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A TECHNIQUE OF BEURLING FOR LOWER ORDER

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Abstract. Suppose that $\phi = u - v$, where u and v are subharmonic in the plane, with u nonconstant. Suppose also that $\underline{\lim}_{r\to\infty}(\log(B(r,u)+B(r,v))/\log r \le \lambda$, for some λ satisfying $0<\lambda<1/2$, and that the deficiency δ of ϕ satisfies $0 \le 1-\delta < \cos\pi\lambda$. Given σ satisfying $\lambda<\sigma<1/2$ and $0\le 1-\delta<\cos\pi\sigma$, we have

$$A(r,\phi)/B(r,\phi) > \kappa = \kappa(\sigma,\delta) := \frac{\cos\pi\sigma-(1-\delta)}{1-(1-\delta)\cos\pi\sigma}$$

for all r in a set of upper logarithmic density at least $1-\lambda/\sigma$. Here $A(r,\phi)=\inf_{|z|=r}\phi(z)$ and $B(r,\phi)=\sup_{|z|=r}\phi(z)$.

1. Introduction

This note concerns δ-subharmonic functions ϕ and the relationship between $A(r,\phi)$ and $B(r,\phi)$, where

$$A(r,\phi)=\inf_{|z|=r}\phi(z), \quad B(r,\phi)=\sup_{|z|=r}\phi(z).$$

If $\phi = u - v$, where u and v are subharmonic in the plane, the deficiency of ϕ is defined to be $\delta := 1-\overline{\lim}_{r\to\infty} N(r,v)/T(r,\phi)$, where $T(r,\phi)=N(r,\phi^+)+N(r,v)$. Here

$$N(r,v)=\frac{1}{2\pi}\int_0^{2\pi} v(re^{i\theta})\,d\theta - v(0),$$

and $N(r,u)$ and $N(r,\phi^+)$ are defined similarly. Let $\mu(E,u)$ be the Riesz measure of u for any Borel set E, and define $\mu^*(t,u) := \mu(\{z : |z|\le t\},u)$, with a similar definition for v. We will prove:

Theorem 1. *Suppose that u and v are subharmonic in the plane, with u nonconstant, and that, for some λ satisfying $0<\lambda<1/2$,*

$$\underline{\lim}_{r\to\infty}\frac{\log(B(r,u)+B(r,v))}{\log r}\le\lambda. \tag{1}$$

Let $\phi = u - v$ and suppose that the deficiency δ of ϕ satisfies $0\le 1-\delta<\cos\pi\lambda$. Given σ satisfying $\lambda<\sigma<1/2$ and $0\le 1-\delta<\cos\pi\sigma$, we have

$$\frac{A(r,\phi)}{B(r,\phi)} > \kappa = \kappa(\sigma,\delta) := \frac{\cos\pi\sigma-(1-\delta)}{1-(1-\delta)\cos\pi\sigma} \tag{2}$$

for all r in a set of upper logarithmic density at least $1-\lambda/\sigma$.

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The upper (or lower) logarithmic density of a set $S \subseteq (0,\infty)$ is the upper (or lower) limit as $r \to \infty$ of $(\log r)^{-1} \int_{S\cap(1,r)} t^{-1}\,dt$. It is known that (2) holds in a set of lower logarithmic density at least $1-\lambda/\sigma$ if the lower limit in (1) is replaced by the upper limit [4, Theorem 1.1].

We remark that if $v \equiv 0$ in Theorem 1 (implying $\delta = 1$), then the result was obtained by Barry [1] as an extension of the classical $\cos\pi\lambda$ theorem for subharmonic functions. (In fact his result holds for $0<\lambda<1$.) Original estimates in Barry's paper involve

$$\int_0^\infty \frac{A(t,u)-\cos\pi\sigma B(t,u)}{t^{\sigma+1}}\,dt.$$

Progressing from these estimates to estimates of the integrand naturally introduces exceptional sets. Hayman [6] subsequently constructed a subharmonic function u_0 of order λ, $0<\lambda<1$, such that, for given σ with $\lambda<\sigma<1$,

$$\frac{A(r,u_0)}{B(r,u_0)} \le \cos\pi\sigma \tag{3}$$

for all r in a set with upper (and lower) logarithmic density equal to λ/σ. This established that the size of the exceptional set in Barry's result is sharp. Similarly, by setting $\phi(z) = u_0(z)-(1-\delta)u_0(-z)$, we see that (2) also fails on E and so the size of the exceptional set is sharp in Theorem 1 as well.

These examples raise the question as to whether we can find classes of functions with no exceptional sets. In the proof of Theorem 2, we will show that the structure of our exceptional set in Theorem 1 is directly related to the behaviour of the Riesz measures of u and v. It will follow that if, for all large values of t, $\mu^*(t,u)$ and $\mu^*(t,v)$ are continuous functions of t of the form

$$\mu^*(t,u) = \varepsilon_1(t)t^\lambda, \quad \mu^*(t,v) = \varepsilon_2(t)t^\lambda, \tag{4}$$

where $\varepsilon_i(t) \downarrow 0$ as $t\to\infty$ for $i=1,2$, then there exists a positive number R such that (2) holds for all $r>R$. Effectively then, there is no exceptional set. As far as we are aware, this is the first time such an observation has been made about a general class of functions.

We will suppose that u and v are modified so as to be harmonic in $|z|<1$, which does not affect the generality of our results. If u has order less than one, then [5, p. 311]

$$u(z) = u(0) + \int_{|\zeta|<\infty} \log|1-z/\zeta|\,d\mu(\zeta,u)$$

and, with

$$u^*(z) := u(0) + \int_0^\infty \log|1+z/t|\,d\mu^*(t,u),$$

we have

$$u^*(-r) \le A(r,u) \le B(r,u) \le u^*(r), \quad 0 \le r < \infty. \tag{5}$$

Theorem 1 is a consequence of the following result, which adapts a technique of Beurling [2, p. 762] to situations in which something is known about lower rather than upper growth.

Theorem 2. *Suppose that* $0 < \nu < 1$, *that* u *is a nonconstant subharmonic function in the plane and that*

$$\varliminf_{r\to\infty} B(r,u)/r^{\nu} = 0. \tag{6}$$

(i) *There are sequences* $\varepsilon_n \downarrow 0$ *and* $R_n \uparrow \infty$ *such that*

$$B(8R_n,u) < \varepsilon_n R_n^{\nu}, \quad N(4R_n,u) < \varepsilon_n R_n^{\nu} \quad \text{and} \quad \mu^*(4R_n,u) < \varepsilon_n R_n^{\nu} \tag{7}$$

for all n.

(ii) *Given* $C \geq 1$ *and* $n \in \mathbf{N}$, *define*

$$U_n(z) = U_n(z,\nu) := u(0) + \int_0^{4R_n} \log|1+z/t|\, d\mu^*(t,u) + 8C\varepsilon_n R_n^{\nu} \log|1+z/(4R_n)|. \tag{8}$$

For any α *satisfying*

$$\alpha \geq \alpha_n := 5C\nu^{-1}\varepsilon_n, \tag{9}$$

let

$$a_{\alpha,n} := \max\{N(r,u) - \alpha r^{\nu} : r \in [0,R_n]\}, \tag{10}$$

and let $r_{\alpha,n}$ *be any point in* $[0,R_n]$ *at which*

$$a_{\alpha,n} = N(r_{\alpha,n},u) - \alpha r_{\alpha,n}^{\nu}. \tag{11}$$

Then $a_{\alpha,n} \geq 0$,

$$A(r_{\alpha,n},u) > U_n(-r_{\alpha,n}) > \pi\nu(\cot\pi\nu)N(r_{\alpha,n},u) + (1-\pi\nu\cot\pi\nu)a_{\alpha,n} \tag{12}$$

and

$$B(r_{\alpha,n},u) < U_n(r_{\alpha,n}) < \pi\nu(\csc\pi\nu)N(r_{\alpha,n},u) + (1-\pi\nu\csc\pi\nu)a_{\alpha,n}. \tag{13}$$

Further, for any $\alpha_0 > 0$, *the part of the set*

$$T = T(\alpha_0) := \bigcup_{n=1}^{\infty}\{r_{\alpha,n} : \alpha_0 \geq \alpha \geq 5\varepsilon_n/\nu\} \tag{14}$$

contained in $[1,\infty)$ *has infinite logarithmic measure. If* u *has lower order* λ, *where* $0 \leq \lambda < \nu$, *then* T *has upper logarithmic density at least* $1-\lambda/\nu$. *It is always possible to choose* α_0 *in such a way that* $a_{\alpha,n} \to \infty$ *as* $r_{\alpha,n} \to \infty$ *in* T.

The following theorem is a lower order analogue of Theorem 3 in [3]. The constant $(\cos\pi\lambda - k)/(1-k\cos\pi\lambda)$ is sharp; see the remark following Theorem 3 in [3].

Theorem 3. *Suppose that* u *and* v *are subharmonic in the plane, with* u *nonconstant, and that, for some* λ *satisfying* $0 < \lambda < 1/2$,

$$\varliminf_{r\to\infty} \frac{B(r,u)+B(r,v)}{r^{\lambda}} = 0$$

and

$$N(r,v) \leq kN(r,u) + O(1) \text{ as } r \to \infty,$$

for some k *such that* $0 \leq k \leq \cos\pi\lambda$. *Let* $\phi(z) = u(z) - v(z)$. *Then, with* $r_{\alpha,n}$ *as defined in* (11),

$$A(r_{\alpha,n},\phi) \geq \frac{\cos\pi\lambda - k}{1-k\cos\pi\lambda} B(r_{\alpha,n},\phi) + b_{r_{\alpha,n}}, \tag{15}$$

where $b_{r_{\alpha,n}} \to \infty$ *as* $r_{\alpha,n} \to \infty$.

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2. Proof of Theorem 2

Concerning the sequences ε_n and R_n, choose ε_n arbitrarily, with $\varepsilon_n \downarrow 0$, and R_n such that $B(8R_1, u) \geq -u(0)$, $N(R_1, u) > 0$, $R_n \uparrow \infty$, and $B(8R_n, u) < \frac{1}{2}\varepsilon_n R_n^\nu \log 2$ for all n, which is possible from (6) and the fact that u is conconstant (so that $B(r, u) \to \infty$ and $N(r, u) \to \infty$ as $r \to \infty$). The first part of (7) is satisfied and, from Jensen's theorem,

$$N(4R_n, u) \leq N(8R_n, u) \leq -u(0) + B(8R_n, u) \leq 2B(8R_n, u),$$

so the second part is satisfied also. For the third part,

$$\mu^*(4R_n, u) \log 2 \leq \int_{4R_n}^{8R_n} \mu^*(t, u)\, dt/t \leq N(8R_n, u).$$

Notice that, since u is harmonic in $|z| < 1$, we have $N(r, u) = 0$ for $0 \leq r \leq 1$, and thus $R_1 > 1$.

According to a result of Kjellberg [7, formulas (6), (8) and (18)], (see also [1, pp. 180–83]), if

$$u_1(z) = u_1(z, R_n) := u(0) + \int_{|\zeta|<4R_n} \log|1 - z/\zeta|\, d\mu(\zeta, u),$$

then

$$|u(z) - u_1(z)| \leq |z| R_n^{-1} B(8R_n, u) \tag{16}$$

for $|z| \leq R_n$. Write

$$u_2(z) = u_2(z, R_n) := u(0) + \int_0^{4R_n} \log|1 + z/t|\, d\mu^*(t, u),$$

so that for all $r \geq 0$,

$$u_2(-r) \leq A(r, u_1) \leq B(r, u_1) \leq u_2(r), \tag{17}$$

from (5) (or by direct computation). Since $B(8R_n, u) < \varepsilon_n R_n^\nu$, we deduce from (16) and (17) that, for $|z| \leq R_n$,

$$u(z) > u_1(z) - |z| R_n^{-1} B(8R_n, u) > u_2(-|z|) - \varepsilon_n |z| R_n^{\nu-1}$$

and therefore

$$A(r, u) > u_2(-r) - \varepsilon_n r R_n^{\nu-1}, \quad 0 \leq r \leq R_n. \tag{18}$$

Similarly

$$B(r, u) < u_2(r) + \varepsilon_n r R_n^{\nu-1}, \quad 0 \leq r \leq R_n. \tag{19}$$

With (8) in view, we have, for $0 \leq r \leq R_n$,

$$\begin{aligned} U_n(-r) &= u_2(-r) + 8C\varepsilon_n R_n^\nu \log(1 - r/(4R_n)) \\ &\leq u_2(-r) - 2C\varepsilon_n r R_n^{\nu-1} < A(r, u), \end{aligned} \tag{20}$$

from (18), and

$$
(21)\qquad \begin{aligned} U_n(r) &= u_2(r) + 8C\varepsilon_n R_n^{\nu}\log(1 + r/(4R_n)) \\ &\geq u_2(r) + 2C\varepsilon_n r R_n^{\nu-1}/\big(1 + r/(4R_n)\big) \geq u_2(r) + C\varepsilon_n r R_n^{\nu-1} > B(r,u), \end{aligned}
$$

from (19).

Now,

$$
(22)\qquad N(r,U_n) = N(r,u_2) = N(r,u_1) = N(r,u), \quad 0 \leq r \leq 4R_n,
$$

since $\mu^*(r,\,\cdot)$ is the same for each function for $0 \leq r \leq 4R_n$, and therefore

$$
(23)\qquad N(r,U_n) \leq N(4R_n,U_n) = N(4R_n,u) < \varepsilon_n R_n^{\nu} < \alpha_n R_n^{\nu} \leq \alpha_n r^{\nu}
$$

for $R_n \leq r \leq 4R_n$, using (7) and (9). Also, since the third term on the right hand side of (8) arises from a point mass $8C\varepsilon_n R_n^{\nu}$ at $-4R_n$, we have, for $r \geq 4R_n$,

$$
(24)\qquad \begin{aligned} N(r,U_n) &= N(4R_n,U_n) + \int_{4R_n}^{r} \frac{\mu^*(t,U_n)}{t}\,dt \\ &= N(4R_n,u) + \big(\mu^*(4R_n,u) + 8C\varepsilon_n R_n^{\nu}\big)\log\big(r/(4R_n)\big) \\ &< N(4R_n,u) + 9C\varepsilon_n R_n^{\nu}\log\big(r/(4R_n)\big), \end{aligned}
$$

using (7). The largest value of $x^{-\nu}\log x$ is $(e\nu)^{-1}$, at $x = e^{1/\nu}$, and therefore

$$
R_n^{\nu}\log\big(r/(4R_n)\big) = 4^{-\nu} r^{\nu}\,(r/(4R_n))^{-\nu}\log\big(r/(4R_n)\big) < (e\nu)^{-1} r^{\nu}.
$$

From this, (7) and (24) we have, for $r \geq 4R_n$,

$$
(25)\qquad N(r,U_n) < (1 + 9C(e\nu)^{-1})\varepsilon_n r^{\nu} < (1 + 4C\nu^{-1})\varepsilon_n r^{\nu} < 5C\nu^{-1}\varepsilon_n r^{\nu} \leq \alpha_n r^{\nu},
$$

using (9).

Suppose now that α satisfies (9). From (10) and (11), $N(r_{\alpha,n},u) - \alpha r_{\alpha_n}^{\nu} \geq N(0,u) = 0$, while from (23) and (25), $N(r,U_n) - \alpha r^{\nu} < 0$ for $R_n \leq r < \infty$. We deduce that

$$
(26)\qquad \begin{aligned} a_{\alpha,n} &= N(r_{\alpha,n},u) - \alpha r_{\alpha,n}^{\nu} = N(r_{\alpha,n},U_n) - \alpha r_{\alpha,n}^{\nu} \\ &= \max\{N(r,U_n) - \alpha r^{\nu} \colon r \in [0,\infty)\}, \end{aligned}
$$

using (22). From (26), it follows [2, p. 764] that

$$
(27)\qquad \begin{aligned} U_n(-r_{\alpha,n}) &> \pi\nu(\cot\pi\nu)N(r_{\alpha,n},U_n) + (1 - \pi\nu\cot\pi\nu)a_{\alpha,n} \\ &= \pi\nu(\cot\pi\nu)N(r_{\alpha,n},u) + (1 - \pi\nu\cot\pi\nu)a_{\alpha,n} \end{aligned}
$$

and

$$
(28)\qquad U_n(r_{\alpha,n}) < \pi\nu(\csc\pi\nu)N(r_{\alpha,n},u) + (1 - \pi\nu\csc\pi\nu)a_{\alpha,n},
$$

which, combined with (20) and (21) at $r = r_{\alpha,n}$, give (12) and (13).

It remains to prove the conclusions about T. Fix a positive integer n and consider, for that n, U_n given by (8). Changing notation somewhat, for any $\alpha > 0$, let t_α be a point at which

$$
\max\{N(t,U_n) - \alpha t^{\nu} : t \in [0,\infty)\}
$$

is attained. Following the argument of [4, pp. 247–248], the points t_α increase as α decreases, and occupy all points of $[0,\infty)$ apart from exceptional intervals of the form (t_α^-, t_α^+), where (recalling that there may be several values of t_α) t_α^- and t_α^+ are the smallest and largest among the possible values of t_α. Further, for all α, $\mu^*(t_\alpha^-,U_n) = \alpha\nu(t_\alpha^-)^{\nu}$, and $\mu^*(t_\alpha^+,U_n) = \alpha\nu(t_\alpha^+)^{\nu}$. As we have shown (cf. (26)), if

$\alpha \geq \alpha_n = 5C\varepsilon_n/\nu$ then $t_\alpha \leq R_n$, so that $\mu^*(t_\alpha, U_n) = \mu^*(t_\alpha, u)$. Returning to our earlier notation, we have $t_\alpha = r_{\alpha,n}$, so that

$$\mu^*(r^-_{\alpha,n}, u) = \alpha\nu(r^-_{\alpha,n})^\nu \text{ and } \mu^*(r^+_{\alpha,n}, u) = \alpha\nu(r^+_{\alpha,n})^\nu. \tag{29}$$

Thus the set $\{r_{\alpha,n} : \alpha_0 \geq \alpha \geq \alpha_n\}$ consists of all points in $(r^-_{\alpha_0,n}, r^+_{\alpha_n,n})$ outside an exceptional set of intervals of total logarithmic measure

$$\begin{aligned}\sum_{\alpha_0\geq\alpha\geq\alpha_n} \nu^{-1}\log\left(\mu^*(r^+_{\alpha,n}, u)/\mu^*(r^-_{\alpha,n}, u)\right) &\leq \nu^{-1}\log(\mu^*(r^+_{\alpha_n,n}, u)/\mu^*(r^-_{\alpha_0,n}, u)) \\ &= \log(r^+_{\alpha_n,n}/r^-_{\alpha_0,n}) + \nu^{-1}\log(\alpha_n/\alpha_0),\end{aligned} \tag{30}$$

using (29). Evidently, since T increases as α_0 increases, we may assume without loss of generality that α_0 is sufficiently small that $\alpha_0 < N(R_1, u)R_1^{-\nu}$. In that case $a_{\alpha,n} > 0$ for all n and all α satisfying $\alpha_0 \geq \alpha \geq \alpha_n$, and, since $N(t,u) - \alpha_0 t^\nu = -\alpha_0 t^\nu \leq 0$ for $0 \leq t \leq 1$, we have $r_{\alpha_0,n} > 1$ for all n. Thus the logarithmic measure of the exceptional points in $[1, r^+_{\alpha_n,n}]$ is at most $\log r^+_{\alpha_n,n} + \nu^{-1}\log(\alpha_n/\alpha_0)$, and consequently the logarithmic measure of the part of T in $[1, r^+_{\alpha_n,n}]$ is at least $\nu^{-1}\log(\alpha_0/\alpha_n) \to \infty$ as $n \to \infty$.

Suppose that u has lower order λ, where $0 \leq \lambda < \nu$, and that ν_1 satisfies $\lambda < \nu_1 < \nu$. There is a sequence of positive numbers $R_n \uparrow \infty$ such that $B(8R_1, u) \geq -u(0)$, $N(R_1, u) > 0$ and

$$B(8R_n, u) < R_n^{\nu_1}, \;\; N(4R_n, u) < R_n^{\nu_1} \text{ and } \mu^*(4R_n, u) < R_n^{\nu_1}, \tag{31}$$

for all n, as we have shown. Thus we have (7), with $\varepsilon_n = R_n^{\nu_1-\nu}$, and (12) and (13) hold.

From (29) and (30), again assuming as we may that $\alpha_0 < N(R_1, u)R_1^{-\lambda}$, the set $\{r_{\alpha,n} : \alpha_0 \geq \alpha \geq \alpha_n\}$ contains all points in $[1, r^+_{\alpha_n,n}]$ outside an exceptional set of logarithmic measure at most $\nu^{-1}\log\mu^*(r^+_{\alpha_n,n}, u) - \nu^{-1}\log(\alpha_0\nu)$. Now, from (29), (9) and the fact that $r^+_{\alpha_n,n} \leq R_n$,

$$\mu^*(r^+_{\alpha_n,n}, u) = \alpha_n\nu(r^+_{\alpha_n,n})^\nu = 5CR_n^{\nu_1-\nu}(r^+_{\alpha_n,n})^\nu \leq 5C(r^+_{\alpha_n,n})^{\nu_1},$$

and therefore the logarithmic measure of the part of T in $[1, r^+_{\alpha_n,n}]$ is at least $(1 - \nu_1/\nu)\log r^+_{\alpha_n,n} - \nu^{-1}\log(5C/(\alpha_0\nu))$. Since $r^+_{\alpha_n,n} \to \infty$ as $n \to \infty$, the upper logarithmic density of T is thus at least $1 - \nu_1/\nu$, for any ν_1 satisfying $\lambda < \nu_1 < \nu$, and so is at least $1 - \lambda/\nu$.

Concerning a choice of α_0 that ensures that $a_{\alpha,n} \to \infty$ as $r_{\alpha,n} \to \infty$ in T, suppose that it is possible to find sequences $\alpha(j)$ and n_j, with $1 \geq \alpha(j) \geq 5\nu^{-1}\varepsilon_{n_j}$, such that $r_{\alpha(j),n_j} \to \infty$, while $a_{\alpha(j),n_j} = O(1)$ as $j \to \infty$. Write $\gamma_j := a_{\alpha(j),n_j}$ and $\gamma := \sup\gamma_j$, so that $0 \leq \gamma < \infty$. From the definition of $a_{\alpha(j),n_j}$, we have

$$N(t,u) \leq \alpha(j)t^\nu + \gamma_j, \quad 0 \leq t \leq R_{n_j}, \tag{32}$$

with equality at $t = r_{\alpha(j),n_j}$. Write $\alpha' := \underline{\lim}_{j\to\infty}\alpha(j)$, so that $0 \leq \alpha' \leq 1$, and suppose that $\alpha' = 0$. In that case, $\alpha(j) \to 0$ on a subsequence, and since $R_{n_j} \geq r_{\alpha(j),n_j} \to \infty$ as $j \to \infty$, we conclude from (32) that $N(t,u) \leq \gamma$, a contradiction, since u is nonconstant and of order less than 1. So $\alpha' > 0$ and, from (32), $N(t,u) \leq \alpha' t^\nu + \gamma$, for $t \geq 0$. Thus, since $N(r_{\alpha(j),n_j}, u) = \alpha(j)r^\nu_{\alpha(j),n_j} + \gamma_j \geq \alpha(j)r^\nu_{\alpha(j),n_j}$, we have

$$\overline{\lim_{t\to\infty}} N(t,u)/t^\nu = \alpha'. \tag{33}$$

It follows that if $0 < \alpha_0 < \alpha'$ then $a_{\alpha,n} \to \infty$ as $r_{\alpha,n} \to \infty$ in T. For otherwise, repeating the above argument, we would obtain $0 \le \overline{\lim}_{t\to\infty} N(t,u)/t^\nu = \alpha''$, for some α'' with $0 < \alpha'' \le \alpha_0$. This completes the proof of Theorem 2.

3. Proofs of Theorems 1 and 3

The argument conflates the proofs of Theorems 1 and 3 in [3]. To prove Theorem 1, given ν satisfying $\lambda < \nu < \sigma$ (so that $1-\delta < \cos \pi\nu$), choose Δ such that $0 < \Delta < \delta$, $1 - \Delta < \cos \pi\nu$ and

$$K := \kappa(\nu, \Delta) > \kappa(\sigma, \delta). \tag{34}$$

This is possible, and in addition Δ can be chosen so that $\Delta \to \delta$ as $\nu \to \sigma$. Following [2, p. 394], we may assume that $u(z) \ge v(z)$ for all z and that $\overline{\lim}_{r\to\infty} N(r,v)/N(r,u) \le 1-\delta$. Thus, for some positive constant c,

$$N(r,v) \le (1-\Delta)N(r,u) + c, \quad r \ge 0. \tag{35}$$

(In what follows, we use c to denote a positive constant that depends only on u, v and the parameters of Theorem 1, not necessarily the same at each occurrence.) From (1), $\underline{\lim}_{r\to\infty}(B(r,u)+B(r,v))/r^\nu = 0$. We may thus find sequences $\varepsilon_n \downarrow 0$ and $R_n \uparrow \infty$ such that the inequalities (7), with $\varepsilon_n/2$ instead of ε_n, hold for both u and v, and define U_n by (8) with $C = 9(1-\Delta)^{-1}/8$, and V_n by (8) with $C = 1$ (and v replacing u). Let

$$w(z) := u(z) + Kv(z),$$

where K is given by (34). Then w is subharmonic, since $K > 0$, and the inequalities (7) hold for w also, since $K < 1$. Also, if W_n is defined by (8), with $C = 9(1-\Delta)^{-1}/8 + K$ and w instead of u, then

$$W_n(z) = U_n(z) + KV_n(z). \tag{36}$$

Write $\psi_n(z) = U_n(z) - V_n(-z)$. For $0 \le r \le R_n$, we have

$$A(r,\phi) \ge A(r,u) - B(r,v) > U_n(-r) - V_n(r) = \psi_n(-r)$$

and

$$B(r,\phi) \le B(r,u) - A(r,v) < U_n(r) - V_n(-r) = \psi_n(r),$$

from (20) and (21), so that, for $0 \le r \le R_n$,

$$\begin{aligned} A(r,\phi) - KB(r,\phi) &> \psi_n(-r) - K\psi_n(r) \\ &= W_n(-r) - KW_n(r) - (1-K^2)V_n(r). \end{aligned} \tag{37}$$

Also, integrating twice by parts,

$$V_n(r) = v(0) + \int_0^\infty \log|1 + r/t|\, d\mu^*(t,V_n) = v(0) + \int_0^\infty N(t,V_n)\frac{r}{(t+r)^2}\,dt, \tag{38}$$

and further

$$N(t,V_n) = N(t,v) \le (1-\Delta)N(t,u) + c = (1-\Delta)N(t,U_n) + c, \quad 0 \le t \le 4R_n,$$

using (35), while for $t \ge 4R_n$, using (35) again and also (7),

$$\begin{aligned} N(t,V_n) &= N(4R_n,v) + (\mu^*(4R_n,v) + 8\varepsilon_n R_n^\nu)\log(t/(4R_n)) \\ &\le (1-\Delta)N(4R_n,u) + 9\varepsilon_n R_n^\nu \log(t/(4R_n)) + c \\ &\le (1-\Delta)N(4R_n,u) + (1-\Delta)\big[\mu^*(4R_n,u) + \end{aligned}$$

$$+ 8(9(1-\Delta)^{-1}/8)\varepsilon_n R_n^\nu] \log(t/(4R_n)) + c = (1-\Delta)N(t,U_n) + c.$$

Thus, from (38) and a similar expression for $U_n(r)$, $V_n(r) \le (1-\Delta)U_n(r) + c$. It follows from (36) that, for all r,

$$W_n(r) \ge ((1-\Delta)^{-1} + K)V_n(r) - c.$$

From (37) then,

$$\begin{aligned} A(r,\phi) - KB(r,\phi) &> W_n(-r) - \frac{K+1-\Delta}{1+(1-\Delta)K} W_n(r) - c \\ &= W_n(-r) - \cos\pi\nu W_n(r) - c, \end{aligned} \tag{39}$$

for $0 \le r \le R_n$. We apply Theorem 2 (ii) to w, with $C = 9(1-\Delta)^{-1}/8 + K$ and α_0 chosen so that $a_{\alpha,n} \to \infty$ as $r_{\alpha,n} \to \infty$. From (12) and (13),

$$W_n(-r_{\alpha,n}) > \pi\nu(\cot\pi\nu)N(r_{\alpha,n}, w) + (1 - \pi\nu\cot\pi\nu)a_{\alpha,n}, \tag{40}$$

$$W_n(r_{\alpha,n}) < \pi\nu(\csc\pi\nu)N(r_{\alpha,n}, w) + (1 - \pi\nu\csc\pi\nu)a_{\alpha,n}. \tag{41}$$

Recalling that $r_{\alpha,n} \le R_n$, and substituting these inequalities into (39), we obtain

$$A(r_{\alpha,n},\phi) - KB(r_{\alpha,n},\phi) > (1-\cos\pi\nu)a_{\alpha,n} - c > 0 \tag{42}$$

for all large $r_{\alpha,n}$. Thus $A(r,\phi)/B(r,\phi) > K > \kappa(\sigma,\delta)$, from (34), for all r in a set of upper logarithmic density at least $1-\lambda/\nu$. Since we may take ν as close to σ as we please, we conclude that (2) holds in a set of upper logarithmic density at least $1-\lambda/\sigma$, which completes the proof of Theorem 1.

To prove that (2) holds for all sufficiently large r under the assumption (4), note that the exceptional set E where (2) fails comes from the exceptional set for $w = u + Kv$. By (4), $\mu^*(t,w) = \varepsilon(t)t^\lambda$ is continuous for all large t, and $\varepsilon(t) \downarrow 0$ as $t \to \infty$. Thus, if α is sufficiently small and $t \in (R,\infty)$ for some positive number R, then, provided R is sufficiently large, the equation $\mu^*(t,w) = \alpha\nu t^\nu$ has precisely one solution. By the argument surrounding (29), we have $r_{\alpha,n}^- = r_{\alpha,n}^+$ for all such α and all large enough n. Thus, since the exceptional set is contained in the intervals $(r_{\alpha,n}^-, r_{\alpha,n}^+)$, $E \cap (R,\infty) = \emptyset$ for sufficiently large R.

The proof of Theorem 3 is almost identical to that of Theorem 1. One needs to set $\nu = \lambda$ in Theorem 2, make use of (14) and set $k = 1-\Delta$ in (35).

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