$I^\alpha\varphi(x) = \frac{1}{\gamma_n(\alpha)} \int\limits_{\mathbf{R}^n} \frac{\varphi(y)\, dy}{|x-y|^{n-\alpha}} = \varphi * k_\alpha(x), \tag{8}$$

where the normalizing constant factor has the form $\gamma_n(\alpha) = \frac{2^\alpha \pi^{n/2}\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)}$. The kernel $k_\alpha(x) = \frac{|x|^{\alpha-n}}{\gamma_n(\alpha)}$ is referred to as the Riesz kernel.

Definition 3. The space $I^\alpha\left(L^{p(\cdot)}\right) = I^\alpha\left(L^{p(\cdot)}(\mathbf{R}^n)\right)$, $0 < \alpha < \frac{n}{p_+}$, called the space of Riesz potentials, is the space of functions f representable as $f = I^\alpha\varphi$ with $\varphi \in L^{p(\cdot)}$, equipped with the norm $\|f\|_{I^\alpha(L^{p(\cdot)})} = \|\varphi\|_{L^{p(\cdot)}}$.

Definition 4. The hypersingular integral operator $\mathbf{D}^\alpha$ of order α, known also as the Riesz derivative, is defined by

$$\mathbf{D}^\alpha f = \lim_{\varepsilon\to 0} \mathbf{D}^\alpha_\varepsilon f = \lim_{\varepsilon\to 0} \frac{1}{d_{n,\ell}(\alpha)} \int\limits_{|y|>\varepsilon} \frac{\left(\Delta^\ell_y f\right)(x)}{|y|^{n+\alpha}}\, dy, \tag{9}$$

where $\alpha > 0$ and $\ell > \alpha$ (see [31, p. 60], for the value of the normalizing constant $d_{n,\ell}(\alpha)$).

It is known that given α, one may choose an arbitrary order $\ell > \alpha$ of the finite difference; the hypersingular integral does not depend on ℓ under this choice, see [31, Chapter 3].

In [1], the following statement was proved.

Proposition 5. *Let $p \in \mathscr{P}(\mathbf{R}^n) \cap \mathbf{P}(\mathbf{R}^n)$ and $0 < \alpha < \frac{n}{p_+}$. Then*

$$\mathbf{D}^\alpha I^\alpha \varphi = \varphi, \quad \varphi \in L^{p(\cdot)}(\mathbf{R}^n),$$

where the hypersingular operator $\mathbf{D}^\alpha$ is taken in the sense of convergence of $L^{p(\cdot)}$-norm.

The characterization of the space $I^\alpha(L^{p(\cdot)}(\mathbf{R}^n))$ is given by the following proposition.

Proposition 6. [2, Theorem 3.2] *Let $0 < \alpha < n$, $p \in \mathscr{P}(\mathbf{R}^n) \cap \mathbf{P}(\mathbf{R}^n)$, $p_+ < \frac{n}{\alpha}$ and let f be a locally integrable function. Then $f \in I^\alpha\left(L^{p(\cdot)}\right)$ if and only if $f \in L^{q(\cdot)}$ with $\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \frac{\alpha}{n}$, and there exists the Riesz derivative $\mathbf{D}^\alpha f$ in the sense of convergence in $L^{p(\cdot)}$.*

Remark 7. Theorem 3.2 in [2] was stated under the assumption that $p(x)$ satisfies the local log-condition and the decay condition at infinity. The analysis of the proof of Theorem 3.2 shows that it is valid under the general assumption $p \in \mathscr{P}(\mathbf{R}^n) \cap \mathbf{P}(\mathbf{R}^n)$ (if one takes into account that $p \in \mathscr{P} \cap \mathbf{P}(\mathbf{R}^n) \Leftrightarrow p' \in \mathscr{P} \cap \mathbf{P}(\mathbf{R}^n)$, see [5, Theorem 8.1]).

By Propositions 5 and 6, for the norm $\|f\|_{I^\alpha(L^{p(\cdot)})} = \|\varphi\|_{L^{p(\cdot)}}$ in the space of Riesz potentials $I^\alpha\left(L^{p(\cdot)}(\mathbf{R}^n)\right)$ we have the following equivalence

$$c_1\left(\|f\|_{L^{q(\cdot)}} + \|\mathbf{D}^\alpha f\|_{L^{p(\cdot)}}\right) \leqq \|f\|_{I^\alpha\left(L^{p(\cdot)}\right)} \leqq c_2\left(\|f\|_{L^{q(\cdot)}} + \|\mathbf{D}^\alpha f\|_{L^{p(\cdot)}}\right), \tag{10}$$

where $\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \frac{\alpha}{n}$ and $c_1 > 0$, $c_2 > 0$ do not depend on f.

2.4. $(\alpha, p(\cdot))$-property of a domain Ω.

Definition 8. A measurable function $g(x)$ is called a pointwise multiplier in the space $I^\alpha\left(L^{p(\cdot)}(\mathbf{R}^n)\right)$, if $\|gI^\alpha\varphi\|_{I^\alpha\left(L^{p(\cdot)}\right)} \leqq C\|\varphi\|_{L^{p(\cdot)}}$.

By equivalence (10), in the case $1 < p_+ < \frac{n}{\alpha}$ the characteristic function $\chi_\Omega(x)$ is a pointwise multiplier in $I^\alpha\left(L^{p(\cdot)}(\mathbf{R}^n)\right)$ if and only if

$$\|\mathbf{D}^\alpha(\chi_\Omega I^\alpha\varphi)\|_{L^{p(\cdot)}(\mathbf{R}^n)} \leqq C\|\varphi\|_{L^{p(\cdot)}(\mathbf{R}^n)} \quad \text{for all} \quad \varphi \in L^{p(\cdot)}(\mathbf{R}^n). \tag{11}$$

We introduce now the following notion related to the property of the characteristic function χ_Ω to be a pointwise multiplier, but weaker than that property. Let $\mathscr{E}_\Omega f(x) = \widetilde{f}(x) = \begin{cases} f(x), & x \in \Omega \\ 0, & x \notin \Omega \end{cases}$ be the zero extension of a function f defined on Ω.

Definition 9. We say that the domain Ω has the $(\alpha, p(\cdot))$-property, if the function $\chi_\Omega(x)$ has the following multiplier property

$$\|\mathbf{D}^\alpha(\chi_\Omega I^\alpha \mathscr{E}_\Omega \varphi)\|_{L^{p(\cdot)}(\Omega)} \leqq C\|\varphi\|_{L^{p(\cdot)}(\Omega)} \quad \text{for all} \quad \varphi \in L^{p(\cdot)}(\Omega). \tag{12}$$

Definition 10. Let $p \in \mathscr{P}(\Omega) \cap w\text{-Lip}(\Omega)$. For brevity we call an extension $p^*(x)$ of $p(x)$ to $\mathbf{R}^n$ regular, if $p^* \in \mathscr{P}(\mathbf{R}^n) \cap \mathbf{P}(\mathbf{R}^n)$, and $p_+(\mathbf{R}^n) = p_+(\Omega)$. Such an extension is always possible, see [4, Theorem 4.2], [26, Lemma 2.2].

Lemma 11. *Let $p \in \mathscr{P}(\Omega) \cap w\text{-Lip}(\Omega)$. If χ_Ω is a pointwise multiplier in the space $I^\alpha\left(L^{p^*(\cdot)}(\mathbf{R}^n)\right)$ under any regular extension $p^*(x)$ of $p(x)$ to $\mathbf{R}^n$, then the domain Ω has the $(\alpha, p(\cdot))$-property.*

Proof. We have to check condition (12), given that $\|\chi_\Omega f\|_{I^\alpha\left(L^{p^*(\cdot)}(\mathbf{R}^n)\right)} \leqq C \cdot \|f\|_{I^\alpha\left(L^{p^*(\cdot)}(\mathbf{R}^n)\right)}$ under some regular extension of the exponent. We have

$$\|\mathbf{D}^\alpha(\chi_\Omega I^\alpha \mathscr{E}_\Omega \varphi)\|_{L^{p(\cdot)}(\Omega)} \leqq \|\mathbf{D}^\alpha(\chi_\Omega I^\alpha \mathscr{E}_\Omega \varphi)\|_{L^{p^*(\cdot)}(\mathbf{R}^n)}.$$

Since the extension $p^*(x)$ is regular, equivalence (10) is applicable so that

$$\begin{aligned} \|\mathbf{D}^\alpha(\chi_\Omega I^\alpha \mathscr{E}_\Omega \varphi)\|_{L^{p(\cdot)}(\Omega)} &\leqq C\|\chi_\Omega I^\alpha \mathscr{E}_\Omega \varphi\|_{I^\alpha\left(L^{p^*(\cdot)}(\mathbf{R}^n)\right)} \\ &\leqq C\|\mathscr{E}_\Omega \varphi\|_{L^{p^*(\cdot)}(\mathbf{R}^n)} = C\|\varphi\|_{L^{p(\cdot)}(\Omega)}, \end{aligned}$$

which completes the proof. □

3. The main result

Theorem 12. *Let Ω be a bounded domain in $\mathbf{R}^n$, $p \in \mathscr{P}(\Omega) \cap w\text{-Lip}(\Omega)$ and $0 < \alpha < \min\left(1, \frac{n}{p_+}\right)$. If the domain Ω has the $(\alpha, p(\cdot))$-property, then the Hardy inequality*

$$\left\| \frac{1}{\delta(x)^\alpha} \int_\Omega \frac{\varphi(y)}{|x-y|^{n-\alpha}}\, dy \right\|_{p(\cdot)} \leqq C\|\varphi\|_{p(\cdot)} \tag{13}$$

holds. If the exterior $\mathbf{R}^n \backslash \overline{\Omega}$ has the cone property, then the $(\alpha, p(\cdot))$-property is equivalent to the validity of the Hardy inequality (13).

4. Proof of Theorem 12

4.1. The principal idea of the proof. The proof of Theorem 12 is based on the observation that the weight $\frac{1}{\delta(x)^\alpha}$ in fact is equivalent to the integral

$$a_\Omega(x) := \int_{\mathbf{R}^n \backslash \Omega} \frac{dy}{|x-y|^{n+\alpha}}, \quad x \in \Omega.$$

Namely, the following statement is valid, see [25, Proposition 3.1].

Proposition 13. *For an arbitrary domain* Ω *there exists a constant* $c_1 > 0$ *(not depending on* $\Omega, c_1 = \frac{1}{\alpha}|S^{n-1}|$*) such that* $a_\Omega(x) \leqq \frac{c_1}{[\delta(x)]^\alpha}$*. If the exterior* $\mathbf{R}^n \backslash \overline{\Omega}$ *has the cone property, then there exists a constant* $c_2 = c_2(\Omega)$ *such that* $\frac{1}{[\delta(x)]^\alpha} \leqq c_2 a_\Omega(x)$*.*

We will prove the following version of Theorem 12.

Theorem 14. *Let* Ω *be a bounded domain in* $\mathbf{R}^n$*,* $p \in \mathscr{P}(\Omega) \cap w\text{-Lip}(\Omega)$ *and* $0 < \alpha < \min\left(1, \frac{n}{p_+}\right)$*. Then the Hardy type inequality*

$$\left\| a_\Omega(x) \int_\Omega \frac{\varphi(y)}{|x-y|^{n-\alpha}}\, dy \right\|_{p(\cdot)} \leqq C\|\varphi\|_{p(\cdot)} \tag{14}$$

holds if and only if the domain Ω *has the* $(\alpha, p(\cdot))$*-property.*

Theorem 12 will immediately follow from Theorem 14 in view of Proposition 13.

4.2. On a hypersingular integral related to Ω**.** As in [25], we define the hypersingular integral (fractional derivative) of order $0 < \alpha < 1$, related to the domain Ω, as the hypersingular integral over $\mathbf{R}^n$ of the extension $\mathscr{E}_\Omega f$:

$$\mathbf{D}_\Omega f(x) := r_\Omega \mathbf{D}^\alpha \mathscr{E}_\Omega f(x) = \frac{1}{d_{n,1}(\alpha)} \int\limits_{\mathbf{R}^n} \frac{f(x) - \widetilde{f}(y)}{|x-y|^{n+\alpha}}\, dy, \quad x \in \Omega,$$

where r_Ω stands for the restriction on Ω. Splitting the integration in the last integral to that over Ω and $\mathbf{R}^n \backslash \overline{\Omega}$, we can easily see that

$$a_\Omega(x) f(x) = d_{n,1}(\alpha) \mathbf{D}^\alpha \mathscr{E}_\Omega f(x) - \int\limits_\Omega \frac{f(x)-f(y)}{|x-y|^{n+\alpha}}\, dy, \quad x \in \Omega. \tag{15}$$

The proof of Theorem 14 will be based on representation (15) and certain known facts from the theory of hypersingular integrals [31].

4.3. Auxiliary functions. Although we will use the auxiliary functions defined below only in the case $\ell = 1$, we give them for an arbitrary integer ℓ as they are presented in [31]. By $\left(\Delta_h^\ell f\right)(x) := \sum_{k=0}^\ell (-1)^k \binom{\ell}{k} f(x-kh)$ we denote the non-centered difference of a function f defined on $\mathbf{R}^n$. We need the non-centered difference

$$\Delta_{\ell,\alpha}(x,h) := \left(\Delta_h^\ell k_\alpha\right)(x) \tag{16}$$

of the Riesz kernel $k_\alpha(x)$ and single out the case of the step $h = e_1 = (1, 0, \ldots, 0)$:

$$k_{\ell,\alpha}(x) := \Delta_{\ell,\alpha}(x, e_1) = \frac{1}{\gamma_n(\alpha)} \sum_{k=0}^\ell (-1)^k \binom{\ell}{k} |x - ke_1|^{\alpha-n}. \tag{17}$$

We will also use the function

$$\mathscr{K}_{\ell,\alpha}(|x|) = \frac{1}{d_{n,\ell}(\alpha)|x|^n} \int\limits_{|y|<|x|} k_{\ell,\alpha}(y)\, dy. \tag{18}$$

The following lemmata can be found in [31, §3.2.1].

Lemma 15. *The function* $\Delta_{\ell,\alpha}(x,h)$*, may be represented via its particular case* $k_{\ell,\alpha}(x)$ *in terms of rotations:*

$$\Delta_{\ell,\alpha}(x,h) = |h|^{\alpha-n} k_{\ell,\alpha}\left(\frac{|x|}{|h|^2} \operatorname{rot}_x^{-1} h\right), \tag{19}$$

where $\mathrm{rot}_x\,\eta$, $\eta \in \mathbf{R}^n$, *denotes any rotation in* $\mathbf{R}^n$ *which transforms* $\mathbf{R}^n$ *onto itself so that* $\mathrm{rot}_x\, e_1 = \frac{x}{|x|}$.

Lemma 16. *The function* $k_{\ell,\alpha}(x)$ *satisfies the condition*

$$|k_{\ell,\alpha}(x)| \leqq c(1+|x|)^{\alpha-n-\ell} \quad \text{when} \quad |x| \geqq \ell+1. \tag{20}$$

Lemma 17. *Let* $\ell > \Re\alpha > 0$. *Then*

$$\int\limits_{\mathbf{R}^n} k_{\ell,\alpha}(y)\, dy = 0. \tag{21}$$

Moreover, in the case when ℓ *is odd and the difference defining* $k_{\ell,\alpha}(x)$ *is non-centered,*

$$\int\limits_{\left|y-\frac{\ell}{2}e_1\right|<N} k_{\ell,\alpha}(y)\, dy = 0 \tag{22}$$

for any $N > 0$.

Lemma 18. *The function* $\mathscr{K}_{\ell,\alpha}(|x|)$, $0<\alpha<1$ *has the bound*

$$|\mathscr{K}_{\ell,\alpha}(|x|)| \leqq C|x|^{\alpha-n} \quad \text{as } |x| \leqq 1. \tag{23}$$

4.4. Proof of Theorem 14. Let $\varphi \in L^{p(\cdot)}(\Omega)$ and $\widetilde{\varphi} = \mathscr{E}_\Omega \varphi(x)$. Substituting

$$f(y) := I^\alpha \widetilde{\varphi} = \frac{1}{\gamma_n(\alpha)} \int\limits_\Omega \frac{\varphi(t)}{|t-y|^{n-\alpha}}\, dt, \quad y \in \mathbf{R}^n,$$

into (15), we have

$$a_\Omega(x) I^\alpha_\Omega \varphi(x) = \mathbf{D}^\alpha \chi_\Omega I^\alpha \mathscr{E}_\Omega \varphi(x) - \mathbf{A}\varphi(x), \quad x \in \Omega, \tag{24}$$

where

$$\mathbf{A}\varphi = \int\limits_\Omega \frac{I^\alpha\tilde{\varphi}(x) - I^\alpha\tilde{\varphi}(y)}{|x-y|^{n+\alpha}}\, dy = \lim_{\varepsilon\to 0} \mathbf{A}_\varepsilon \varphi(x)$$

and

$$\mathbf{A}_\varepsilon \varphi(x) = \int\limits_{\substack{y\in\Omega \\ |x-y|>\varepsilon}} \frac{I^\alpha\tilde{\varphi}(x) - I^\alpha\tilde{\varphi}(y)}{|x-y|^{n+\alpha}}\, dy.$$

The $(\alpha, p(\cdot))$-property of Ω, by the definition of this property and equivalence in (10), is nothing else but the boundedness in $L^{p(\cdot)}(\Omega)$ of the operator $\mathbf{D}^\alpha \chi_\Omega I^\alpha \mathscr{E}_\Omega$. Thus, in the case of bounded domains Ω, the required equivalence of the Hardy inequality to the $(\alpha, p(\cdot))$-property will follow from (24), if the operator $\mathbf{A}$ is bounded.

Lemma 19. *Let* $0<\alpha<1$ *and* Ω *be a bounded domain. The operators* $\mathbf{A}_\varepsilon$ *are uniformly dominated by the maximal operator:*

$$|\mathbf{A}_\varepsilon \varphi(x)| \leqq C \mathscr{M}\varphi(x), \quad x \in \Omega, \tag{25}$$

for any $\varphi \in L^1(\Omega)$, *where* $C>0$ *does not depend on* x *and* ε. *Consequently, the operator* $\mathbf{A}$ *is bounded in the space* $L^{p(\cdot)}(\Omega)$ *whenever* $p \in \mathbf{P}(\Omega)$.

Proof. We make use of the known representation

$$I^\alpha\tilde{\varphi}(x) - I^\alpha\tilde{\varphi}(x-y) = \int\limits_{\mathbf{R}^n} \Delta_{1,\alpha}(\xi, y)\tilde{\varphi}(x-\xi)\, d\xi$$

for the differences of the Riesz potential, see [31, formula (3.64)], and get

$$\begin{aligned}\mathbf{A}_\varepsilon\varphi(x) &= \int\limits_{\substack{y\in\Omega_x\\ |y|>\varepsilon}} \frac{dy}{|y|^{n+\alpha}} \int\limits_{\mathbf{R}^n} \tilde{\varphi}(x-\xi)\Delta_{1,\alpha}(\xi,y)\,d\xi \\ &= \int\limits_{\mathbf{R}^n} \tilde{\varphi}(x-\xi)\,d\xi \int\limits_{\substack{y\in\Omega_x\\ |y|>\varepsilon}} \frac{\Delta_{1,\alpha}(\xi,y)}{|y|^{n+\alpha}}\,dy,\end{aligned} \tag{26}$$

where $\Omega_x = \{y\in\mathbf{R}^n\colon x-y\in\Omega\}$, the interchange of the order of integration being easily justified by Fubini's theorem whenever $\varepsilon>0$. By (19) we then have

$$\begin{aligned}\mathbf{A}_\varepsilon\varphi(x) &= \int\limits_{\mathbf{R}^n} \tilde{\varphi}(x-\xi)\,d\xi \int\limits_{\substack{y\in\Omega_x\\ |y|>\varepsilon}} \frac{k_{1,\alpha}\left(\frac{|\xi|}{|y|^2}\operatorname{rot}_\xi^{-1}\, y\right)}{|y|^{2n}}\,dy = \int\limits_{\mathbf{R}^n} \frac{\tilde{\varphi}(x-\xi)}{|\xi|^n}\,d\xi \int\limits_{\substack{z\in\Omega(x,\xi)\\ |z|<\frac{|\xi|}{\varepsilon}}} k_{1,\alpha}(z)\,dz \\ &= \int\limits_{\mathbf{R}^n} \frac{\tilde{\varphi}(x-\varepsilon\xi)}{|\xi|^n}\,d\xi \int\limits_{\substack{z\in\Omega(x,\varepsilon\xi)\\ |z|<|\xi|}} k_{1,\alpha}(z)\,dz = \int\limits_{\mathbf{R}^n} \tilde{\varphi}(x-\varepsilon\xi)V_\varepsilon(x,\xi)\,d\xi,\end{aligned} \tag{27}$$

where

$$\Omega(x,\xi) = \left\{ z\in\mathbf{R}^n\colon |\xi|\operatorname{rot}_\xi \frac{z}{|z|^2}\in\Omega_x \right\}$$

and we denoted

$$V_\varepsilon(x,\xi) = \frac{1}{|\xi|^n} \int\limits_{\substack{z\in\Omega(x,\varepsilon\xi)\\ |z|<|\xi|}} k_{1,\alpha}(z)dz$$

for brevity. We split $\mathbf{A}_\varepsilon\varphi(x)$ in the following way

$$\mathbf{A}_\varepsilon\varphi(x) = \left(\int_{|\xi|<2} + \int_{|\xi|>2}\right) \tilde{\varphi}(x-\varepsilon\xi)V_\varepsilon(x,\xi)d\xi =: J_{1,\varepsilon}\varphi(x) + J_{2,\varepsilon}\varphi(x). \tag{28}$$

For $J_{1,\varepsilon}\varphi(x)$ we have

$$\begin{aligned}|J_{1,\varepsilon}\varphi(x)| &\leqq \int\limits_{|\xi|<2} |\tilde{\varphi}(x-\varepsilon\xi)|\frac{d\xi}{|\xi|^n} \int\limits_{|z|<|\xi|} |k_{1,\alpha}(z)|\,dz \\ &\leqq C \int\limits_{|\xi|<2} \frac{|\tilde{\varphi}(x-\varepsilon\xi)|}{|\xi|^{n-\alpha}}\,d\xi = C\,|\tilde{\varphi}| * \psi_\varepsilon(x),\end{aligned} \tag{29}$$

where $\psi(\xi) = \begin{cases} |\xi|^{\alpha-n}, & |\xi|<2,\\ 0, & |\xi|\geqq 2,\end{cases}$ and $\psi_\varepsilon(x) = \varepsilon^{-n}\psi(x/\varepsilon)$.

When $|\xi|>2$, the key moment in the estimation is the usage of property (22) of the Riesz kernel:

$$V_\varepsilon(x,\xi) = \frac{1}{|\xi|^n}\left(\int\limits_{B(0,|\xi|)\cap\Omega(x,\varepsilon\xi)} - \int\limits_{\left|z-\frac{e_1}{2}\right|<|\xi|-1} \right) k_{1,\alpha}(z)\,dz = \frac{1}{|\xi|^n} \int\limits_{\Theta(x,\varepsilon)} k_{1,\alpha}(z)\,dz,$$

where

$$\Theta(x,\varepsilon)=\{z: z\in B(0,|\xi|)\cap\Omega(x,\varepsilon\xi)\}\setminus\left\{z:\left|z-\frac{e_1}{2}\right|<|\xi|-1\right\}.$$

Since $\Theta(x,\varepsilon)$ is embedded in the annulus $|\xi|-\frac{3}{2}\leqq|z|\leqq|\xi|$, we have

$$|V_\varepsilon(x,\xi)|\leqq\frac{1}{|\xi|^n}\int\limits_{|\xi|-\frac{3}{2}\leqq|z|\leqq|\xi|}|k_{1,\alpha}(z)|\,dz$$

and by (20)

$$|V_\varepsilon(x,\xi)|\leqq\frac{C}{|\xi|^n}\left||\xi|^{\alpha-1}-\left(|\xi|-\frac{3}{2}\right)^{\alpha-1}\right|\leqq\frac{C}{|\xi|^{n+2-\alpha}}. \tag{30}$$

The estimation of $J_{2,\varepsilon}\varphi(x)$ is then given by

$$|J_{2,\varepsilon}\varphi(x)|\overset{(30)}{\leqq}C\,|\tilde{\varphi}|*\phi_\varepsilon(x), \tag{31}$$

where $\phi(\xi)=\begin{cases}2^{\alpha-n-2}, & |\xi|<2,\\ |\xi|^{\alpha-n-2}, & |\xi|\geqq 2,\end{cases}$ and $\phi_\varepsilon(x)=\varepsilon^{-n}\phi(x/\varepsilon)$.

Since the kernels ψ, ϕ are radially decreasing and integrable, we can use the well known estimation of convolutions with such kernels via the maximal function, which yields

$$J_{i,\varepsilon}\varphi(x)\leqq C\mathscr{M}(|\varphi|),\quad i=1,2,\forall\varepsilon>0, \tag{32}$$

and implies (25) after gathering (28), (29), (31) and (32). This completes the proof. □

4.5. Corollaries. As a corollary of Theorem 12 we obtain an estimate in classical $L^p(\Omega)$ spaces, but first we need the following definition.

Definition 20. Let Ω be an open set in $\mathbf{R}^n$. We say that Ω satisfies the *Strichartz condition* if there exist a coordinate system in $\mathbf{R}^n$ and an integer $N>0$ such that almost every line parallel to the axes intersects Ω in at most N components.

Lemma 21. [24, 36], [27, p. 244] *The characteristic function χ_Ω of a domain Ω satisfying the Strichartz condition is a pointwise multiplier in the space $I^\alpha\left(L^p(\mathbf{R}^n)\right)$ when $1<p<1/\alpha$.*

Corollary 22. *The Hardy inequality*

$$\left\|\frac{1}{\delta(x)^\alpha}\int\limits_\Omega\frac{\varphi(y)}{|x-y|^{n-\alpha}}\,dy\right\|_p\leqq C\|\varphi\|_p,\quad 1<p<1/\alpha$$

holds for any bounded open set $\Omega\subset\mathbf{R}^n$ satisfying the Strichartz condition.

Proof. By Lemma 11 and Lemma 21 we have that Ω has the $(\alpha,p(\cdot))$-property and then the results follows from Theorem 12. □

References

[1] Almeida, A.: Inversion of the Riesz potential operator on Lebesgue spaces with variable exponent. - Fract. Calc. Appl. Anal. 6:3, 2003, 311–327.

[2] Almeida, A., and S. Samko: Characterization of Riesz and Bessel potentials on variable Lebesgue spaces. - J. Funct. Spaces Appl. 4:2, 2006, 113–144.

[3] Diening, L.: Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$. - Math. Inequal. Appl. 7:2, 2004, 245–253.

[4] Diening, L.: Riesz potential and Sobolev embeddings on generalized Lebesgue and Sobolev spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$. - Math. Nachr. 268:2, 2004, 31–43.

[5] Diening, L.: Maximal function on Musielak–Orlicz spaces and generalized Lebesgue spaces. - Bull. Sci. Math. 129:8, 2005, 657–700.

[6] Diening, L., P. Hästö, and A. Nekvinda: Open problems in variable exponent Lebesgue and Sobolev spaces. - In: Function Spaces, Differential Operators and Nonlinear Analysis, Proceedings of the Conference held in Milovy, Bohemian–Moravian Uplands, May 28 – June 2, 2004, Math. Inst. Acad. Sci. Czech Republick, Praha, 2005, 38–58.

[7] Diening, L., and S. Samko: Hardy inequality in variable exponent Lebesgue spaces. - Fract. Calc. Appl. Anal. 10:1, 2007, 1–18.

[8] Edmunds, D. E., and W. D. Evans: Hardy operators, function spaces and embeddings. - Springer, 2004.

[9] Edmunds, D. E., V. M. Kokilashvili, and A. Meskhi: Bounded and compact integral operators. - Kluwer, Dordrecht, 2002.

[10] Hajłasz, P.: Pointwise Hardy inequalities. - Proc. Amer. Math. Soc. 127:2, 1999, 417–423.

[11] Harjulehto, P., P. Hästö, and M. Koskenoja: Hardy's inequality in a variable exponent Sobolev space. - Georgian Math. J. 12:3, 2005, 431–442.

[12] Kinnunen, J., and O. Martio: Hardy's inequalities for Sobolev functions. - Math. Res. Lett. 4:4, 1997, 489–500.

[13] Kokilashvili, V.: On a progress in the theory of integral operators in weighted Banach function spaces. - In: Function Spaces, Differential Operators and Nonlinear Analysis, Proceedings of the Conference held in Milovy, Bohemian-Moravian Uplands, May 28 – June 2, 2004, Math. Inst. Acad. Sci. Czech Republick, Praha, 2005.

[14] Kokilashvili, V., and S. Samko: Singular integrals in weighted Lebesgue spaces with variable exponent. - Georgian Math. J. 10, 2003, 145–156.

[15] Kokilashvili, V., and S. Samko: Maximal and fractional operators in weighted $L^{p(x)}$ spaces. - Rev. Mat. Iberoamericana 20:2, 2004, 493–515.

[16] Kovăčik, O., and J. Rákosnǐk: On spaces $L^{p(x)}$ and $W^{k,p(x)}$. - Czech. Math. J. 41:116, 1991, 592–618.

[17] Kufner, A.: Weighted Sobolev spaces. - John Wiley & Sons, 1985.

[18] Kufner, A., L. Maligranda, and L.-E. Persson: The Hardy inequality. About its history and some related results. - Pilsen, 2007

[19] Mashiyev, R. A., B. Çekiç, F. I. Mamedov, and S. Ogras: Hardy's inequality in power-type weighted $L^{p(\cdot)}(0,\infty)$ spaces. - J. Math. Anal. Appl. 334:1, 2007, 289–298.

[20] Mashiyev, R. A., B. Çekiç, and S. Ogras: On Hardy's inequality in $L^{p(x)}(0,\infty)$. - JIPAM. J. Inequal. Pure Appl. Math. 7:3, Article 106, 2006, 1–5.

[21] Maz'ya, V. G., and T. O. Shaposhnikova: Theory of multipliers in spaces of differentiable functions. - Pitman, 1985.

[22] Maz'ya, V. G., and T. O. Shaposhnikova: Multipliers in spaces of differentiable functions. - Leningrad. Univ., Leningrad, 1986 (in Russian).

[23] Něcas, J.: Sur une méthode pour résoudre les équations aux dérivées partielles du type elliptique, voisine de la variationnelle. - Ann. Sc. Norm. Super. Pisa 3:16, 1962, 305–326.

[24] Nogin, V. A., and B. S. Rubin: Boundedness of the operator of multiplication by a characteristic function of the Strichartz domain in spaces of Riesz potentials. - Izv. Sev.-Kavk. Nauchn. Tsentra Vyssh. Shkoly Estestv. Nauki 2, 1986, 62–66 (in Russian).

[25] Rafeiro, H., and S. Samko: On multidimensional analogue of Marchaud formula for fractional Riesz-type derivatives in domains in $\mathbf{R}^n$. - Fract. Calc. Appl. Anal. 8:4, 2005, 393–401.

[26] Rafeiro, H., and S. Samko: Characterization of the range of one-dimensional fractional integration in the space with variable exponent. - Operator Algebras, Operator Theory and Applications Series: Operator Theory: Advances and Applications 181, edited by A. Bastos et al., 2007, 393–416.

[27] Rubin, B. S.: Fractional integrals and potentials. - Pitman Monographs and Surveys in Pure and Applied Mathematics 82, 1996.

[28] Runst, T., and W. Sickel: Sobolev spaces of fractional order, Nemytskij operators and nonlinear partial differential equations. - Gruyter, 1996.

[29] Růžička, M.: Electroreological fluids: modeling and mathematical theory. - Lecture Notes in Math. 1748, Springer, 2000.

[30] Samko, S.: Differentiation and integration of variable order and the spaces $L^{p(x)}$. - In: Proceed. Intern. Conference "Operator Theory and Complex and Hypercomplex Analysis", December 12–17, 1994, Mexico City, Contemp. Math. 212, 1998, 203–219.

[31] Samko, S. G.: Hypersingular integrals and their applications. - Taylor & Francis, London, 2002.

[32] Samko, S.: Hardy inequality in the generalized Lebesgue spaces. - Fract. Calc. Appl. Anal. 6:4, 2003, 355–362.

[33] Samko, S.: Hardy–Littlewood–Stein–Weiss inequality in the Lebesgue spaces with variable exponent. - Fract. Calc. Appl. Anal. 6:4, 2003, 421–440.

[34] Samko, S.: On a progress in the theory of Lebesgue spaces with variable exponent: maximal and singular operators. - Integral Transforms Spec. Funct. 16:5-6, 2005, 461–482.

[35] Sharapudinov, I. I.: The topology of the space $l^{p(t)}([0, 1])$. - Mat. Zametki 26:4, 1979, 613–632; English transl. in Math. Notes 26:3-4, 1979, 796–806.

[36] Strichartz, R. S.: Multipliers on fractional Sobolev spaces. - J. Math. Mech. 16, 1967, 1031–1060.

[37] Triebel, H.: Function spaces in Lipschitz domains and on Lipschitz manifolds. Characteristic functions as pointwise multipliers. - Rev. Math. Complut. 15:2, 2002, 475–524.

[38] Triebel, H.: Non-smooth atoms and pointwise multipliers in function spaces. - Ann. Mat. Pura Appl. (4) 182:4, 2003, 457–486.

[39] Wannebo, A.: Hardy inequalities. - Proc. Amer. Math. Soc. 109:1, 1990, 85–95.

[40] Wannebo, A.: Hardy inequalities and imbeddings in domains generalizing $C^{0,\lambda}$ domains. - Proc. Amer. Math. Soc. 122:4, 1994, 1181–1190.

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