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NEVANLINNA CLASS CONTAINS FUNCTIONS WHOSE SPHERICAL DERIVATIVES GROW ARBITRARILY FAST

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Abstract. It is shown that for any given increasing function $\varphi\colon [0,1) \to (0,\infty)$ there exists a meromorphic function f_φ of bounded Nevanlinna characteristic such that its spherical derivative $f_\varphi^\#(z) = |f_\varphi'(z)|/(1+|f_\varphi(z)|^2)$ satisfies $\limsup_{|z|\to 1^-} f_\varphi^\#(z)/\varphi(|z|) = \infty$. Such a function is constructed by using Blaschke products and the desired property is proved by normal family arguments. This study is inspired by results on non-normal Dirichlet and Blaschke quotients due to Yamashita.

1. Introduction and results

The class $\mathscr{N}$ of normal functions consists of those meromorphic functions f in the unit disc $\mathbf{D} := \{z : |z| < 1\}$ for which the family $\{f \circ \tau\}$, where τ is a Möbius transformation of $\mathbf{D}$, is normal in the sense of Montel (i.e. ∞ is a permitted limit). Lehto and Virtanen [4] showed that a meromorphic function f is normal if and only if its spherical derivative $f^\#(z) := |f'(z)|/(1+|f(z)|^2)$ satisfies

$$\sup_{z\in\mathbf{D}} f^\#(z)(1-|z|^2) < \infty.$$

The Nevanlinna class N consists of those meromorphic functions f in $\mathbf{D}$ for which the Nevanlinna characteristic $T(r,f)$ remains bounded as $r \to 1^-$. It is well known that every such function can be represented as a quotient of two bounded analytic functions, and therefore the zeros and poles of functions in N are neatly characterized by the Blaschke condition [2]. For a given sequence $\{z_n\}_{n=1}^\infty$ of points in $\mathbf{D}$ for which $\sum_{n=1}^\infty (1-|z_n|^2)$ converges (with the convention $z_n/|z_n| = 1$ for $z_n = 0$), the Blaschke product associated with the sequence $\{z_n\}_{n=1}^\infty$ is defined as

$$B(z) := \prod_{n=1}^\infty \frac{|z_n|}{z_n}\frac{z_n - z}{1-\overline{z}_n z}.$$

Lehto and Virtanen [4] showed that every $f \in \mathscr{N}$ satisfies

$$T(r,f) = \mathscr{O}\left(\log\frac{1}{1-r}\right), \quad r \to 1^-,$$

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so that the order of growth of any normal function is zero. In contrast to this, Yamashita [7] constructed non-normal functions in the Nevanlinna class via Dirichlet and Blaschke quotients. The purpose of this note is to show that for any given increasing function $\varphi\colon [0,1) \to (0,\infty)$ there is a function f_φ in the Nevanlinna class such that its spherical derivative $f_\varphi^{\#}(z)$ exceeds the growth of $\varphi(|z|)$ as $|z| \to 1^-$.

Theorem 1. *Let $\varphi\colon [0,1) \to (0,\infty)$ be an increasing function. Then there exists a function f_φ in the Nevanlinna class N such that*

$$\limsup_{|z|\to 1^-} \frac{f_\varphi^{\#}(z)}{\varphi(|z|)} = \infty. \tag{1}$$

Theorem 1 is proved by constructing Blaschke products B_1 and B_2 with real positive zeros $\{z_n\}_{n=1}^\infty$ and $\{w_n\}_{n=1}^\infty$ such that the distance between z_n and w_n tends to zero sufficiently fast depending on the given function φ. The faster the $\varphi(r)$ grows as $r \to 1^-$, the faster the points z_n and w_n must approach to each other when $n \to \infty$. The property (1) for the quotient $f_\varphi := B_1/B_2$ is then established by normal family arguments. The density of zeros is not essential for the construction, so B_1 and B_2 can be chosen such that their zero-sequences $\{z_n\}_{n=1}^\infty$ and $\{w_n\}_{n=1}^\infty$ are both separated. In fact, it is shown that B_1 and B_2 can be chosen such that they both belong to the Möbius invariant Q_p-space for all $p > 0$. For $0 < p < \infty$, the Q_p-space [6] consists of those analytic functions f in $\mathbf{D}$ for which

$$\|f\|_{Q_p}^2 := \sup_{a\in\mathbf{D}} \int_{\mathbf{D}} |f'(z)|^2 \left(\log\left|\frac{1-\bar{a}z}{a-z}\right|\right)^p dA(z) < \infty.$$

Since the zeros of B_1 and B_2 are real and positive, the functions $f_i(z) := (1-z)^2 B_i(z)$, $i = 1,2$, satisfy $\sup_{z\in\mathbf{D}} |f_i'(z)| \le C$ for some positive constant C. Therefore f_1 and f_2 both belong to the classical Besov space B^p for all $1 < p < \infty$. Recall that the Besov space B^p consists of those analytic functions f in $\mathbf{D}$ for which

$$\int_{\mathbf{D}} |f'(z)|^p (1-|z|^2)^{p-2}\, dA(z) < \infty.$$

This observation for $p = 2$ shows that there are Dirichlet quotients which are not just non-normal as Yamashita [7] showed, but whose spherical derivatives exceed the pregiven increasing function φ in growth. In particular, for any $\alpha > 1$, there are non-α-normal Dirichlet and Blaschke quotients.

The rest of this note is devoted to the proof of Theorem 1.

2. Proof of Theorem 1

The first step in the proof is the following lemma.

Lemma 2. *Let $\varphi\colon [0,1) \to (0,\infty)$ be an increasing function. Then there exists an increasing twice differentiable function $\Phi\colon (0,1) \to (0,\infty)$ such that $1/\Phi$ is convex and $\lim_{r\to 1^-} \varphi(r)/\Phi(r) = 0$.*

Proof. Let $\varphi\colon [0,1) \to (0,\infty)$ be increasing. Consider the functions

$$\psi(r) := \frac{1}{\varphi(r)}, \quad \psi_1(r) := r\int_r^1 \frac{\psi(s)}{s^2}\, ds \quad \text{and} \quad \psi_2(r) := r\int_r^1 \frac{\psi_1(s)}{s^2}\, ds,$$

and define $\Phi := 1/\psi_2$. Then $\psi_1'(r) \le -\psi(r)$, $\psi_2'(r) \le -\psi_1(r)$ and $\psi_2''(r) = -\psi_1'(r)/r$. Therefore $\Phi\colon (0,1) \to (0,\infty)$ is increasing and twice differentiable such that $1/\Phi$ is convex. Moreover,

$$\frac{\varphi(r)}{\Phi(r)} = \frac{\psi_2(r)}{\psi_1(r)}\frac{\psi_1(r)}{\psi(r)} = \frac{r}{\psi_1(r)}\int_r^1 \frac{\psi_1(s)}{s^2}\,ds\,\frac{r}{\psi(r)}\int_r^1 \frac{\psi(s)}{s^2}\,ds$$
$$\leq \left(r\int_r^1 \frac{ds}{s^2}\right)^2 = (1-r)^2 \to 0, \quad r \to 1^-,$$

as desired. □

By Lemma 2 we may assume that $\varphi\colon [0,1) \to (0,\infty)$ is differentiable and increasing such that $1/\varphi$ is convex. Without loss of generality, we may also assume that $\lim_{r\to 1^-}\varphi(r)(1-r) = \infty$. For such a φ, let $\mathscr{N}^\varphi$ denote the set of those meromorphic functions f in $\mathbf{D}$ for which

$$f^\#(z) = \mathscr{O}(\varphi(|z|)), \quad |z| \to 1^-.$$

The second step in the proof of Theorem 1 is the following characterization of functions in $\mathscr{N}^\varphi$ in terms of normal families. For analogous results for normal and α-normal functions, see [4] and [3, 5].

Lemma 3. *Let f be a meromorphic function in $\mathbf{D}$, and let $\varphi\colon [0,1) \to (0,\infty)$ be a differentiable increasing function such that $1/\varphi$ convex and $\lim_{r\to 1^-}\varphi(r)(1-r) = \infty$. Then $f \in \mathscr{N}^\varphi$ if and only if the family $\{f(z_n + z/\varphi(|z_n|)) : n \in \mathbf{N}\}$ is normal in $\mathbf{C}$ for any sequence $\{z_n\}_{n=1}^\infty$ of points in $\mathbf{D}$ such that $\lim_{n\to\infty}|z_n| = 1$.*

Proof. Let first $f \in \mathscr{N}^\varphi$ and let $\{z_n\}_{n=1}^\infty$ be a sequence of points in $\mathbf{D}$ such that $\lim_{n\to\infty}|z_n| = 1$. Let $z \in D(0,r) := \{w : |w| \leq r\}$, and define $\phi_a(z) := a + z/\varphi(|a|)$ for $a \in \mathbf{D}$. Since $f \in \mathscr{N}^\varphi$ and $\lim_{r\to 1^-}\varphi(r)(1-r) = \infty$, there exists a positive constant C and an $N_r \in \mathbf{N}$ such that

$$(f \circ \phi_{z_n})^\#(z) = f^\#(\phi_{z_n}(z))\,(\varphi(|z_n|))^{-1} \leq C\varphi(|\phi_{z_n}(z)|)\,(\varphi(|z_n|))^{-1}$$

for all $n \geq N_r$ and $z \in D(0,r)$. Denote $\psi := 1/\varphi$ so that $\psi\colon (0,1) \to (0,\infty)$ is differentiable, decreasing and convex by the assumptions. Then

$$\lim_{n\to\infty}\sup_{|z|\leq r}\varphi(|\phi_{z_n}(z)|)\,(\varphi(|z_n|))^{-1} \leq \lim_{n\to\infty}\frac{\psi(|z_n|)}{\psi(|z_n| + r\psi(|z_n|))}$$
$$\leq \lim_{n\to\infty}\frac{1}{1+\psi'(|z_n|)r} = 1,$$

and it follows that $(f \circ \phi_{z_n})^\#(z)$ is uniformly bounded in $D(0,r)$ for all $n \geq N_r$. Therefore Marty's theorem implies that the family $\{f \circ \phi_{z_n} : n \in \mathbf{N}\}$ is normal in $\mathbf{C}$.

Assume now that $\{f \circ \phi_{z_n}\}$ is normal for any sequence $\{z_n\}_{n=1}^\infty$ of points in $\mathbf{D}$ such that $\lim_{n\to\infty}|z_n| = 1$. Assume on the contrary to the assertion that $f \notin \mathscr{N}^\varphi$. Then there exists a sequence $\{w_n\}_{n=1}^\infty$ of points in $\mathbf{D}$ such that $\lim_{n\to\infty}|w_n| = 1$ and

$$\frac{f^\#(w_n)}{\varphi(|w_n|)} \to \infty, \quad n \to \infty.$$

By Marty's theorem there exists a positive constant C such that

$$\frac{f^\#(w_n)}{\varphi(|w_n|)} = (f \circ \phi_{w_n})^\#(0) \leq C$$

for all $n \in \mathbf{N}$. This is clearly a contradiction, and so $f \in \mathscr{N}^\varphi$. □

To prove Theorem 1, let the sequences $\{z_n\}_{n=1}^\infty$ and $\{w_n\}_{n=1}^\infty$ be defined by $z_n := 1-2^{-n}$ and $w_n := z_n + \exp(-\varphi(1-2^{-n}))$. Then

$$\sum_{n=1}^\infty (1-|w_n|) \le \sum_{n=1}^\infty (1-|z_n|) = \sum_{n=1}^\infty 2^{-n} = 1,$$

so the Blaschke products B_1 and B_2 associated with the sequences $\{z_n\}_{n=1}^\infty$ and $\{w_n\}_{n=1}^\infty$ converge. Since

$$\lim_{n\to\infty} \frac{1-|w_n|}{1-|w_{n+1}|} = 2 > 1 \quad \text{and} \quad \frac{1-|z_n|}{1-|z_{n+1}|} = 2 > 1$$

for all $n \in \mathbf{N}$, the sequences $\{1-|z_n|\}_{n=1}^\infty$ and $\{1-|w_n|\}_{n=1}^\infty$ are not asymptotically concentrated, and therefore B_1 and B_2 both belong to $\bigcap_{p>0} Q_p$ by [1, Theorem 1]. Define $f_i(z) := (1-z)^2 B_i(z)$ for $i = 1, 2$. Then $|f_i'(z)|$ is uniformly bounded in $\mathbf{D}$ for $i = 1, 2$, and therefore f_1 and f_2 both are bounded analytic functions and belong to $\bigcap_{p>1} B^p$. Consider the quotient $f := f_1/f_2 = B_1/B_2$. By Lemma 3 it suffices to show that the family $\{f \circ \phi_{z_n} : n \in \mathbf{N}\}$ is not normal in a neighborhood of the origin. Consider the sequence $\{Z_n\}_{n=1}^\infty$ defined by $Z_n := (w_n - z_n)\varphi(|z_n|)$. Clearly, $|Z_n| = \exp(-\varphi(1-2^{-n}))\varphi(1-2^{-n}) \to 0$, as $n \to \infty$, so, for a given $0 < r < 1$, there exists an $N_r \in \mathbf{N}$ such that the points Z_n belong to $D(0,r)$ for all $n \ge N_r$. Now $B_1(z_n) = 0$ for all $n \in \mathbf{N}$, and therefore $(f \circ \phi_{z_n})(0) = f(z_n) = 0$ for all $n \in \mathbf{N}$. On the other hand, $B_2(w_n) = 0$, and therefore $(f \circ \phi_{z_n})(Z_n) = \infty$ for all $n \in \mathbf{N}$. It follows that $\{f \circ \phi_{z_n} : n \in \mathbf{N}\}$ is not normal in any neighborhood of the origin, and thus $f \notin \mathscr{N}^\varphi$ by Lemma 3. Therefore

$$\limsup_{|z|\to 1^-} \frac{f^\#(z)}{\varphi(|z|)} = \infty,$$

and we are done.

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