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MEAN ERGODIC OPERATORS IN FRÉCHET SPACES

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Abstract. Classical results of Pelczynski and of Zippin concerning bases in Banach spaces are extended to the Fréchet space setting, thus answering a question posed by Kalton almost 40 years ago. Equipped with these results, we prove that a Fréchet space with a basis is reflexive (resp. Montel) if and only if every power bounded operator is mean ergodic (resp. uniformly mean ergodic). New techniques are developed and many examples in classical Fréchet spaces are exhibited.

1. Introduction and statement of the main results

A continuous linear operator T in a Banach space X (or a locally convex Hausdorff space, briefly lcHs) is called *mean ergodic* if the limits

$$Px := \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} T^m x, \quad x \in X, \tag{1.1}$$

exist (in the topology of X). von Neumann (1931) proved that unitary operators in Hilbert space are mean ergodic. Ever since, intensive research has been undertaken concerning mean ergodic operators and their applications; for the period up to the 1980's see [19, Ch. VIII Section 4], [25, Ch. XVIII], [33, Ch. 2], and the references therein. A continuous linear operator T in X (the space of all such operators is denoted by $\mathscr{L}(X)$) is called *power bounded* if $\sup_{m\geq 0} \|T^m\| < \infty$. Such a T is mean ergodic if and only if

$$X = \{u \in X : u = Tu\} \oplus \overline{\operatorname{Im}(I-T)}, \tag{1.2}$$

where $\operatorname{Im}(I-T)$ denotes the range of $I-T$ and the bar denotes "closure in X".

It quickly became evident that there was an intimate connection between geometric properties of the underlying Banach space X and mean ergodic operators on X. The space X itself is called *mean ergodic* if every power bounded operator $T \in \mathscr{L}(X)$ satisfies (1.1). As a sample, F. Riesz (1938) showed that all L^p-spaces $(1 < p < \infty)$ are mean ergodic. In 1939 Lorch proved that all reflexive Banach spaces

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are mean ergodic. Sucheston, [51], posed the following question, concerning the converse of Lorch's result: *If every contraction in a Banach space X is mean ergodic, is X reflexive*? In 1997, Emel'yanov showed that every mean ergodic Banach lattice is reflexive, [20]. A major breakthrough came in 2001 in the penetrating paper of Fonf, Lin and Wojtaszczyk, [23]. By using the theory of bases in Banach spaces, they were able to establish (amongst other things) the following characterizations for a Banach space X *with a basis*:

(i) *X is finite dimensional if and only if every power bounded operator is uniformly mean ergodic* (*i.e. the limit* (1.1) *exists uniformly on the unit ball of* X).
(ii) *X is reflexive if and only if every power bounded operator is mean ergodic* (*i.e. if and only if X is mean ergodic*).

Essential to their arguments are two classical results on bases in Banach spaces. Namely, a result of Pelczynski, [15, p. 54], [39, p. 77], stating that if X is non-reflexive, then it possesses a non-reflexive (separable), closed subspace with a basis, and a result of Zippin, [55], stating that if a non-reflexive Banach space has a basis, then it has a non-shrinking basis. The first result for (a special class of) power bounded operators T on certain lcHs' X is due to Altman, [4]. The restriction on T that Altman imposed (a weak compactness condition) was later removed by Yosida [54, Ch. VIII]. For quasi-compact operators and R-endomorphisms in lcHs', a mean ergodic theorem due to Pietsch is also available, [42]. Our aim is to present several results in Fréchet spaces X (some being in the spirit of [23]) which connect geometric/analytic properties of the underlying space X to mean ergodicity of operators acting on X. To be more precise, $T \in \mathscr{L}(X)$ is called *power bounded* if $\{T^m\}_{m=0}^{\infty}$ is an equicontinuous subset of $\mathscr{L}(X)$. Since the requirement (1.1) is not dependent on X being normable, an operator $T \in \mathscr{L}(X)$ is again called *mean ergodic* whenever it satisfies (1.1).

Technical terms concerning lcHs' X and various types of bases will be formally defined later. A general reference is [40]. Let us recall that if Γ_X is a system of continuous seminorms generating the topology of X, then the strong operator topology τ_s in $\mathscr{L}(X)$ is determined by the seminorms

$$q_x(S) := q(Sx), \quad S \in \mathscr{L}(X), \tag{1.3}$$

for each $x \in X$ and $q \in \Gamma_X$ (in which case we write $\mathscr{L}_s(X)$), and the uniform operator topology τ_b in $\mathscr{L}(X)$ is defined by the family of seminorms

$$q_B(S) := \sup_{x \in B} q(Sx), \quad S \in \mathscr{L}(X), \tag{1.4}$$

for each $q \in \Gamma_X$ and bounded set $B \subseteq X$ (in which case we write $\mathscr{L}_b(X)$). One refers to τ_b as the topology of uniform convergence on the bounded sets of X. If X is a Banach space, then τ_b is precisely the operator norm topology on $\mathscr{L}(X)$. A *Fréchet space* is a complete metrizable locally convex space. If X is a Fréchet space, then Γ_X can be taken countable. The result of Pelczynski and that of Zippin mentioned above, both crucial if any headway is to be made in the non-normable setting, have been extended, each one being of independent interest.

Theorem 1.1. *Every non-reflexive Fréchet space contains a non-reflexive, closed subspace with a basis.*

Theorem 1.2. *A complete, barrelled lcHs with a basis is reflexive if and only if every basis is shrinking if and only if every basis is boundedly complete.*

Theorem 1.2 clearly implies Zippin's result in the Banach space setting. It should be noted that Theorem 1.2 provides a positive answer to a question posed by Kalton almost 40 years ago, [28, p. 265].

Given any $T \in \mathscr{L}(X)$, X an arbitrary lcHs, let us introduce the notation

$$T_{[n]} := \frac{1}{n}\sum_{m=1}^{n} T^m, \quad n \in \mathbf{N}, \tag{1.5}$$

for the Cesàro means of T. The operator T is called *mean ergodic* when $\{T_{[n]}\}_{n=1}^{\infty}$ is a convergent sequence in $\mathscr{L}_s(X)$. In case X is barrelled it is enough to assume that $\lim_{n\to\infty} T_{[n]}x$ exists in X for every $x \in X$; see (1.1). If $\{T_{[n]}\}_{n=0}^{\infty}$ happens to be convergent in $\mathscr{L}_b(X)$, then T will be called *uniformly mean ergodic*. The space X itself is called uniformly mean ergodic if every power bounded operator on X is uniformly mean ergodic. Equipped with Theorems 1.1 and 1.2 it is possible to establish the following two facts.

Theorem 1.3. *Let X be a Fréchet space with a basis. Then X is Montel if and only if every power bounded, mean ergodic operator on X is uniformly mean ergodic, that is, if and only if X is uniformly mean ergodic.*

Theorem 1.3 can be interpreted as an extension of (i) above. The following extension of (ii) above is also presented.

Theorem 1.4. *A Fréchet space X with a basis is reflexive if and only if every power bounded operator on X is mean ergodic.*

The mean ergodicity and uniform mean ergodicity of such classical Fréchet spaces as the Köthe echelon spaces $\lambda_0(A)$ and $\lambda_p(A)$, for $1 \le p \le \infty$, the sequence spaces $\ell^{p+} := \bigcap_{q>p} \ell^q$ for $1 \le p < \infty$, the Lebesgue spaces $L_{p-} := \bigcap_{1\le r<p} L^r([0,1])$ for $p \in (1,\infty)$ and the spaces $L^p_{\ell oc}(\mathbf{R})$ of measurable locally p-th integrable functions, for $1 \le p < \infty$, are completely determined.

For the following fact in Banach spaces, see [23, Theorem 1].

Theorem 1.5. *Let X be any Fréchet space which admits a non-shrinking Schauder decomposition. Then there exists a power bounded operator on X which is not mean ergodic* (*i.e. X is not mean ergodic*).

Our final sample result is the following one. For the Banach space case it follows from a result of Emel'yanov and Wolff, [21, Theorem 3.1].

Theorem 1.6. *Let X be a Fréchet space which contains an isomorphic copy of the Banach space c_0. Then there exists a power bounded operator on X which is not mean ergodic* (*i.e. X is not mean ergodic*).

2. Preliminary results and some examples

Given any lcHs X and $T \in \mathscr{L}(X)$ we always have the identities

$$(I-T)T_{[n]} = T_{[n]}(I-T) = \frac{1}{n}(T - T^{n+1}), \quad n \in \mathbf{N}, \tag{2.1}$$

and also (setting $T_{[0]} := I$, the identity operator on X) that

$$\frac{1}{n}T^n = T_{[n]} - \frac{(n-1)}{n}T_{[n-1]}, \quad n \in \mathbf{N}. \tag{2.2}$$

Some authors prefer to use $\frac{1}{n}\sum_{m=0}^{n-1} T^m$ in place of $T_{[n]}$; since

$$T_{[n]} = T\Big(\frac{1}{n}\sum_{m=0}^{n-1} T^m\Big) = \frac{1}{n}(T^n - I) + \frac{1}{n}\sum_{m=0}^{n-1} T^m, \quad n \in \mathbf{N},$$

this leads to identical results. Recall the notation $\ker(T) := \{x \in X : Tx = 0\}$. The following result is due to Yosida, [54, Ch. VIII, §3].

Proposition 2.1. *Let X be a lcHs and $T \in \mathscr{L}(X)$ be power bounded. Then*

$$\overline{\mathrm{Im}(I-T)} = \{x \in X : \lim_{n\to\infty} T_{[n]}x = 0\} \tag{2.3}$$

and so, in particular,

$$\overline{\mathrm{Im}(I-T)} \cap \ker(I-T) = \{0\}. \tag{2.4}$$

A subset A of a lcHs X is called *relatively sequentially $\sigma(X,X')$-compact* if every sequence in A contains a subsequence which is $\sigma(X,X')$-convergent to some element of X. As a consequence of Proposition 2.1, Yosida established the following mean ergodic theorem, [54, Ch. VIII; §3].

Theorem 2.2. *Let X be a lcHs and $T \in \mathscr{L}(X)$ be a power bounded operator such that*

$$\{T_{[n]}x\}_{n=1}^{\infty} \text{ is relatively sequentially } \sigma(X,X')\text{-compact}, \ \forall x \in X. \tag{2.5}$$

Then T is mean ergodic and the operator $P = \lim_{n\to\infty} T_{[n]}$ (limit existing in $\mathscr{L}_s(X)$) is a projection satisfying $\mathrm{Im}(P) = \ker(I-T)$ *and* $\ker(P) = \overline{\mathrm{Im}(I-T)}$ *with*

$$X = \overline{\mathrm{Im}(I-T)} \oplus \ker(I-T). \tag{2.6}$$

For X a Banach space, the above result is the customary one, that is, T is required to satisfy $\sup_n \|T^n\| < \infty$, which clearly implies

$$\lim_{n\to\infty} \frac{1}{n}T^n = 0, \quad \text{in } \mathscr{L}_s(X). \tag{2.7}$$

By the principle of uniform boundedness, every mean ergodic operator T necessarily satisfies

$$\sup_n \|T_{[n]}\| < \infty, \tag{2.8}$$

that is, T is *Cesàro bounded* (in the terminology of [33, Ch. 2]). On the other hand, (2.8) is also sufficient for mean ergodicity of T whenever T additionally satisfies (2.5) and (2.7), [33, p. 72]. Hille exhibited a classical kernel operator T in $L^1([0,1])$ which fails to be power bounded (actually, $\|T^n\| = 0(n^{1/4})$) but, nevertheless, is mean ergodic (and hence, satisfies (2.8)), [24, §6]. Actually, for this example, it is shown in [24] that $T_{[n]} \to 0$ in $\mathscr{L}_s(X)$ as $n \to \infty$ which then implies, via (2.2), that (2.7) holds. There also exist operators $T \in \mathscr{L}(X)$ which satisfy (2.8) but fail to satisfy (2.7), even with X a finite dimensional space, [22, Example 4.7]. In view of such features (see also [33, p. 85], [43], for further relevant comments), it is not surprising that some authors take the viewpoint that "it is not the power boundedness of T

but, rather the existence of the limit (2.7), which is essential in ergodic theory"; this viewpoint is explicitly stated in [18, p. 186], [38, p. 214], for example. The proof of the following version of Proposition 2.1 is routine.

Proposition 2.3. *Let X be a barrelled lcHs. Let the operator $T \in \mathscr{L}(X)$ satisfy* (2.7) *and the condition*

$$\{T_{[n]}x\}_{n=1}^{\infty} \text{ is a bounded set in } X, \quad \forall x \in X. \tag{2.9}$$

Then T satisfies both (2.3) *and* (2.4).

Given $T \in \mathscr{L}(X)$, its dual operator $T^t\colon X' \to X'$ is defined by $\langle Tx, x'\rangle = \langle x, T^t x'\rangle$ for all $x \in X$ and $x' \in X'$. Let us denote X equipped with its weak topology $\sigma(X, X')$ by X_σ.

Theorem 2.4. *Let X be a barrelled lcHs and $T \in \mathscr{L}(X)$. Then T is mean ergodic if and only if it satisfies both* (2.5) *and* (2.7).

Setting $P := \lim_{n\to\infty} T_{[n]}$ (existence of the limit in $\mathscr{L}_s(X)$), the operator P is a projection which commutes with T and satisfies $\mathrm{Im}(P) = \ker(I - T)$ *and* $\ker(P) = \overline{\mathrm{Im}(I - T)}$. *Moreover, X satisfies* (2.6).

Proof. If T is mean ergodic, then it follows from (2.2) that (2.7) holds. Moreover, (2.5) is also satisfied. Conversely, suppose that T satisfies (2.5) and (2.7). Fix $x \in X$. By (2.5) there exists $x_0 \in X$ and a subsequence such that $T_{[n_k]}x \to x_0$ in X_σ. It follows from (2.1) and (2.7) that

$$\lim_{n\to\infty} (TT_{[n]} - T_{[n]})x = \lim_{n\to\infty} \Big(\frac{(n+1)}{n} \cdot \frac{1}{(n+1)} T^{n+1}x - \frac{1}{n}Tx\Big) = 0, \tag{2.10}$$

in X, and hence, that $(TT_{[n_k]} - T_{[n_k]})x \to 0$ in X_σ. So, $TT_{[n_k]}x \to x_0$ in X_σ. According to (2.10), for each $x' \in X'$, we then have

$$\lim_{k\to\infty} \langle TT_{[n_k]}x, x'\rangle = \lim_{k\to\infty} \langle T_{[n_k]}x, x'\rangle = \langle x_0, x'\rangle.$$

Using the identities

$$\langle TT_{[n_k]}x, x'\rangle = \langle T_{[n_k]}x, T^t x'\rangle, \quad k \in \mathbf{N},$$

it follows that $\langle x_0, x'\rangle = \langle x_0, T^t x'\rangle = \langle Tx_0, x'\rangle$. Since $x' \in X'$ is arbitrary, we have $Tx_0 = x_0$ and hence, via (1.5), that

$$T_{[n]}x_0 = x_0, \quad n \in \mathbf{N}. \tag{2.11}$$

Accordingly, we have

$$x - x_0 = x - \lim_{k\to\infty} T_{[n_k]}x = \lim_{k\to\infty} (x - T_{[n_k]}x) \tag{2.12}$$

in X_σ. Now, since $x - T_{[n]}x = (I - T)\sum_{m=1}^{n} \frac{1}{n}(I + T + \ldots + T^{m-1})x \in \mathrm{Im}(I - T), n \in \mathbf{N}$, we conclude that $\{x - T_{[n_k]}x\}_{k=1}^{\infty} \subseteq \mathrm{Im}(I - T)$. That is, $x - x_0$ belongs to the closure of $\mathrm{Im}(I - T)$ in X_σ which, of course, coincides with $\overline{\mathrm{Im}(I - T)}$, [31, p. 236]. Since (2.5) implies (2.9), Proposition 2.3 is available and hence, (2.3) holds. It follows, because $x - x_0 \in \overline{\mathrm{Im}(I - T)}$, that $T_{[n]}(x - x_0) \to 0$ in X. Combined with (2.11) this yields $\lim_{n\to\infty} T_{[n]}x = x_0$ (in X). So, we can define a linear map $P\colon X \to X$ by

$$Px := \lim_{n\to\infty} T_{[n]}x, \quad x \in X. \tag{2.13}$$

Since X is barrelled, actually $P \in \mathscr{L}(X)$, [32, p. 141]; in particular, $T_{[n]} \to P$ in $\mathscr{L}_s(X)$, that is, T is mean ergodic.

Note that $Px = x_0$ implies that $TPx = Px$ for each $x \in X$, that is, $TP = P$. It then follows that $T_{[n]}P = P$ for all $n \in \mathbf{N}$ and hence, that $P^2x = \lim_{n\to\infty} T_{[n]}(Px) = Px$, for $x \in X$, and P is a projection. On the other hand, (2.1) and (2.7) imply that $T_{[n]} - T_{[n]}T = \frac{1}{n}(T - T^{n+1}) \to 0$ in $\mathscr{L}_s(X)$, that is, $P = PT$ (see (2.13)). So, P and T commute.

According to (2.3) and (2.13) we have $\overline{\mathrm{Im}(I-T)} = \ker(P)$. It is routine to verify that $\ker(I-T) = \mathrm{Im}(P)$. Since P is a projection, (2.6) follows. □

Corollary 2.5. *Let X be a barrelled lcHs such that each bounded set in X is relatively sequentially $\sigma(X, X')$-compact. If $T \in \mathscr{L}(X)$ satisfies* (2.7) *and* (2.9), *then T is mean ergodic.*

Proof. Under the given hypothesis on X, condition (2.9) implies (2.5) and hence, Theorem 2.4 applies. □

Remark 2.6. (i) In any lcHs X, conditions (2.7) and (2.9) are satisfied by *every* power bounded operator $T \in \mathscr{L}(X)$. Indeed, fix $q \in \Gamma_X$. By the assumed equicontinuity of $\{T^n\}_{n=0}^\infty$ there exists $p \in \Gamma_X$ satisfying

$$q(T^n x) \le p(x), \quad x \in X,\ n \in \mathbf{N}. \tag{2.14}$$

It follows that $q(T_{[n]}x) \le \frac{1}{n}\sum_{m=1}^n q(T^m x) \le p(x)$, for $x \in X$, $n \in \mathbf{N}$, which clearly implies (2.9). Moreover, (2.14) gives that $q(\frac{1}{n}T^n x) \le \frac{1}{n}p(x)$ for all $x \in X$ and $n \in \mathbf{N}$ which yields (2.7).

(ii) Reflexive lcHs' which are either Fréchet, (DF)- or (LF)-spaces satisfy the hypotheses of Corollary 2.5; see [10, Theorem 11, Examples 1,2].

Corollary 2.7. *Every reflexive lcHs which is a* (DF)- *or* (LF)-*space* (*in particular, any reflexive Fréchet space*) *is mean ergodic.*

Proposition 2.8. *Let X be any* (DF)- *or* (LF)-*space* (*in particular, any Fréchet space*) *which is Montel. Then X is uniformly mean ergodic.*

Proof. Let $T \in \mathscr{L}(X)$ be power bounded. Since X is reflexive, [31, p. 369], Corollary 2.7 implies that T is mean ergodic. Set $P := \lim_{n\to\infty} T_{[n]}$, with the limit existing in $\mathscr{L}_s(X)$. Since X is barrelled and $H := \{P\} \cup \{T_{[n]}\}_{n=1}^\infty$ is bounded in $\mathscr{L}_s(X)$, it follows that H is an equicontinuous subset of $\mathscr{L}(X)$, [32, (2) p. 137]. So, τ_s and the topology τ_c in $\mathscr{L}(X)$ of uniform convergence on the precompact sets of X coincide on H, [32, (2) p. 139]. But, bounded sets in Montel spaces are relatively compact (per definition) and so $T_{[n]} \to P$ in $\mathscr{L}_b(X)$, that is, T is uniformly mean ergodic. □

Let Λ be an index set, always assumed to be countable. Any increasing sequence $A = (a_n)_n$ of functions $a_n\colon \Lambda \to (0,\infty)$ is called a *Köthe matrix* on Λ, where by increasing we mean $0 < a_n(i) \le a_{n+1}(i)$ for all $i \in \Lambda$ and $n \in \mathbf{N}$. To each $p \in [1,\infty)$ is associated the linear space

$$\lambda_p(A) := \{x \in \mathbf{C}^\Lambda : q_n^{(p)}(x) := \Big(\sum_{i\in\Lambda} |a_n(i)x_i|^p\Big)^{1/p} < \infty, \quad \forall n \in \mathbf{N}\}. \tag{2.15}$$

We also require the linear space

$$\lambda_\infty(A) := \{x \in \mathbf{C}^\Lambda : q_n^{(\infty)}(x) := \sup_{i\in\Lambda} a_n(i)|x_i| < \infty,\ \forall n \in \mathbf{N}\} \tag{2.16}$$

and its closed subspace (equipped with the relative topology)

$$\lambda_0(A) := \{x \in \mathbf{C}^\Lambda : \lim_{i\to\infty} a_n(i)x_i = 0, \quad \forall n \in \mathbf{N}\}.$$

Elements $x \in \mathbf{C}^\Lambda$ are denoted by $x = (x_i)$. The spaces $\lambda_p(A)$, for $p \in [1,\infty]$, are called *Köthe echelon spaces* (of order p); they are all Fréchet spaces (separable if $p \neq \infty$ and reflexive if $p \neq 1, \infty$) relative to the increasing sequence of seminorms $q_1^{(p)} \leq q_2^{(p)} \leq \ldots$. For the general theory of such spaces we refer to [31], [40], for example.

Proposition 2.9. *Let A be a Köthe matrix and $1 < p < \infty$. The reflexive Fréchet space $\lambda_p(A)$ is Montel if and only if it is uniformly mean ergodic.*

Proof. Suppose that $\lambda_p(A)$ is not Montel. According to the proof of Proposition 2.5(ii) in [8] there exists a direct sum decomposition $\lambda_p(A) = Y \oplus Z$ with Z isomorphic to the Banach space ℓ^p. Yahdi exhibited a power bounded operator on ℓ^2 which fails to be uniformly mean ergodic, [53, Example 2.4]. It is routine to check that the "same operator", say S, now considered to be acting in ℓ^p, is also power bounded but not uniformly mean ergodic. Denote the operator S, when transferred to Z, by R. Then $T\colon Y \oplus Z \to Y \oplus Z$ given by $T(y,z) := (0, Rz)$, for $(y,z) \in Y \oplus Z$, is a power bounded operator on $\lambda_p(A)$ which fails to be uniformly mean ergodic. In particular the Köthe echelon space $\lambda_p(A)$ is not uniformly mean ergodic. $\square$

Recall the separable Fréchet spaces $L_{p-} := \bigcap_{1\leq r<p} L^r([0,1])$, for $p \in (1,\infty)$, equipped with the seminorms

$$q_{p,\beta(m)}(f) := \Big(\int_0^1 |f(t)|^{\beta(m)}\,dt\Big)^{1/\beta(m)}, \quad f \in L_{p-}, \tag{2.17}$$

for any increasing sequence $1 \leq \beta(m) \uparrow p$ as $m \to \infty$. These spaces, which are all reflexive, have been studied in [11].

Lemma 2.10. *Each Fréchet space L_{p-}, for $1 < p < \infty$, contains a complemented subspace which is a Banach space isomorphic to ℓ^2.*

Proof. Consider the Rademacher functions $(r_i)_{i=0}^\infty$ on $[0,1]$ which are orthonormal in $L^2([0,1])$. For every $1 \leq p < \infty$, Khinchine's inequality ensures the existence of constants $A_p, B_p > 0$ such that

$$A_p\Big(\sum_{i=0}^n |\alpha_i|^2\Big)^{1/2} \leq \Big(\int_0^1 \Big|\sum_{i=0}^n \alpha_i r_i(t)\Big|^p\,dt\Big)^{1/p} \leq B_p\Big(\sum_{i=0}^n |\alpha_i|^2\Big)^{1/2}, \tag{2.18}$$

for all choices of scalars $(\alpha_i)_{i=0}^n$ and $n \in \mathbf{N}_0$, [16, pp.13-14], [36, Theorem 2.b.3]. It follows from (2.18) that on the closed subspace Z, of L_{p-}, which is spanned by $(r_i)_{i=0}^\infty$ the relative topology from L_{p-} is a norm topology and that Z endowed with this topology is canonically isomorphic to ℓ^2.

Fix $p \in (1,\infty)$. Given $f \in L_{p-}$, set $\langle f, r_i\rangle := \int_0^1 f(t)r_i(t)\,dt$ for $i = 0,1,2,\ldots$. Then, in the notation of (2.17), we have for each $n \in \mathbf{N}_0$ (by (2.18) and Hölder's inequality) that

$$\sum_{i=0}^{n}|\langle f,r_i\rangle|^2=\int_0^1\overline{f(t)}\Big(\sum_{i=0}^{n}\langle f,r_i\rangle r_i(t)\Big)\,dt\le q_{q,\beta(m)}(f)\cdot\Big\|\sum_{i=0}^{n}\langle f,r_i\rangle r_i\Big\|_{L^{\beta(m)'}([0,1])}$$
$$\le q_{p,\beta(m)}(f)B_{\beta(m)'}\cdot\Big(\sum_{i=0}^{n}|\langle f,r_i\rangle|^2\Big)^{1/2},$$

where $1<\beta(m)'<\infty$ satisfies $\frac{1}{\beta(m)}+\frac{1}{\beta(m)'}=1$. Then, for n large enough,

$$\Big(\sum_{i=0}^{n}|\langle f,r_i\rangle|^2\Big)^{1/2}\le B_{\beta(m)'}q_{p,\beta(m)}(f)$$

and hence, by letting $n\to\infty$, we see that

$$\Big(\sum_{i=0}^{\infty}|\langle f,r_i\rangle|^2\Big)^{1/2}\le B_{\beta(m)'}q_{p,\beta(m)}(f),\quad f\in L_{p-}. \tag{2.19}$$

Accordingly, the linear map $P\colon L_{p-}\to L_{p-}$ defined by

$$Pf:=\sum_{i=0}^{\infty}\langle f,r_i\rangle r_i,\quad f\in L_{p-},$$

is well defined. To see that P is continuous, fix $m\in\mathbf{N}$. Then, for each $f\in L_{p-}$, it follows from (2.18) and (2.19) that

$$q_{p,\beta(m)}(Pf)=q_{p,\beta(m)}\Big(\sum_{i=0}^{\infty}\langle f,r_i\rangle r_i\Big)\le B_{\beta(m)}\Big(\sum_{i=0}^{\infty}|\langle f,r_i\rangle|^2\Big)^{1/2}$$
$$\le B_{\beta(m)}B_{\beta(m)'}q_{p,\beta(m)}(f).$$

That P is a projection follows from the fact that $(r_i)_{i=0}^{\infty}$ is an orthonormal sequence. So, $P\in\mathscr{L}(L_{p-})$ is a projection of L_{p-} onto Z and hence, Z is a complemented subspace of L_{p-} which is a Banach space isomorphic to ℓ^2. □

Since ℓ^2 is not uniformly ergodic, an argument as in the proof of Proposition 2.9 yields the following result.

Proposition 2.11. *Let $1<p<\infty$. Then the reflexive Fréchet space L_{p-} fails to be uniformly mean ergodic.*

As usual ω will denote $\mathbf{C}^{\mathbf{N}}$ when equipped with the seminorms

$$q_n(x):=\max_{1\le i\le n}|x_i|,\quad x\in\omega, \tag{2.20}$$

for each $n\in\mathbf{N}$, and

$$s=\{x\in\mathbf{C}^{\mathbf{N}}:p_n(x):=\sup_{i\in\mathbf{N}}i^n|x_i|<\infty,\quad\forall n\in\mathbf{N}\} \tag{2.21}$$

is the space of all rapidly decreasing sequences. Both are nuclear Fréchet spaces and hence, also Montel. For $A:=(a_n)_n$ with $a_n(i):=i^n$, for $i\in\mathbf{N}$, it is known that $s=\lambda_p(A)$ for all $p\in[1,\infty]$, [40, p. 360]. For each $p\in[1,\infty)$, the sequence space $\ell^{p+} := \bigcap_{q>p} \ell^q$ is a separable, reflexive Fréchet space when equipped with the seminorms

$$q_{n,p}(x) := \Big(\sum_{i=1}^{\infty} |x_i|^{\beta(n)}\Big)^{1/\beta(n)}, \quad x \in \ell^{p+}, \tag{2.22}$$

where $\beta(n) := p + \frac{1}{n}$ for $n \in \mathbf{N}$. This family of spaces was studied in [41].

For Banach spaces, the following criterion is due to Sine, [49].

Theorem 2.12. *Let X be a lcHs and $T \in \mathscr{L}(X)$ be power bounded such that* $\ker(I - T) = \{0\}$. *Then T is mean ergodic if and only if* $\ker(I - T^t) = \{0\}$.

Proof. Let T be mean ergodic. Suppose that $\ker(I - T^t) \neq \{0\}$. Then there exist $u \in \ker(I - T^t) \subseteq X'$ and $x \in X$ satisfying $\langle u, x\rangle = 1$. It follows from $u = T^t u$ that $\langle T^m x, u\rangle = \langle x, (T^t)^m u\rangle = \langle x, u\rangle$ for all $m \in \mathbf{N}$ and hence, via (1.5), that $\langle T_{[n]}x, u\rangle = \langle x, u\rangle = 1$ for all $n \in \mathbf{N}$. That is, $\lim_{n\to\infty}\langle T_{[n]}x, u\rangle = 1$. The mean ergodicity of T ensures that $x_0 := \lim_{n\to\infty} T_{[n]}x$ exists. It follows from (1.5) that

$$Tx_0 = \lim_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} T^{m+1}x = \lim_{n\to\infty}\Big(T_{[n]}x - \frac{1}{n}Tx + \frac{1}{n}T^{n+1}x\Big).$$

Clearly $\frac{1}{n}Tx \to 0$ and also $\frac{1}{n}T^{n+1}x \to 0$ as T is mean ergodic; see (2.2). So, $x_0 = Tx_0$ and hence, $x_0 = 0$ because $\ker(I - T) = \{0\}$. That is, $T_{[n]}x \to 0$ and hence, $\langle T_{[n]}x, u\rangle \to 0$; contradiction. So, $\ker(I - T^t) = \{0\}$.

Now suppose that $\ker(I - T^t) = \{0\}$. If $X \neq \overline{\operatorname{Im}(I - T)}$, then there exists $x' \in X' \setminus \{0\}$ whose restriction to $\operatorname{Im}(I - T)$ is zero, that is,

$$\langle x, (I - T^t)x'\rangle = \langle (I - T)x, x'\rangle = 0, \quad \forall x \in X.$$

Accordingly, x' is a non-zero element of $\ker(I - T^t)$; contradiction. So $X = \overline{\operatorname{Im}(I - T)}$. According to Proposition 2.1 we have

$$X = \overline{\operatorname{Im}(I - T)} = \{x \in X : \lim_{n\to\infty} T_{[n]}x = 0\},$$

that is, $T_{[n]} \to 0$ in $\mathscr{L}_s(X)$. Hence, T is mean ergodic. □

Proposition 2.13. *Let A be any Köthe matrix. Then the following assertions are equivalent.*

(i) $\lambda_\infty(A)$ *is mean ergodic.*
(ii) $\lambda_\infty(A)$ *is uniformly mean ergodic.*
(iii) $\lambda_\infty(A)$ *is a Montel space.*
(iv) $\lambda_\infty(A)$ *does not contain an isomorphic copy of* ℓ^∞.
(v) $\lambda_0(A)$ *is mean ergodic.*
(vi) $\lambda_0(A)$ *is uniformly mean ergodic.*
(vii) $\lambda_0(A)$ *is a Montel space.*
(viii) $\lambda_0(A)$ *does not contain an isomorphic copy of* c_0.
(ix) $\lambda_1(A)$ *is mean ergodic.*
(x) $\lambda_1(A)$ *is uniformly mean ergodic.*
(xi) $\lambda_1(A)$ *is Montel.*
(xii) $\lambda_1(A)$ *does not contain an isomorphic copy of* ℓ^1.

Proof. (iii) $\Rightarrow$ (ii) by Proposition 2.8 and (ii) $\Rightarrow$ (i) is obvious.

Suppose that $\lambda_\infty(A)$ is not Montel. According to [9, Proposition 2.3] we have $\lambda_\infty(A) = Y \oplus Z$ with Z isomorphic to ℓ^∞. Hence, there exists a power bounded operator on $\lambda_\infty(A)$ which is not mean ergodic. This establishes (i) $\Rightarrow$ (iii) and also (iv) $\Rightarrow$ (iii) was established along the way.

(iii) $\Rightarrow$ (iv) is trivial.

(iii) $\Leftrightarrow$ (vii) is known and is equivalent to $\lambda_0(A) = \lambda_\infty(A)$; see for example, [40, Theorems 27.9 and 27.15].

Again (vii) $\Rightarrow$ (vi) by Proposition 2.8 and (vi) $\Rightarrow$ (v) is clear.

Suppose that $\lambda_0(A)$ is not Montel. Then $\lambda_0(A)$ contains a sectional subspace which is *complemented* in $\lambda_0(A)$ and is isomorphic to c_0; see, for example, the proof of Proposition 2.5(ii) in [8] which also applies when $p = 0$ there. Thus, we have established that (v) $\Rightarrow$ (vii) and (viii) $\Rightarrow$ (vii).

(vii) $\Rightarrow$ (viii). Since c_0 is not reflexive, the same argument used for the proof of (iii) $\Rightarrow$ (iv) is again applicable.

(iii) $\Leftrightarrow$ (xi) is immediate from [40, Proposition 27.9].

(xi) $\Rightarrow$ (xii). Since ℓ^1 is not reflexive, the same argument used for the proof of (iii) $\Rightarrow$ (iv) is again applicable.

Suppose that $\lambda_1(A)$ is not Montel. Then $\lambda_1(A)$ contains a sectional subspace which is *complemented* in $\lambda_1(A)$ and is isomorphic to ℓ^1. This implies (ix) $\Rightarrow$ (xi) and (xii) $\Rightarrow$ (xi).

Finally, (xi) $\Rightarrow$ (x) by Proposition 2.8 and (x) $\Rightarrow$ (ix) is clear. □

Remark 2.14. (i) A lcHs which contains an isomorphic copy of ℓ^∞ contains it as a complemented subspace, [27, Corollary 7.4.6]. Therefore *every* lcHs which contains a copy of ℓ^∞ fails to be mean ergodic.

(ii) An analogue of part (i) is also available for c_0 and is based on the following version of

Sobczyk's Theorem. *Let X be a separable lcHs which contains an isomorphic copy of c_0. Then X contains a complemented copy of c_0.*

Its proof proceeds as follows: one modifies in an obvious way the last part of the proof of the classical Banach space version of Sobczyk's theorem, as presented in [27, p. 160], by replacing there the use of Theorem 7.4.4 of [27] with Corollary 7.4.5 of [27].

Each Fréchet space ℓ^{p+}, for $1 \leq p < \infty$, has *no* closed subspace isomorphic to any (infinite dimensional) Banach space, [41, p. 10]. So, an argument as in the proof of Proposition 2.11 *cannot* be used for ℓ^{p+}.

The following class of multiplication operators will be useful. Let X denote any one of the sequence spaces $\omega, s, \ell^p, c_0, \ell^{p+}$ or $\lambda_p(A)$. Given any sequence of numbers $0 < \mu_i < 1$, for $i \in \mathbf{N}$, define a linear operator $T^{(\mu)} \in \mathscr{L}(X)$ by

$$T^{(\mu)}x := (\mu_i x_i), \quad x \in X. \tag{2.23}$$

Direct calculation shows, for each $n \in \mathbf{N}$, that

$$T^{(\mu)}_{[n]}x = \frac{1}{n}\Big(\frac{\mu_i}{(1-\mu_i)} \cdot (1-\mu_i^n)x_i\Big)_i \quad x \in X. \tag{2.24}$$

Suppose, in addition, that $\mu_i \uparrow 1$. Then there exists an increasing sequence of integers $n_i \to \infty$ satisfying $n_i \leq \frac{1}{(1-\mu_i)} < 1 + n_i$, for $i \in \mathbf{N}$, and hence,

$$\frac{1}{n_i}\frac{\mu_i}{(1-\mu_i)} \cdot (1-\mu_i^{n_i}) \geq \mu_1\Big(1 - \big(1-(n_i+1)^{-1}\big)^{n_i}\Big), \quad i \in \mathbf{N}. \tag{2.25}$$

Proposition 2.15. *For each $1 \leq p < \infty$, the reflexive Fréchet space ℓ^{p+} is mean ergodic but, not uniformly mean ergodic.*

Proof. First note that the Banach space ℓ^p is continuously included in ℓ^{p+} and hence, the unit ball B of ℓ^p is a bounded set in ℓ^{p+}. Let $0 < \mu_i \uparrow 1$ and define the power bounded operator $T^{(\mu)} \in \mathscr{L}(\ell^{p+})$ via (2.23). Let q denote any one of the seminorms (2.22). If $e_i \in B$ denotes the standard i-th unit basis vector of ℓ^p, then it follows from the definition of q and (2.24) that

$$q\Big(T_{[n]}^{(\mu)} e_i\Big) = \frac{1}{n}\frac{\mu_i}{(1-\mu_i)} \cdot (1-\mu_i^n), \quad i, n \in \mathbf{N}.$$

Since ℓ^{p+} is mean ergodic, there is $P \in \mathscr{L}(\ell^{p+})$ with $T_{[n]}^{(\mu)} \to P$ in $\mathscr{L}_s(\ell^{p+})$ as $n \to \infty$. By the above identities, $P(e_i) = 0$ for all $i \in \mathbf{N}$. But, $(e_i)_{i\in\mathbf{N}}$ is a basis for ℓ^{p+}, [41, p. 8], and so $P = 0$. By (2.25) and the fact that its right-hand side converges to $\mu_1(1-e^{-1}) > 0$ we can conclude that

$$\lim_{n\to\infty} \sup_{x \in B} q\Big(T_{[n]}^{(\mu)} x\Big) \neq 0,$$

that is, $\big(T_{[n]}^{(\mu)}\big)$ fails to converge to 0 in $\mathscr{L}_b(\ell^{p+})$ as $n \to \infty$. Hence, $T^{(\mu)}$ is not uniformly mean ergodic. □

For X a Banach space, the next result is due to Lin, [35].

Proposition 2.16. *Let X be a Fréchet space and $T \in \mathscr{L}(X)$ satisfy the conditions* $\ker(I-T) = \{0\}$ *and $\frac{1}{n}T^n \to 0$ in $\mathscr{L}_b(X)$. Consider the following statements.*

(i) *$I - T_{[n]}$ is surjective for some $n \in \mathbf{N}$.*
(ii) *$I - T$ is surjective.*
(iii) *$T_{[n]} \to 0$ in $\mathscr{L}_b(X)$ as $n \to \infty$.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). *If X is a Banach space, then also* (iii) $\Rightarrow$ (i).

Proof. (i) $\Rightarrow$ (ii). Suppose that $I - T_{[n]}$ is surjective. It is routine to verify that $I - T_{[n]} = (I-T)g_n(T)$, where $g_n(T) := \frac{1}{n}\sum_{r=0}^{n-1}(\sum_{j=0}^{r} T^j)$. Given $y \in X$ there is, by hypothesis, $x \in X$ such that $(I - T_{[n]})x = y$ and hence, $(I-T)g_n(T)x = y$. So, $(I-T)$ is surjective.

(ii) $\Rightarrow$ (iii). Fix a bounded set $B \subseteq X$ and $q \in \Gamma_X$. Since $(I-T) : X \to X$ is continuous and bijective and X is a Fréchet space, it follows that $(I-T)$ is a bicontinuous isomorphism. So, $C := (I-T)^{-1}B$ is a bounded set in X and $B = (I-T)C$. In the notation of (1.4) it follows from (2.1) that

$$\begin{aligned} q_B(T_{[n]}) &= \sup_{x\in C} q\left((I-T)T_{[n]}\right) = \sup_{x \in C} q\Big(\frac{1}{n}(T - T^{n+1})x\Big) \\ &\leq \frac{1}{n} q_C(T) + 2q_C\Big(\frac{1}{(n+1)} \cdot T^{n+1}\Big). \end{aligned}$$

Since we are assuming that $\frac{1}{n}T^n \to 0$ in $\mathscr{L}_b(X)$, it follows that $q_B(T_{[n]}) \to 0$. But, q and B are arbitrary and so $T_{[n]} \to 0$ in $\mathscr{L}_b(X)$. Hence, (iii) holds.

Suppose now, in addition, that X is a Banach space. Given (iii), it follows that $\|T_{[n]}\| < 1$ for some $n \in \mathbf{N}$. It is then classical that $(I - T_{[n]})$ is invertible in $\mathscr{L}(X)$, [19, Ch. VII, §3], and so (i) holds. □

We now show that (iii) ⇒ (ii) may *fail* in the non-normable setting.

Example 2.17. Define $T \in \mathscr{L}(s)$ by

$$Tx := \left((1-2^{-i})x_i\right), \quad x = (x_i) \in s.$$

It is routine to check that $\ker(I-T) = \{0\}$ and that $y := (2^{-i}) \in s$ does not belong to $\mathrm{Im}(I-T)$, that is, $(I-T)$ is not surjective. So (ii) of Proposition 2.16 *fails* to hold.

Since $T^m x = ((1-2^{-i})^m x_i)$ for $x \in s$ and all $m \in \mathbf{N}$, it follows from (2.21) (with the notation from there) that

$$p_n\left(T^m x\right) \le p_n(x), \quad x \in s, \tag{2.26}$$

for all $n, m \in \mathbf{N}$. Given any bounded set $B \subseteq s$, it follows that

$$(p_n)_B\Big(\frac{1}{m}T^m\Big) = \sup_{x\in B} p_n\Big(\frac{1}{m}T^m x\Big) \le \frac{1}{m}\sup_{x\in B} p_n(x), \quad m \in \mathbf{N}.$$

Hence, $(p_n)_B\left(\frac{1}{m}T^m\right) \to 0$ as $m \to \infty$. That is, $\frac{1}{m}T^m \to 0$ in $\mathscr{L}_b(s)$.

It remains to verify condition (iii) of Proposition 2.16. The dual space of s is given by

$$s' = \{\xi \in \mathbf{C}^{\mathbf{N}} : \exists k \in \mathbf{N} \text{ with } \sup_i \frac{|\xi_i|}{i^k} < \infty\}$$

and direct calculation shows that

$$T^t\xi = \left((1-2^{-i})\xi_i\right), \quad \xi \in s'.$$

It follows that $\ker(I - T^t) = \{0\}$ and, by (2.26), we see that T is power bounded. According to Theorem 2.12 the operator T is mean ergodic. Since s is Montel, T is actually uniformly mean ergodic (see Proposition 2.8). So, there exists $P \in \mathscr{L}(s)$ such that $T_{[n]} \to P$ in $\mathscr{L}_b(s)$. In particular, $T_{[n]} \to P$ in $\mathscr{L}_s(s)$. Fix $r \in \mathbf{N}$ and let e_r denote the element of s with a 1 in the r-th coordinate and 0's elsewhere. Then

$$T^n e_r = (1-2^{-r})^n e_r \to 0 \quad \text{as } n \to \infty$$

and hence, also $T_{[n]}e_r = \frac{\mu(1-\mu)^n}{n(1-\mu)}e_r \to 0$ as $n \to \infty$ (where $\mu := 1 - 2^{-r}$). But, $T_{[n]}e_r \to Pe_r$ as $n \to \infty$ and we conclude that $Pe_r = 0$. Since $r \in \mathbf{N}$ is arbitrary, it follows that $P = 0$ and hence, $T_{[n]} \to 0$ in $\mathscr{L}_b(s)$ which is precisely condition (iii).

We conclude this section by exhibiting an interesting family of power bounded operators. Fix $1 < p < \infty$. A Borel measurable function $\varphi\colon [0,1] \to \mathbf{C}$ defines a continuous multiplication operator $M_\varphi \in \mathscr{L}(L_{p-})$ via $f \mapsto \varphi f$, for $f \in L_{p-}$, if and only if $\varphi \in \bigcap_{1\le q<\infty} L^q([0,1])$, [7, Proposition 18].

Proposition 2.18. *Let $1 < p < \infty$ and $\varphi \in \bigcap_{1\le q<\infty} L^q([0,1])$. The following assertions for the multiplication operator $M_\varphi \in \mathscr{L}(L_{p-})$ are equivalent.*

(i) $\varphi \in L^\infty([0,1])$ *with* $\|\varphi\|_\infty \le 1$.
(ii) M_φ *is power bounded.*
(iii) M_φ *is mean ergodic.*
(iv) M_φ *is uniformly mean ergodic.*

Proof. Observe that always $(M_\varphi)^n = M_{\varphi^n}$, for $n \in \mathbf{N}$, and so we have

$$q_{p,m}((M_\varphi)^n f) = q_{p,m}(\varphi^n f), \quad f \in L_{p-}, \tag{2.27}$$

for each $m \in \mathbf{N}$ and all $n \in \mathbf{N}$ (the notation is from (2.17)).

(i) $\Rightarrow$ (ii). Fix $m \in \mathbf{N}$. It follows from (2.27) and (2.17) that $q_{p,m}((M_\varphi)^n f) \leq \|\varphi^n\|_\infty \cdot q_{p,m}(f)$ for all $n \in \mathbf{N}$ and $f \in L_{p-}$. Since (i) implies that $\|\varphi^n\|_\infty \leq 1$ for all $n \in \mathbf{N}$, we see that

$$q_{p,m}((M_\varphi)^n f) \leq q_{p,m}(f), \quad f \in L_{p-},\ n \in \mathbf{N},$$

that is, $\{(M_\varphi)^n\}_{n=1}^\infty$ is equicontinuous and so M_φ is power bounded.

(ii) $\Rightarrow$ (iii). This is immediate from the reflexivity of L_{p-} and Corollary 2.7.

(iii) $\Rightarrow$ (iv). Set $T := M_\varphi$. By assumption there exists $Q \in \mathscr{L}(L_{p-})$ such that $T_{[n]} \to Q$ in $\mathscr{L}_s(L_{p-})$ and hence, (2.2) implies that $\frac{1}{n}T^n = \frac{1}{n}M_{\varphi^n} \to 0$ in $\mathscr{L}_s(L_{p-})$ as $n \to \infty$. For the choice $f := \chi_{[0,1]} \in L_{p-}$ we can conclude that $\frac{\varphi^n}{n} \to 0$ in L_{p-}. But, L_{p-} is continuously included in $L^1([0,1])$ and so $\frac{\varphi^n}{n} \to 0$ in $L^1([0,1])$. Accordingly, there exists a subsequence $\frac{\varphi^{n(k)}}{n(k)} \to 0$ a.e. in $[0,1]$ as $k \to \infty$, which implies that $|\varphi| \leq 1$ a.e. For each $z \in \mathbf{C}$ with $|z| \leq 1$, it is routine to check that $\frac{1}{n}\sum_{k=1}^n z^k \to 1$ if $z = 1$ and $\frac{1}{n}\sum_{k=1}^n z^k \to 0$ otherwise (as $n \to \infty$). Hence, $\|\varphi\|_\infty \leq 1$ implies that

$$\frac{1}{n}\sum_{k=1}^n \varphi^k \to \chi_{_A} \ \text{ a.e. on } [0,1], \tag{2.28}$$

as $n \to \infty$, where $A := \{t \in [0,1] : \varphi(t) = 1\}$.

For each Borel set $E \subseteq [0,1]$, define the projection $P(E) := M_{\chi_E} \in \mathscr{L}(L_{p-})$. Then $E \mapsto P(E)$ is a spectral measure in L_{p-} (see Section 4 for the definition) which is boundedly σ-additive, [7, Proposition 6(iii)]. Since the lcHs $\mathscr{L}_b(L_{p-})$ in which P takes its values is quasicomplete, [32, p. 144], it is known that all bounded measurable functions are P-integrable, [34, p. 161]. In particular, $\frac{1}{n}\sum_{k=1}^n \varphi^k$ is P-integrable for each $n \in \mathbf{N}$. It follows from (2.28) and the dominated convergence theorem for the $\mathscr{L}_b(L_{p-})$-valued vector ($=$ spectral) measure P, [34, Theorem 2.2], that $\int_{[0,1]}(\frac{1}{n}\sum_{k=1}^n \varphi^k)\, dP \to P(A)$ in $\mathscr{L}_b(L_{p-})$, that is, $T_{[n]} \to P(A) = Q$ in $\mathscr{L}_b(L_{p-})$ as $n \to \infty$. Hence, $M_\varphi = T$ is uniformly mean ergodic.

(iv) $\Rightarrow$ (i). If M_φ is uniformly mean ergodic, then it is mean ergodic. Hence, $\varphi \in L^\infty([0,1])$ with $\|\varphi\|_\infty \leq 1$; see the proof of (iii) $\Rightarrow$ (iv) above. □

Since the containment $L^\infty([0,1]) \subseteq \bigcap_{1\leq q<\infty} L^q([0,1])$ is *proper*, Proposition 2.18 implies that there exist multiplication operators $M_\varphi \in \mathscr{L}(L_{p-})$ which fail to be power bounded.

3. On bases and reflexivity in lcHs'

The main aim of this section is to establish Theorems 1.1 and 1.2.

A sequence $(x_n)_n$ in a lcHs X is called a *basis* if, for every $x \in X$, there is a unique sequence $(\alpha_n)_n \subseteq \mathbf{C}$ such that the series $\sum_{n=1}^\infty \alpha_n x_n$ converges to x in X. By setting $f_n(x) := \alpha_n$ we obtain a linear form $f_n\colon X \to \mathbf{C}$ which is called the *n-th coefficient functional associated to* $(x_n)_n$. The functionals f_n, $n \in \mathbf{N}$, are uniquely determined by $(x_n)_n$ and $\{(x_n, f_n)\}_{n=1}^\infty$ is a biorthogonal sequence (i.e. $\langle x_n, f_m\rangle = \delta_{mn}$

for $m, n \in \mathbf{N}$). For each $n \in \mathbf{N}$, the map $P_n \colon X \to X$ defined by

$$P_n \colon x \mapsto \sum_{i=1}^{n} f_i(x)x_i = \sum_{i=1}^{n} \langle x, f_i \rangle x_i, \quad x \in X, \tag{3.1}$$

is a linear projection with range equal to the finite dimensional space $\mathrm{span}(x_i)_{i=1}^n$. If, in addition, $(f_n)_n \subseteq X'$, then the basis $(x_n)_n$ is called a *Schauder basis* for X. In this case, $(P_n)_n \subseteq \mathscr{L}(X)$ and each dual operator

$$P_n^t \colon x' \mapsto \sum_{i=1}^{n} \langle x_i, x' \rangle f_i, \quad x' \in X', \tag{3.2}$$

for $n \in \mathbf{N}$, is a projection with range equal to $\mathrm{span}(f_i)_{i=1}^n$. Moreover, for every $x' \in X'$ the series $\sum_{i=1}^{\infty} \langle x_i, x' \rangle f_i$ converges to f in X'_σ (the space X' equipped with the topology $\sigma(X', X)$). For this reason, $(f_n)_n$ is also referred to as the *dual basis* of the Schauder basis $(x_n)_n$. The terminology "X has a Schauder basis" will also be abbreviated simply to saying that "X has a basis". A sequence $(x_n)_n$ in a lcHs X is called a *basic sequence* if it is a Schauder basis for the closed linear hull $\overline{\mathrm{span}}(x_n)_n$ of $(x_n)_n$ in X.

Let X be a Fréchet space with a fundamental sequence of seminorms $\Gamma_X = (q_n)_n$. Then X_n denotes the local Banach space generated by q_n, that is, X_n is the completion of the quotient normed space $(X/q_n^{-1}(\{0\}), q_n)$. Let $\pi_n : X \to X_n$ denote the canonical map. Then $X = \mathrm{proj}_n X_n$ is the (reduced) projective limit of the sequence of Banach spaces $(X_n)_n$, [31, p. 232].

Now the promised extension of Pelcynski's result. We wish to thank Professor J. C. Díaz for some useful discussions on this topic.

Proof of Theorem 1.1. Let X be a non-reflexive Fréchet space and $\Gamma_X = (q_n)_n$ be an increasing, fundamental sequence of seminorms. If X contains an isomorphic copy of ℓ^1, then it surely contains a non-reflexive, closed subspace with a basis.

So, suppose that X does *not* contain an isomorphic copy of ℓ^1. According to [31, p. 303 & pp. 312–313], the non-reflexivity of X ensures the existence of a bounded sequence in X with no weakly convergent subsequence and hence, by Rosenthal's dichotomy theorem for Fréchet spaces [13, Lemma 3], X contains a sequence $(x_k)_k$ which is Cauchy but, not convergent in X_σ. For each $n \in \mathbf{N}$, the sequence $(\pi_n(x_k))_k$ is $\sigma(X_n, X_n')$-Cauchy in the Banach space X_n. Moreover, there exists $n(0) \in \mathbf{N}$ such that $(\pi_n(x_k))_k$ is not convergent in $(X_n)_\sigma$ for all $n \geq n(0)$. Indeed, if the contrary were the case, then there would exist positive integers $n(m) \uparrow \infty$ such that $(\pi_{n(m)}(x_k))_k$ converges in $(X_{n(m)})_\sigma$, for all $m \in \mathbf{N}$. Since $X = \mathrm{proj}_m X_{n(m)}$, it follows that $(x_k)_k$ is convergent in X_σ which contradicts the choice of $(x_k)_k$.

Now, $(\pi_{n(0)}(x_k))_k$ is Cauchy but not convergent in $(X_{n(0)})_\sigma$. According to [15, p. 54 Ex. 10(ii)], there exists a subsequence $(\pi_{n(0)}(x_k^0))_k$ which is a basic sequence in $X_{n(0)}$. Then the sequence $(\pi_{1+n(0)}(x_k^0))_k \subseteq X_{1+n(0)}$ is Cauchy but not convergent in $(X_{1+n(0)})_\sigma$. If it were convergent then, being a subsequence of the weak Cauchy sequence $(\pi_{1+n(0)}(x_k))_k$, also this latter sequence would converge in $(X_{1+n(0)})_\sigma$; contradiction! So, we can again apply [15, p. 54 Ex. 10(ii)] to select a subsequence $(\pi_{1+n(0)}(x_k^1))_k$ of $(\pi_{1+n(0)}(x_k^0))_k$ which is a basic sequence in $X_{1+n(0)}$. Continue this procedure inductively and, finally, select the diagonal subsequence $(x_k^k)_{k \geq n(0)}$. According to [13, Lemma 1], we see that $(x_k^k)_{k \geq n(0)}$ is a basic sequence in X. Since this sequence is $\sigma(X, X')$-Cauchy but, has no convergent subsequence in X_σ, it follows that $\overline{\mathrm{span}}(x_k^k)_{k\geq n(0)}$ is not reflexive. □

The strong topology on X (resp. X') is denoted by $\beta(X, X')$ (resp. $\beta(X', X)$) and we write X_β (resp. X'_β). For a Schauder basis $(x_n)_n \subseteq X$, we denote by H the subspace of X' consisting of all $x' \in X'$ such that $P_n^t x' \mapsto x'$ in X'_β as $n \to \infty$ and endow it with the topology $\beta(H, X)$ induced by $\beta(X', X)$. If $H = X'$ (i.e. the dual basis $(f_n)_n$ is a Schauder basis for X'_β), then $(x_n)_n$ is called *shrinking.* A Schauder basis $(x_n)_n \subseteq X$ is called *boundedly complete* if the series $\sum_{n=1}^\infty \alpha_n x_n$ converges in X whenever $(\sum_{i=1}^n \alpha_i x_i)_n$ is a bounded sequence in X. For Banach spaces, these notions are due to James, [26], and for lcHs' they are due to Dubinsky and Retherford, [17], [44].

Corollary 3.1. *Let X be a non-reflexive Fréchet space. Then X contains a closed subspace with a basis which is not shrinking.*

Proof. According to Theorem 1.1, X contains a basic sequence $(x_n)_n$ such that $E := \overline{\mathrm{span}}(x_n)_n$ is non-reflexive. Then [28, Theorem 3.3(ii)] implies that E (hence, also X) has a basic sequence $(u_k)_k$ which is not shrinking. So, $\overline{\mathrm{span}}(u_k)_k$ is a closed subspace of X with a basis which is not shrinking. □

A Schauder basis $(x_n)_n$ of a lcHs X is called *regular* if there exists a neighbourhood V of zero such that $x_n \notin V$ for all $n \in \mathbf{N}$. Equivalently, there exists $q \in \Gamma_X$ such that $\inf_n q(x_n) > 0$. If, in addition, the sequence $(x_n)_n$ is bounded in X, then it is called *normalized.* Even if X is a Fréchet space, it does not necessarily follow that a given basis can be regularized, [30].

Proposition 3.2. *Let X be a Fréchet space which is not Montel. Then X contains a closed subspace which is not Montel and has a basis.*

Proof. According to [5, Proposition 2.2], the space X possesses a (infinite) normalized basic sequence, say $(x_n)_n$; a proof of this result which avoids non-standard analysis is also available (see [5, p. 205 Remark (2)]). Hence, $Y := \overline{\mathrm{span}}(x_n)_n$ is a Fréchet space with a normalized Schauder basis. But, it is routine to check that such a basis cannot exist in any infinite dimensional Montel Fréchet space. So, Y is not Montel. □

Given a lcHs X, a biorthogonal sequence $\{(x_n, f_n)\}_{n=1}^\infty$ in $X \times X'$ is said to be of *type P* if there exists a neighbourhood V of zero in X such that $x_n \notin V$, for all $n \in \mathbf{N}$, and if the sequence of partial sums $(\sum_{i=1}^n x_i)_n$ is bounded in X. We say $\{(x_n, f_n)_n\}$ is of *type* P^* if $(x_n)_n$ is bounded in X and $(\sum_{i=1}^n f_i)_n$ is bounded in X'_β. Bases and basic sequences of types P and P^* were introduced and studied in Banach spaces by Singer, [50]. The definition was extended by Dubinsky and Retherford, [17], to bases and basic sequences in lcHs' and by Kalton, [28], to biorthogonal sequences.

The behavior of bases and basic sequences in reflexive Banach spaces has been completely characterized. Recall that James proved that a Banach space with a Schauder basis is reflexive if and only if the basis is both shrinking and boundedly complete, [26]. Zippin showed, for a Banach space X with a basis, that if every basis is boundedly complete or if every basis is shrinking, then X is reflexive, [55]. James's characterization of reflexivity was generalized to lcHs' by Dubinsky and Retherford [17], [44] (see also [27, Theorem 14.5.1]). Furthermore, Kalton extended both Singer's and Zippin's results to cover lcHs' more general than Banach spaces,

[28]. In particular, he proved that a sequentially complete lcHs with a Schauder basis in which every basic sequence is boundedly complete or every basic sequence is shrinking is necessarily semi-reflexive. He also showed that a complete barrelled lcHs with a normalized Schauder basis in which every normalized basis is boundedly complete, or in which every normalized basis is shrinking, is reflexive. As noted in the Introduction, Kalton raised the question of whether the last result remains true without the restriction of normalization on the basis. By using ideas of [28] and [55], we now proceed to show that the answer is positive. As usual, we begin with some auxiliary results.

Remark 3.3. Let X be a complete barrelled lcHs and $(x_n)_n$ be a sequence in $X \setminus \{0\}$ for which $X = \overline{\mathrm{span}}(x_n)_n$. Then $(x_n)_n$ is a Schauder basis for X if and only if for each $p \in \Gamma_X$ there exist $M_p > 0$ and $q \in \Gamma_X$ such that

$$p\left(\sum_{i=1}^{r} \alpha_i x_i\right) \le M_p q\left(\sum_{i=1}^{s} \alpha_i x_i\right) \tag{3.3}$$

for arbitrary positive integers $r \le s$ and arbitrary scalars $\alpha_1, \ldots, \alpha_s$; (see [45, Theorems 3.1–3.2] or [27, Theorem 14.3.6]). By setting

$$\tilde{p}(x) := \sup_{r \in \mathbf{N}} p\left(\sum_{i=1}^{r} \langle x, f_i \rangle x_i\right),$$

we obtain from (3.3) that

$$p(x) \le \tilde{p}(x) \le M_p q(x) \le M_p \tilde{q}(x), \quad x \in X, \tag{3.4}$$

for all $p \in \Gamma_X$. Hence, $\widetilde{\Gamma}_X := \{\tilde{p} :\ p \in \Gamma_X\}$ is also a system of continuous seminorms generating the topology of X and we have

$$\tilde{p}\left(\sum_{i=1}^{r} \alpha_i x_i\right) \le \tilde{p}\left(\sum_{i=1}^{s} \alpha_i x_i\right)$$

for every $\tilde{p} \in \widetilde{\Gamma}_X$ and for arbitrary positive integers $r \le s$ and arbitrary scalars $\alpha_1, \ldots, \alpha_s$. That is, $(x_n)_n$ is a *monotone basis* with respect to $\widetilde{\Gamma}_X$. $\square$

Lemma 3.4. *Let X be a complete barrelled lcHs with a Schauder basis $(x_n)_n$. Then $X = \mathrm{proj}_{\,i\in I} X_i$ is the projective limit of a system $(X_i)_{i\in I}$ of local Banach spaces such that $(\pi_i(x_n))_n$ is a Schauder basis in X_i for every $i \in I$, where π_i denotes the canonical projection of X into X_i.*

Proof. According to Remark 3.3, we can select a system Γ_X of continuous seminorms defining the topology of X with respect to which the basis $(x_n)_n$ is monotone, that is, for every $p \in \Gamma_X$ and for arbitrary positive integers $r \le s$ and arbitrary scalars $\alpha_1, \ldots, \alpha_s$ we have

$$p\left(\sum_{i=1}^{r} \alpha_i x_i\right) \le p\left(\sum_{i=1}^{s} \alpha_i x_i\right). \tag{3.5}$$

In the case that X admits a continuous norm, we can assume that each $p \in \Gamma_X$ is a norm. Then the completion X_p of the normed space (X, p) is a Banach space. Clearly, $X = \mathrm{proj}_{\,p\in\Gamma_X} X_p$ with X continuously embedded in every X_p, i.e., $\pi_p(x) = x$.

In particular, by (3.5) and Remark 3.3 we see that $(x_n)_n$ is a Schauder basis in every X_p.

Suppose now that X does not admit a continuous norm. Denote by $(f_n)_n$ the dual basis of $(x_n)_n$. For each $p \in \Gamma_X$, the space $\mathrm{Ker}(p)$ is then infinite dimensional and

$$\mathrm{Ker}(p) = \{x \in X : p(x) = 0\} = \bigcap_{n \in J(p)} \mathrm{Ker}(f_n), \tag{3.6}$$

where $J(p) := \{n \in \mathbf{N} : p(x_n) \neq 0\}$. Indeed, if $n \in J(p)$ (hence, $p(x_n) \neq 0$), then we obtain from (3.5) that

$$|f_n(x)| = \frac{1}{p(x_n)} p(\langle x, f_n \rangle x_n) \leq \frac{2}{p(x_n)} p(x), \quad x \in X,$$

thereby implying that, if $x \in \mathrm{Ker}(p)$, then $\langle x, f_n \rangle = 0$ for all $n \in J(p)$ and hence, $x \in \bigcap_{n \in J(p)} \mathrm{Ker}(f_n)$. Conversely, if $x \in \bigcap_{n \in J(p)} \mathrm{Ker}(f_n)$, then $x = \sum_{i \notin J(p)} \langle x, f_i \rangle x_i$ and hence, $0 = p(\sum_{i=1, i \notin J(p)}^n \langle x, f_i \rangle x_i) \to p(x)$, i.e., $p(x) = 0$ because $(x_n)_{n \notin J(p)} \subset \mathrm{Ker}(p)$.

Denote by π_p the canonical quotient map from X onto $X/\ker(p)$ and by $\hat{p}$ the quotient norm on $X/\ker(p)$ given by

$$\hat{p}(\pi_p(x)) := \inf\{p(y) : \pi_p(y) = \pi_p(x)\}.$$

Then the local Banach space X_p is the completion of the normed space $(X/\ker(p), \hat{p})$ and $X = \mathrm{proj}_{p \in \Gamma_X} X_p$. It remains to show that $(\pi_p(x_n))_{n \in \mathbf{N}} = (\pi_p(x_n))_{n \in J(p)}$ is a Schauder basis in X_p.

Suppose that $J(p) = (n(i))_i$, where $(n(i))_i$ is either a finite sequence or an increasing sequence of positive integers. If $(n(i))_i$ is a finite sequence, then it is routine to check that $(\pi_p(x_{n(i)}))_i$ is a linearly independent set and $X/\ker(p) = \overline{\mathrm{span}}(x_{n(i)})_i$.

Suppose then that $(n(i))_i$ is an increasing sequence of positive integers. Let $\hat{p}\left(\sum_{i=1}^{m+r} \alpha_i \pi_p(x_{n(i)})\right) = 1$. For any $\varepsilon > 0$ there exists $x = \sum_{n=1}^{\infty} \beta_n x_n$ such that $p(x) \leq 1 + \varepsilon$ and $\pi_p(x) = \sum_{n=1}^{\infty} \beta_n \pi_p(x_n) = \sum_{i=1}^{m+r} \alpha_i \pi_p(x_{n(i)})$. This means that

$$\sum_{n=1}^{\infty} \beta_n x_n - \sum_{i=1}^{m+r} \alpha_i x_{n(i)} \in \mathrm{Ker}(p) = \bigcap_i \mathrm{Ker}(f_{n(i)}).$$

In view of the biorthogonality, we obtain that $\beta_{n(i)} = 0$ if $i > m + r$, and $\beta_{n(i)} = a_i$ if $i \in \{1, \ldots, m + r\}$. Therefore, by (3.5) we have

$$\begin{aligned} \hat{p}\Big(\sum_{i=1}^{m} \alpha_i \pi_p(x_{n(i)})\Big) &= \hat{p}\Big(\sum_{i=1}^{m} \beta_{n(i)} \pi_p(x_{n(i)})\Big) \leq p\Big(\sum_{i=1}^{m} \beta_{n(i)} x_{n(i)}\Big) \\ &\leq p\Big(\sum_{i=1}^{m+r} \beta_{n(i)} x_{n(i)}\Big) = p\Big(\sum_{n=1}^{\infty} \beta_n x_n\Big) \leq 1 + \varepsilon. \end{aligned} \tag{3.7}$$

The last equality follows from $p\left(\sum_{i=1}^{m+r} \beta_{n(i)} x_{n(i)}\right) = p\left(\sum_{n=1}^{s} \beta_n x_n\right) \to p(x)$, which is a consequence of

$$p\left(\sum_{n=1}^{s} \beta_n x_n\right) = p\left(\sum_{n=1, n \in J(p)}^{s} \beta_n x_n + \sum_{n=1, n \notin J(p)}^{s} \beta_n x_n\right) = p\left(\sum_{n=1, n \in J(p)}^{s} \beta_n x_n\right)$$

because $p(x_n) = 0$ if $n \notin J(p)$. Since (3.7) holds for all m, r and ε, we can conclude that $(\pi_p(x_{n(i)}))_i$ is a Schauder basis for the Banach space X_p. In particular, $(f_{n(i)})_i$ is the dual basis of $(\pi_p(x_{n(i)}))_i$. □

Let X be a lcHs. For each $p \in \Gamma_X$, we set $U_p := \{x \in X : p(x) \leq 1\}$ and define the *dual seminorm* p' of p by

$$p'(u) := \sup\{|\langle x, u\rangle| : p(x) \leq 1\} = \sup\{|\langle x, u\rangle| : p(x) = 1\}, \quad u \in X',$$

that is, p' is the gauge of the polar U_p° in X'. Let $X'_p := \{u \in X' : p'(u) < \infty\}$. Then (X'_p, p') is a Banach space and the transpose map π_p^t of the canonical quotient map π_p is an isometry from the strong dual of the Banach space X_p (i.e. the completion of $(X/\ker(p), \hat{p})$) onto (X'_p, p'). Therefore, every $v \in (X/\ker(p), \hat{p})'$ defines a continuous linear functional $u = v \circ \pi_p \in X'$ with $p'(u) < \infty$.

We now formulate a useful criterion for turning algebraic bases into Schauder bases. This criterion is due to Zippin in the Banach space setting, [55, Lemma 2], and was later extended by Robinson to complete barrelled lcHs', [48, Theorem 1.1].

Lemma 3.5. *Let X be a complete barrelled lcHs and $(x_n)_n$ be a Schauder basis of X.*

For each $k \in \mathbf{N}$, let $(y_i)_{i=p(k)+1}^{p(k+1)}$ be a basis of $\operatorname{span}(x_i)_{i=p(k)+1}^{p(k+1)}$, where $(p(k))_k$ is an increasing sequence of positive integers with $p(1) = 0$. Suppose, for each $q \in \Gamma_X$, that there exist $M_q > 0$ and $r \in \Gamma_X$ such that

$$q\left(\sum_{i=p(k)+1}^{m} \alpha_i y_i\right) \leq M_q r\left(\sum_{i=p(k)+1}^{n} \alpha_i y_i\right) \tag{3.8}$$

for all $k \in \mathbf{N}$, for all integers m, n with $p(k) < m \leq n \leq p(k+1)$ and for all scalars $(\alpha_i)_{i=p(k)+1}^{n}$.

Then the sequence $(y_i)_i$ is a Schauder basis of X.

Remark 3.6. If the basis $(x_n)_n$ in Lemma 3.5 is monotone relative to Γ_X and (3.8) holds with q in the place of r in its right-hand side, then it can be shown that, for each $q \in \Gamma_X$, there exists $C_q > 0$ such that

$$q\left(\sum_{i=1}^{r} \alpha_i y_i\right) \leq C_q q\left(\sum_{i=1}^{s} \alpha_i y_i\right) \tag{3.9}$$

for arbitrary positive integers $r \leq s$ and arbitrary scalars $\alpha_1, \ldots, \alpha_s$.

We now present a criterion for extending Schauder block sequences to Schauder bases for the whole space.

Lemma 3.7. *Let X be a complete barrelled lcHs and $(x_n)_n$ be a Schauder basis of X.*

Assume that $0 \neq y_k = \sum_{i=p(k)+1}^{p(k+1)} \alpha_i x_i$ where $(p(k))_k$ is an increasing sequence of positive integers with $p(1) = 0$ so that, for some $p_0 \in \Gamma_X$,

$$\inf_{k \in \mathbf{N}} p_0(y_k) = d > 0 \tag{3.10}$$

and, for every $q \in \Gamma_X$,

$$\sup_{k \in \mathbf{N}} q(y_k) \leq M_q < \infty. \tag{3.11}$$

Then there exists a Schauder basis $(z_i)_i$ in X such that, for each $k \in \mathbf{N}$, we have $z_i = y_k$ for some $i \in \{p(k)+1, \dots, p(k+1)\}$.

Proof. As X is barrelled, we can suppose that $(x_n)_n$ is a monotone basis with respect to Γ_X. Indeed, by Remark 3.3 there exists a system $\widetilde{\Gamma}_X$ of continuous seminorms defining the topology of X such that $(x_n)_n$ is a monotone basis with respect to $\widetilde{\Gamma}$. Therefore, conditions (3.10) and (3.11) continue to hold. In particular, condition (3.10) is satisfied with $\tilde{p}_0 \in \widetilde{\Gamma}_X$; see (3.4). According to Lemma 3.4, $(\pi_{p_0}(x_n))_n$ is a Schauder basis in the local Banach space X_{p_0}. In particular, by setting $J := \{n \in \mathbf{N} : \ p_0(x_n) \neq 0\}$ we have $(\pi_{p_0}(x_n))_n = (\pi_{p_0}(x_n))_{n \in J}$ and $\pi_{p_0}(x_n) = 0$ for all $n \notin J$. Moreover, by (3.6) and (3.10) we have $0 \neq \pi_{p_0}(y_k) = \sum_{i=p(k)+1, i \in J}^{p(k+1)} \alpha_i \pi_{p_0}(x_i)$.

Fix $k \in \mathbf{N}$ and set $E_k = \operatorname{span}(x_i)_{i=p(k)+1}^{p(k+1)}$, in which case $\pi_{p_0}(E_k) = \operatorname{span}(\pi_{p_0}(x_i))_{i=p(k)+1, i \in J}^{p(k+1)}$. Since $0 \neq \pi_{p_0}(y_k) \in \pi_{p_0}(E_k)$, there exists $i_k \in \{p(k)+1, \dots, p(k+1)\} \cap J$ so that $\pi_{p_0}(y_k) \notin \operatorname{span}(\pi_{p_0}(x_i))_{i=p(k)+1, i \in J, i \neq i_k}^{p(k+1)}$. To see this, observe that if for every $i \in \{p(k)+1, \dots, p(k+1)\} \cap J$ we have $\pi_{p_0}(y_k) \in \operatorname{span}(\pi_{p_0}(x_j))_{j=p(k)+1, j \in J, j \neq i}^{p(k+1)}$, then $\alpha_i = 0$ for all $i \in \{p(k)+1, \dots, p(k+1)\} \cap J$, thereby implying that $\pi_{p_0}(y_k) = 0$ and hence, that $p_0(y_k) = \hat{p}_0(\pi_{p_0}(y_k)) = 0$ which contradicts (3.6).

So, the set $\{\pi_{p_0}(y_k)\} \cup \{\pi_{p_0}(x_i)\}_{i \in \{p(k)+1, \dots, p(k+1)\} \cap J, i \neq i_k}$ is linearly independent in the local Banach space X_{p_0}. Accordingly, there exists $w_k \in X'_{p_0}$ for which $\langle \pi_{p_0}(y_k), w_k \rangle = 1$ and $\langle \pi_{p_0}(x_i), w_k \rangle = 0$ for all $i \in \{p(k)+1, \dots, p(k+1)\} \cap J$ and $i \neq i_k$. Note that $p'_0(w_k) \leq 1/d$ as $p_0(y_k) > d$. In particular, by (3.6) we also have $\langle \pi_{p_0}(x_i), w_k \rangle = 0$ for all $i \in \{p(k)+1, \dots, p(k+1)\} \setminus J$. Clearly, the set defined by

$$z_i = \begin{cases} x_i & \text{if } i \in \{p(k)+1, \dots, p(k+1)\} \text{ and } i \neq i_k, \\ y_k & \text{if } i = i_k, \end{cases}$$

is also linearly independent. Thus, $(z_i)_{i=p(k)+1}^{p(k+1)}$ is a basis of E_k.

Define a linear map $P_k \colon E_k \to \operatorname{span}(z_i)_{i=p(k)+1}^{p(k+1)}$ by $P_k z = \langle z, v_k \rangle z_{i_k}$, where $v_k := w_k \circ \pi_{p_0} \in X'$ so that $p'_0(v_k) \leq 1/d$. Then $P_k z_i = 0$ if $i \neq i_k$ and $P_k z_{i_k} = z_{i_k}$. Moreover, by (3.11) we obtain, for each $q \in \Gamma_X$ with $q \geq p_0$, that

$$q(P_k z) \leq p'_0(v_k) p_0(z) q(z_{i_k}) \leq \frac{M_q}{d} q(z) \,. \tag{3.12}$$

For such a q, fix $p(k) < r < s \leq p(k+1)$ and scalars $(\beta_i)_{i=p(k)+1}^{p(k+1)}$. If $r \geq i_k$, then (3.5) and (3.12) imply that

$$\begin{aligned} q\Big(\sum_{i=p(k)+1}^{r} \beta_i z_i\Big) &\leq q\left(\sum_{i=p(k)+1, i \neq i_k}^{r} \beta_i x_i\right) + q(\beta_{i_k} z_{i_k}) \\ &\leq q\left(\sum_{i=p(k)+1, i \neq i_k}^{s} \beta_i x_i\right) + q(\beta_{i_k} z_{i_k}) \end{aligned}$$

$$\leq q\left(\sum_{i=p(k)+1}^{s}\beta_i z_i\right)+2q(\beta_{i_k}z_{i_k}) \tag{3.13}$$
$$= q\left(\sum_{i=p(k)+1}^{s}\beta_i z_i\right)+2q\left(P_k\left(\sum_{i=p(k)+1}^{s}\beta_i z_i\right)\right)$$
$$\leq \left(1+2\frac{M_q}{d}\right)q\left(\sum_{i=p(k)+1}^{s}\beta_i z_i\right).$$

If r satisfies $p(k) < r < i_k \leq s$, then again by (3.5) and (3.12) we have

$$q\left(\sum_{i=p(k)+1}^{r}\beta_i z_i\right) \leq q\left(\sum_{i=p(k)+1, i\neq i_k}^{s}\beta_i z_i\right)$$
$$\leq q\left(\sum_{i=p(k)+1}^{s}\beta_i z_i\right)+q(\beta_{i_k}z_{i_k}) \tag{3.14}$$
$$\leq \left(1+\frac{M_q}{d}\right)q\left(\sum_{i=p(k)+1}^{s}\beta_i z_i\right).$$

Finally, if r, s satisfy $p(k) < r < s < i_k$, then by (3.5) we have

$$q\left(\sum_{i=p(k)+1}^{r}\beta_i z_i\right) \leq q\left(\sum_{i=p(k)+1}^{s}\beta_i z_i\right). \tag{3.15}$$

Inequalities (3.13), (3.14) and (3.15) allow us to conclude that there exists $\widetilde{M}_q := (1+2\frac{M_q}{d})$, for each $q \in \Gamma_X$, such that the sequence $(z_i)_i$ satisfies the inequality (3.8) for every $q \in \Gamma_X$. According to Lemma 3.5, $(z_i)_i$ is then a Schauder basis of X. □

Theorem 1.2 will now be presented as two separate results.

Theorem 3.8. *Let X be a complete barrelled lcHs with a Schauder basis $(x_n)_n$. Assume that all the bases in X are shrinking. Then X is reflexive.*

Proof. Suppose that X is not reflexive. Then, by a result of Retherford [44, Theorem 2.3] (see, also, [12, Theorem 4] or [27, Theorem 14.5.1]), $(x_n)_n$ is not boundedly complete. Let Γ_X be a system of continuous seminorms defining the topology of X with respect to which the Schauder basis $(x_n)_n$ is monotone. Since $(x_n)_n$ is not boundedly complete, there exists a sequence of scalars $(\alpha_i)_i$ such that

$$\sup_{n\in\mathbf{N}} q\left(\sum_{i=1}^{n}\alpha_i x_i\right) = M_q < \infty, \quad q \in \Gamma_X, \tag{3.16}$$

and $\sum_{i=1}^{\infty} \alpha_i x_i$ does not converge. Accordingly, there exist $p_0 \in \Gamma_X$ and an increasing sequence $(p(k))_k$ of positive integers with $p(1) = 0$ satisfying

$$\inf_{k\in\mathbf{N}} p_0\left(\sum_{i=p(k)+1}^{p(k+1)} \alpha_i x_i\right) = d > 0 \tag{3.17}$$

and

$$\sup_{k\in\mathbf{N}} q\left(\sum_{i=p(k)+1}^{p(k+1)} \alpha_i x_i\right) \le 2M_q < \infty, \quad q \in \Gamma_X. \tag{3.18}$$

For each $k \in \mathbf{N}$, let $y_k = \sum_{i=p(k)+1}^{p(k+1)} \alpha_i x_i$. Then, by (3.17), $p_0(y_k) \ge d$ for all $k \in \mathbf{N}$ and, by (3.16),

$$q\left(\sum_{s=1}^{k} y_s\right) = q\left(\sum_{i=1}^{p(k+1)} \alpha_i x_i\right) \le M_q\,, \quad k \in \mathbf{N}\,,\ q \in \Gamma_X. \tag{3.19}$$

By (3.17) and (3.18) we can apply Lemma 3.7 (and Remark 3.6) to conclude that there exist an increasing sequence $(i_k)_k$ of integers with $i_k \in \{p(k)+1, \dots, p(k+1)\}$, for $k \in \mathbf{N}$, and a Schauder basis $(z_i)_i$ in X given by

$$z_i = \begin{cases} x_i & \text{if } i \ne i_k \text{ for all } k, \\ y_k & \text{if } i = i_k \end{cases} \tag{3.20}$$

such that, for every $q \in \Gamma_X$ with $q \ge p_0$, there exists $C_q > 0$ such that

$$q\left(\sum_{i=1}^{r} \beta_i z_i\right) \le C_q q\left(\sum_{i=1}^{s} \beta_i z_i\right), \tag{3.21}$$

for arbitrary positive integers $r \le s$ and arbitrary scalars $\beta_1, \dots, \beta_s$.

Denote by $(f_i)_i$ the dual basis of $(z_i)_i$. Observe also that, if $(v_n)_n$ denotes the dual basis of $(x_n)_n$, then $\operatorname{span}(f_i)_{i=1}^{\infty} = \operatorname{span}(v_n)_{n=1}^{\infty}$ by (3.20). It follows from (3.17), (3.19) and (3.21) that the sequence $(z_{i_k}, f_{i_k})_k$ is of type P and

$$p_0'(f_{i_k}) \le 2\frac{C_{p_0}}{d}, \quad k \in \mathbf{N}. \tag{3.22}$$

For each $i \in \mathbf{N}$, define

$$u_i = \begin{cases} z_i & \text{if } i \ne i_k \text{ for all } k, \\ \sum_{r=1}^{k} z_{i_r} & \text{if } i = i_k \end{cases} \tag{3.23}$$

and

$$g_i = \begin{cases} f_i & \text{if } i \ne i_k \text{ for all } k, \\ f_{i_k} - f_{i_{k+1}} & \text{if } i = i_k. \end{cases} \tag{3.24}$$

Then $\overline{\operatorname{span}}(u_i)_{i=1}^{\infty} = X$ and $(u_i, g_i)_i$ is a biorthogonal sequence; see, [50, Proposition 2], for example. Actually, $(u_i)_i$ is also a Schauder basis in X. To show this we can proceed either as in [28, Theorem 3.1] or as follows. Let $U_n(x) = \sum_{i=1}^{n} \langle x, g_i\rangle u_i$ for any $x \in X$

and $n \in \mathbf{N}$. If $p(k) \leq n < p(k+1)$ for some $k \in \mathbf{N}$, then (3.19) and (3.21), (3.22) imply that

$$\begin{aligned}
q\left(U_n(x)\right) &= q\left(\sum_{i=1, i\notin\{i_1,i_2,\dots,i_k\}}^{n} \langle x, f_i\rangle z_i + \sum_{r=1}^{k}\langle x, f_{i_k} - f_{i_{k+1}}\rangle \sum_{s=1}^{r} z_{i_s}\right)\\
&= q\left(\sum_{i=1, i\notin\{i_1,i_2,\dots,i_k\}}^{n} \langle x, f_i\rangle z_i + \sum_{s=1}^{k} z_{i_s}\sum_{r=s}^{k}\langle x, f_{i_k} - f_{i_{k+1}}\rangle\right)\\
&= q\left(\sum_{i=1, i\notin\{i_1,i_2,\dots,i_k\}}^{n} \langle x, f_i\rangle z_i + \sum_{s=1}^{k}\langle x, f_{i_s}\rangle z_{i_s} - \sum_{s=1}^{k}\langle x, f_{i_{k+1}}\rangle z_{i_s}\right)\\
&\leq q\left(\sum_{i=1}^{n}\langle x, f_i\rangle z_i\right) + q\left(-\sum_{s=1}^{k-1}\langle x, f_{i_{k+1}}\rangle z_{i_s} - \langle x, f_{i_{k+1}}\rangle z_{i_k}\right)\\
&\leq C_q q(x) + |\langle x, f_{i_{k+1}}\rangle| q\left(\sum_{s=1}^{k} z_{i_s}\right)\\
&\leq (C_q + p_0'(f_{i_{k+1}})M_q)q(x) \leq (C_q + 2\frac{C_{p_0}}{d}M_q)q(x),
\end{aligned}$$

for all $q \in \Gamma_X$ with $q \geq p_0$ and all $x \in \mathbf{N}$. As X is a complete lcHs, this means that $(u_i)_i$ is indeed a Schauder basis of X (with dual basis $(g_i)_i$). In particular, the sequence $(u_{i_k}, g_{i_k})_k$ is of type P^* as the sequences $(u_{i_k})_k$ and $(\sum_{h=1}^{k} g_{i_h})_k = (f_{i_1} - f_{i_{k+1}})_k$ are bounded by (3.19) and (3.22).

But, for every $k \in \mathbf{N}$, we have that

$$\langle u_{i_k}, f_{i_1}\rangle = \langle\sum_{r=1}^{k} z_{i_r}, f_{i_1}\rangle = 1,$$

which implies that $(u_i)_i$ is *not* shrinking. For, if $(u_i)_i$ is shrinking, then $f_{i_1} = \sum_{i=1}^{\infty}\langle u_i, f_{i_1}\rangle g_i$ in X'_β because $f_{i_1} \in X'$. Hence, $\langle u_i, f_{i_1}\rangle g_i \to 0$ in X'_β and so also $\langle u_{i_k}, f_{i_1}\rangle g_{i_k} \to 0$, thereby implying that $1 = \sup_{x\in(u_{i_s})_s} |\langle x, g_{i_k}\rangle| \to 0$ as $(u_{i_k})_k$ is bounded. So, $(u_i)_i$ is not shrinking and the theorem is proved. □

Theorem 3.9. *Let X be a complete barrelled lcHs with a Schauder basis $(x_n)_n$. If all the bases in X are boundedly complete, then X is reflexive.*

Proof. Denote by $(f_n)_n$ the dual basis of $(x_n)_n$. Recall that H denotes the subspace of X' consisting of all $f \in X'$ such that $\lim_{n\to\infty}\sum_{i=1}^{n}\langle x_i, f\rangle f_i = f$ in X'_β, endowed with the topology $\beta(H, X)$ induced by $\beta(X', X)$. Hence, $(f_n)_n$ is a Schauder basis for H.

By assumption $(x_n)_n$ is boundedly complete. So, by [29, Lemma 6.2] applied to the space $H = \overline{\operatorname{span}}(f_n)_{n=1}^{\infty}$, with the closure taken in X'_β, we can conclude that $\beta(X', X) = \beta(H, X) = \beta(H, H')$ and, by [29, Proposition 5.3 and Theorem 6.3], H is barrelled with $H'_\beta = (H', \beta(H', H)) = X$ algebraically and topologically. Moreover, $(f_n)_n$ is a shrinking basis for H whose dual basis is clearly $(x_n)_n$; see [29, Corollary 3] or [17, Theorem 1.6].

Suppose that X is not reflexive. Again by a result of Retherford [44, Theorem 2.3] (see also [12, Theorem 4] or [27, Theorem 14.5.1]), $(x_n)_n$ is not shrinking in X and so $(f_n)_n$ is not a boundedly complete basis for H, [29, Proposition 5.5]. Since H is a complete barrelled lcHs and $(f_n)_n$ is not a boundedly complete Schauder basis of H, we can proceed as in the proof of Theorem 3.8 to establish the following fact: the space H has another Schauder basis $(g_i)_i$, with dual basis $(w_i)_i$, such that $(g_i)_i$ contains a biorthogonal subsequence $(g_{i_k}, w_{i_k})_k$ of type P and

$$\mathrm{span}(w_i)_{i=1}^{\infty} = \mathrm{span}(x_n)_{n=1}^{\infty}. \tag{3.25}$$

Therefore the sequence $(h_i)_i$ in H defined by

$$h_i = \begin{cases} g_i & \text{if } i \neq i_k \text{ for all } k, \\ \sum_{r=1}^{k} g_{i_r} & \text{if } i = i_k \end{cases} \tag{3.26}$$

is a non shrinking Schauder basis for H with dual basis $(t_i)_i$ given by

$$t_i = \begin{cases} w_i & \text{if } i \neq i_k \text{ for all } k, \\ w_{i_k} - w_{i_{k+1}} & \text{if } i = i_k. \end{cases} \tag{3.27}$$

In particular, the biorthogonal sequence $(h_{i_k}, t_{i_k})_k$ is of type P^*. Thus, $(t_i)_i$ is a Schauder basis for $\overline{\mathrm{span}}(t_i)_{i=1}^{\infty}$ in $H'_\beta = X$.

Now, set $t_0 = w_{i_1}$. By (3.27) and (3.25) we obtain that

$$\mathrm{span}(t_i)_{i=0}^{\infty} = \mathrm{span}(w_i)_{i=1}^{\infty} = \mathrm{span}(x_n)_{n=1}^{\infty}\,.$$

If $t_0 = w_{i_1} \in \overline{\mathrm{span}}(t_i)_{i=1}^{\infty}$, then $w_{i_1} = \sum_{k=1}^{\infty} t_{i_k}$ with the series converging in $H'_\beta = X$. But, this is impossible because $(t_{i_k})_k$ does not $\beta(H', H)$-converge to 0 (indeed, $1 = \sup_{x\in(h_{i_s})_s} |\langle x, t_{i_k}\rangle|$, where $(h_{i_s})_s$ is $\beta(H, H')$-bounded because $(h_{i_k}, t_{i_k})_k$ is of type P^*). Thus, $X = H'_\beta = \overline{\mathrm{span}}(x_n)_{n=1}^{\infty} = [t_0] \oplus \overline{\mathrm{span}}(t_i)_{i=1}^{\infty}$ which implies that the extension $(t_i)_{i=0}^{\infty}$ of $(t_i)_{i=1}^{\infty}$ by the element t_0 is also a Schauder basis of X. Hence, $(t_i)_{i=0}^{\infty}$ must be boundedly complete. However, $(\sum_{h=1}^{k} t_{i_h})_k$ is bounded and does not converge in $X = H'_\beta$; contradiction! The proof is thereby complete. □

Finally, the validity of the converse of both Theorem 3.8 and Theorem 3.9 has already been noted earlier (see e.g. [27, Theorem 14.5.1]). The proof of Theorem 1.2 is thereby completely established.

4. Schauder decompositions

A decomposition of a lcHs X is a sequence $(E_n)_n$ of closed, non-trivial subspaces of X such that each $x \in X$ can be expressed uniquely in the form $x = \sum_{i=1}^{\infty} y_i$ with $y_i \in E_i$ for each $i \in \mathbf{N}$. This induces a sequence $(Q_n)_n$ of projections defined by $Q_n x := y_n$ where $x = \sum_{i=1}^{\infty} y_i$ with $y_i \in E_i$ for each $i \in \mathbf{N}$. These projections are pairwise orthogonal (i.e. $Q_n Q_m = 0$ if $n \neq m$) and $Q_n(X) = E_n$ for $n \in \mathbf{N}$. If, in addition, each $Q_n \in \mathscr{L}(X)$, for $n \in \mathbf{N}$, then we speak of a *Schauder decomposition* of X. For a basis $(x_n)_n$ of X, the projection Q_n takes the form $Q_n x = \langle x, f_n\rangle$ with $(f_n)_n$ being the sequence of coefficient functionals associated to (x_n); see Section 3. Then $Q_n \in \mathscr{L}(X)$ if and only if $f_n \in X'$. So, for the 1-dimensional spaces $E_n := \{\lambda x_n : \lambda \in \mathbf{C}\}$, for $n \in \mathbf{N}$, we see that $(E_n)_n$ is a Schauder decomposition of X if and only if $(x_n)_n$ is a Schauder basis for X.

Let $(E_n)_n$ be a Schauder decomposition for a lcHs X. As observed above, $(E_n)_n$ induces a sequence $(Q_n)_n \subseteq \mathscr{L}(X)$ of non-zero projections satisfying $Q_n Q_m = 0$ (if

$n \neq m$) and $x = \sum_{n=1}^{\infty} Q_n x$ for each $x \in X$. Conversely, if $(Q_n)_n \subseteq \mathscr{L}(X)$ is any sequence of projections satisfying these two conditions, then $(Q_n(X))_n$ is a Schauder decomposition of X. By setting $P_n := \sum_{i=1}^{n} Q_i$, for $n \in \mathbf{N}$, we arrive at the following equivalent definition, preferred by some authors in certain situations; see [9], [37], [38]. A sequence $(P_n)_n \subseteq \mathscr{L}(X)$ of projections is called a *Schauder decomposition* of X if it satisfies:

(S1) $P_n P_m = P_{\min\{m,n\}}$ for all $m, n \in \mathbf{N}$,
(S2) $P_n \to I$ in $\mathscr{L}_s(X)$ as $n \to \infty$, and
(S3) $P_n \neq P_m$ whenever $n \neq m$.

By setting $Q_1 := P_1$ and $Q_n := P_n - P_{n-1}$ for $n \geq 2$ we arrive at the more traditional formulation of a Schauder decomposition as given above. Let $(P_n)_n \subseteq \mathscr{L}(X)$ be a Schauder decomposition of X. Then the dual projections $(P_n^t)_n \subseteq \mathscr{L}(X'_\sigma)$ always form a Schauder decomposition of X'_σ, [29, p. 378]. Note that necessarily $(P_n^t)_n \subseteq \mathscr{L}(X'_\beta)$, [32, p. 134]. If, in addition, $(P_n^t)_n$ is a Schauder decomposition for X'_β, then the original sequence $(P_n)_n$ is called *shrinking*, [29, p. 379]. Since (S1) and (S3) clearly hold for $(P_n^t)_n$, this means precisely that $P_n^t \to I$ in $\mathscr{L}_s(X'_\beta)$; see (S2).

In dealing with the uniform mean ergodicity of operators the following notion, due to Díaz and Miñarro, [14, p. 194], is rather useful. A Schauder decomposition $(P_n)_n$ in a lcHs X is said to have *property* (M) if $P_n \to I$ in $\mathscr{L}_b(X)$ as $n \to \infty$. Since every non-zero projection P in a Banach space satisfies $\|P\| \geq 1$, it is clear that no Schauder decomposition in any Banach space can have property (M). For non-normable spaces we will see below that the situation is quite different.

Remark 4.1. Let $(P_n)_n \subseteq \mathscr{L}(X)$ be a Schauder decomposition of X with property (M). If X is quasi-barrelled, then it is routine to verify that $P_n^t \to I$ in $\mathscr{L}_b(X'_\beta)$. In particular, $P_n^t \to I$ in $\mathscr{L}_s(X'_\beta)$. That is, $(P_n)_n$ is necessarily a *shrinking* Schauder decomposition of X and $(P_n^t)_n$ is a Schauder decomposition of X'_β with property (M).

We now formulate a result which provides a systematic method for producing examples of Schauder decompositions *without* property (M).

Let X be a lcHs and (Ω, Σ) be a measurable space. A finitely additive set function $P\colon \Sigma \to \mathscr{L}(X)$ is called a *spectral measure* in X if it satisfies $P(\emptyset) = 0$ and $P(\Omega) = I$, is multiplicative (i.e. $P(A \cap B) = P(A)P(B)$ for all $A, B \in \Sigma$) and is σ-additive in $\mathscr{L}_s(X)$, i.e. $A_n \downarrow \emptyset$ in Σ implies that $P(A_n) \to 0$ in $\mathscr{L}_s(X)$. If, in addition, P is also σ-additive in $\mathscr{L}_b(X)$, then it is called *boundedly σ-additive*. Non-trivial examples of boundedly σ-additive spectral measures can only occur in non-normable spaces. For examples of spectral measures in classical Fréchet spaces, some of which are boundedly σ-additive and others which are not, we refer to [7], [8], [6], [9], [46], [47].

We denote by $\mathscr{B}(X)$ the collection of all bounded subsets of a lcHs X.

Proposition 4.2. *Let X be a Fréchet space. Suppose that there exists a spectral measure in X which fails to be boundedly σ-additive. Then X admits a Schauder decomposition without property (M).*

Proof. Let $P\colon \Sigma \to \mathscr{L}(X)$ be any spectral measure which fails to be boundedly σ-additive. Let $q_1 \leq q_2 \leq \dots$ be a sequence of seminorms generating the topology of X and having the property that

$$q_k(P(A)x) \leq q_k(x), \quad x \in X,\ A \in \Sigma, \tag{4.1}$$

for each $k \in \mathbf{N}$; see Proposition 2.3 in [52] and the discussion following its proof. Since P fails to be boundedly σ-additive, there exists a sequence $(A_n)_n \subseteq \Sigma$ with $A_n \uparrow \Omega$ such that $P(A_n) \not\to I$ in $\mathscr{L}_b(X)$. It follows that there exists a sequence of positive integers $n(k) \uparrow \infty$ such that $P(A_{n(k)}) \neq P(A_{n(k+1)})$ for $k \in \mathbf{N}$. Set $P_k := P(A_{n(k)})$ for $k \in \mathbf{N}$, in which case it is routine to check (using the σ-additivity of P in $\mathscr{L}_s(X)$) that $(P_k)_k$ is a Schauder decomposition of X.

The proof is completed by showing that $P_k \not\to I$ in $\mathscr{L}_b(X)$. On the contrary, suppose that $P_k \to I$ in $\mathscr{L}_b(X)$ as $k \to \infty$. According to (1.4), the topology of $\mathscr{L}_b(X)$ is determined by the seminorms

$$q_{B,k}(T) := \sup_{x\in B} q_k(Tx), \quad T \in \mathscr{L}(X),$$

for all $k \in \mathbf{N}$ and $B \in \mathscr{B}(X)$. So, fix such a k and B. Given $\varepsilon > 0$, there exists $M \in \mathbf{N}$ such that

$$\sup_{x\in B} q_k((I - P(A_{n(m)})x) = q_{B,k}(I - P_m) \le \varepsilon, \quad m \ge N. \tag{4.2}$$

Let $n \ge n(M)$. Then there exists $m \ge M$ satisfying $n(m) \le n < n(m+1)$. Hence, by property (S1) of a Schauder decomposition we have $I - P(A_n) = (I - P(A_{n(m)}))(I - P(A_n))$. It follows from this identity, (4.1) and (4.2) that $\sup_{x\in B} q_k((I - P(A_n))x) \le \varepsilon$, that is,

$$q_{B,k}((I - P(A_n))) \le \varepsilon, \quad n \ge n(M).$$

This shows that $P(A_n) \to I$ in $\mathscr{L}_b(X)$; contradiction to the choice of $(P(A_n))_n$! Hence, $P_k \not\to I$ in $\mathscr{L}_b(X)$. $\square$

Remark 4.3. Explicit examples of Fréchet spaces which admit spectral measures which *fail* to be boundedly σ-additive are surely known. For the spaces $L^p_{\ell oc}(\mathbf{R})$, $1 \le p < \infty$, we refer to [7, Proposition 6.2(ii)] and for ℓ^{p+}, $1 \le p < \infty$, see [8, Corollary 3.2(i)]. For those Köthe echelon spaces $\lambda_p(A)$ with $p \in \{0\} \cup [1, \infty)$ which are *not* Montel see [8, Corollary 3.2(ii)]. The same conclusion holds for Köthe function spaces over non-atomic measure spaces, [6, Proposition 2.14].

We conclude with two technical results needed later. The first result is an extension of a lemma in [23, p. 149] to the Fréchet space setting.

Lemma 4.4. *Let X be a Fréchet space which admits a non-shrinking Schauder decomposition. Then there exist a Schauder decomposition $(P_j)_j \subseteq \mathscr{L}(X)$ of X, a functional $\xi \in X'$ and a bounded sequence $(z_j)_j \subseteq X$ with $z_j \in (P_{j+1} - P_j)(X)$ such that $|\langle z_j, \xi\rangle| > \frac{1}{2}$ for all $j \in \mathbf{N}$.*

Proof. Let $(R_n)_n \subseteq \mathscr{L}(X)$ be a non-shrinking Schauder decomposition of X. It follows that there exist a set $B \in \mathscr{B}(X)$, a functional $\xi \in X'$ and a sequence of positive integers $n(j) \uparrow \infty$ such that

$$\sup_{x\in B} |\langle x, (I - R^t_{n(j)})\xi\rangle| > 1, \quad j \in \mathbf{N}. \tag{4.3}$$

Select $q \in \Gamma_X$ such that

$$|\langle x, \xi\rangle| \le q(x), \quad x \in X. \tag{4.4}$$

By (4.3), there exists $x^1 \in B$ with $|\langle (I - R_{n(1)})x^1, \xi\rangle| > 1$. Set $m(1) := n(1)$. Since $R_n x^1 \to x^1$ in X as $n \to \infty$, we can select $m(2) \in (n(k))_k$ with $m(2) > n(1)$ such that $q((I-R_{m(2)})x^1) < \frac{1}{2}$. Moreover, (4.4) then implies that $|\langle (I-R_{m(2)})x^1, \xi\rangle| < \frac{1}{2}$. Define $z_1 := (R_{m(2)} - R_{m(1)})x^1$ and set $P_1 := R_{m(1)}$ and $P_2 := R_{m(2)}$. Observe that

$$\begin{aligned} |\langle z_1, \xi\rangle| &= |\langle (I-R_{m(1)})x^1 - (I-R_{m(2)})x^1, \xi\rangle| \\ &\geq |\langle (I-R_{m(1)})x^1, \xi\rangle| - |\langle (I-R_{m(2)})x^1, \xi\rangle| > \left(1-\frac{1}{2}\right) = \frac{1}{2}. \end{aligned}$$

According to (4.3), there exists $x^2 \in B$ with $|\langle (I-R_{m(2)})x^2, \xi\rangle| > 1$. Select $m(3) \in (n(k))_k$ with $m(3) > m(2)$ such that $q((I-R_{m(3)})x^2) < \frac{1}{2}$. Again (4.4) implies that $|\langle (I-R_{m(3)})x^2, \xi\rangle| < \frac{1}{2}$. Define $z_2 := (R_{m(3)} - R_{m(2)})x^2$ and set $P_3 := R_{m(3)}$. As above,

$$|\langle z_2, \xi\rangle| = |\langle (I-R_{m(2)})x^2 - (I-R_{m(3)})x^2, \xi\rangle| > \frac{1}{2}.$$

Proceeding in this way we get $m(j) \uparrow \infty$ and a sequence $(x^j)_j \subseteq B$ such that $|\langle z_j, \xi\rangle| \geq \frac{1}{2}$ for all $j \in \mathbf{N}$, where $z_j := (R_{m(j+1)} - R_{m(j)})x^j$. Setting $P_j := R_{m(j)}$, for $j \in \mathbf{N}$, we see that $(P_j)_j$ is a Schauder decomposition of X and that $z_j \in (P_{j+1} - P_j)(X)$ for all $j \in \mathbf{N}$. Since $B \in \mathscr{B}(X)$ and $(P_{j+1} - P_j)_j$ is equicontinuous in $\mathscr{L}(X)$, it follows that $D := \bigcup_{j\in\mathbf{N}} (P_{j+1} - P_j)(B)$ is a bounded set in X. But, $(z_j)_j \subseteq D$ and hence, $(z_j)_j$ is bounded in X. □

Lemma 4.5. *Let X be a Fréchet space which admits a Schauder decomposition without property (M). Then there exist a Schauder decomposition $(P_j)_j$ of X, a seminorm $q \in \Gamma_X$ and a bounded sequence $(z_j)_j \subseteq X$ with $z_j \in (P_{j+1} - P_j)(X)$ such that $q(z_j) > \frac{1}{2}$ for all $j \in \mathbf{N}$.*

Proof. Let $(R_n)_n \subseteq \mathscr{L}(X)$ be a Schauder decomposition of X which does not have property (M). Hence, there exist a set $B \in \mathscr{B}(X)$, a seminorm $q \in \Gamma_X$ and positive integers $n(j) \uparrow \infty$ such that

$$\sup_{x\in B} q((I-R_{n(j)})x) > 1, \quad j \in \mathbf{N}. \tag{4.5}$$

Select $x^1 \in B$ with $q((I-R_{n(1)})x^1) > 1$. Set $m(1) := n(1)$. Since $R_n x^1 \to x^1$ in X as $n \to \infty$, there exists $m(2) \in (n(k))_k$ such that $q((I-R_{m(2)})x^1) < \frac{1}{2}$. Set $z_1 := (R_{m(2)} - R_{m(1)})x^1$ and define $P_1 := R_{m(1)}$ and $P_2 := R_{m(2)}$. As in the proof of Lemma 4.4 we can conclude that

$$q(z_1) = q\left((I-R_{m(1)})x^1 - (I-R_{m(2)})x^1\right) > \frac{1}{2}.$$

According to (4.5), there exists $x^2 \in B$ with $q((I-R_{m(2)})x^2) > 1$. Select $m(3) \in (n(k))_k$ with $m(3) > m(2)$ such that $q((I-R_{m(3)})x^2) < \frac{1}{2}$. Define $z_2 := (R_{m(3)} - R_{m(2)})x^2$ and $P_3 := R_{m(3)}$ and note that $q(z_2) > \frac{1}{2}$. Proceed in this way to get $m(j) \uparrow \infty$ and a sequence $(x^j)_j \subseteq B$ such that $z_j := (R_{m(j+1)} - R_{m(j)})x^j$ satisfies $q(z_j) > \frac{1}{2}$, for each $j \in \mathbf{N}$. Put $P_j := R_{m(j)}$, for $j \in \mathbf{N}$, and then complete the argument as in the proof of Lemma 4.4. □

5. Mean ergodic operators

Using the results of previous sections we can now establish Theorems 1.3–1.6. Various examples and consequences of these results are also given.

We begin with a useful observation. Let $(P_n)_n \subseteq \mathscr{L}(X)$ be any Schauder decomposition in the Fréchet space X. As noted in [14, p. 192], there exists an increasing sequence of seminorms $(q_k)_k$ determining the topology of X such that

$$q_k(P_jx) \le q_k(x), \quad x \in X,\ j \in \mathbf{N}, \tag{5.1}$$

for each $k \in \mathbf{N}$. Indeed, if $(p_m)_m$ is any increasing sequence of seminorms determining the topology of X, then the seminorms $q_k(x) := \sup_{j\in\mathbf{N}} p_k(P_jx)$, for $x \in X$ and $k \in \mathbf{N}$, have the desired property (after noting that X is barrelled and hence, $(P_n)_n \subseteq \mathscr{L}(X)$ is equicontinuous).

Proof of Theorem 1.5. Let $(P_j)_j \subseteq \mathscr{L}(X)$ denote a Schauder decomposition as given by Lemma 4.4 and define projections $Q_j := P_j - P_{j-1}$ (with $P_0 := 0$) and closed subspaces $X_j := Q_j(X)$ of X, for $j \in \mathbf{N}$. By Lemma 4.4 there also exist a bounded sequence $(z_j)_j \subseteq X$ with $z_j \in X_{j+1}$ and $\xi \in X'$ such that $|\langle z_j, \xi\rangle| > \frac{1}{2}$, for all $j \in \mathbf{N}$. Set $e_j := z_j/\langle z_j, \xi\rangle \in Q_{j+1}(X)$, in which case $(e_j)_j$ is a bounded sequence in X with $\langle e_j, \xi\rangle = 1$, for all $j \in \mathbf{N}$. Let $(q_k)_k$ be any increasing sequence of seminorms giving the topology of X, satisfying (5.1) and such that $|\langle x, \xi\rangle| \le q_1(x)$ for all $x \in X$.

As in [23, p. 150], take an arbitrary sequence of positive numbers $a = (a_j)_j$ with $\sum_{j=1}^\infty a_j = 1$ and set $A_n := \sum_{j=1}^n a_j$, for $n \in \mathbf{N}$. Let $x \in X$. For integers $m > n \ge 2$ we have

$$\sum_{k=n}^m A_kQ_kx = \Big(\sum_{j=1}^{n-1} a_j\Big)\Big(\sum_{k=n}^m Q_kx\Big) + \sum_{j=n}^m a_j\Big(\sum_{k=j}^m Q_kx\Big).$$

Since $\sum_{k=1}^\infty Q_kx = x$, we see that $(\sum_{k=1}^m A_kQ_kx)_m$ is Cauchy and hence, convergent in X. Moreover, for each $s \in \mathbf{N}$, we have (cf. (5.1))

$$\begin{aligned} q_s\Big(\sum_{k=1}^m A_kQ_kx\Big) &= q_s\Big(\sum_{j=1}^m a_j(P_m - P_{j-1})x\Big) \\ &\le \sum_{j=1}^m a_j(q_s(P_mx) + q_s(P_{j-1}x)) \le 2q_s(x), \end{aligned}$$

for all $m \in \mathbf{N}$. Define a linear map $T_a \colon X \to X$ by

$$T_ax := \sum_{k=1}^\infty A_kQ_kx + \sum_{j=2}^\infty \langle P_{j-1}x, \xi\rangle a_je_j, \quad x \in X. \tag{5.2}$$

Using the previous inequalities we see that

$$q_s(T_ax) \le 2q_s(x) + \sum_{j=2}^\infty a_jq_1(P_{j-1}x)q_s(e_j), \quad x \in X.$$

Setting $M_s := \sup_j q_s(e_j) < \infty$ (recall that $(e_j)_j$ is bounded in X) and noting that $q_1(P_{j-1}x) \le q_1(x) \le q_s(x)$, it follows that

$$q_s(T_ax) \le (2 + M_s)q_s(x), \quad x \in X,\ s \in \mathbf{N},$$

with M_s independent of a. In particular, $T_a \in \mathscr{L}(X)$.

To show that T_a is power bounded it suffices to show, for arbitrary sequences $a = (a_j)_j$ and $b = (b_j)_j$ of positive numbers with $\sum_{j=1}^\infty a_j = 1 = \sum_{j=1}^\infty b_j$, that the composition T_aT_b is also of the same type (say, T_c for an appropriate $c = (c_j)_j$). But, this is precisely the Claim on p. 150 in [23] which is proved there (on p. 151) by purely "algebraic computations" and hence, carries over to our setting here. So, T_a is indeed power bounded.

To deduce that T_a is *not* mean ergodic we apply Theorem 2.12 by verifying that $\ker(I - T_a) = \{0\}$ and that $\xi \in X'$ belongs to $\ker(I - T_a^t)$.

Let $x \in \ker(I - T_a)$. It follows from (5.2) and $x = \sum_{k=1}^{\infty} Q_k x$ that

$$\sum_{k=1}^{\infty} Q_k x = x = \sum_{k=1}^{\infty} A_k Q_k x + \sum_{k=2}^{\infty} \langle P_{k-1}x, \xi\rangle a_k e_k. \tag{5.3}$$

Moreover, (5.2) together with the identities

$$Q_k Q_j = 0, \quad k \neq j, \tag{5.4}$$

which imply that $Q_1 e_k = 0$ for $k \geq 2$, yield

$$Q_1 x = Q_1 T_a x = A_1 Q_1 x.$$

Since $0 < A_1 = a_1 < 1$, we see that $Q_1 x = 0$. For $k > 1$, apply Q_k to (5.3) to conclude that

$$Q_k x = A_k Q_k x + \langle P_{k-1}x, \xi\rangle a_k e_k. \tag{5.5}$$

Now, $P_1 x = Q_1 x + P_0 x = 0$ and so, by substituting $k = 2$ into (5.5), we see that $Q_2 x = A_2 Q_2 x$ with $0 < A_2 < 1$. Hence, $Q_2 x = 0$. Proceed inductively (via (5.5) and the formula $P_{k-1}x = \sum_{j=1}^{k-1} Q_j x$) to conclude that $Q_k x = 0$ for all $k \in \mathbf{N}$. Then $x = \sum_{k=1}^{\infty} Q_k x = 0$, that is, $\ker(I - T_a) = \{0\}$.

It remains to verify that $\xi \in \ker(I - T_a^t)$. To this end, fix $k \in \mathbf{N}$ and $y \in X_k$. By (5.2), (5.4), the equalities $\langle e_j, \xi\rangle = 1$ for all $j \in \mathbf{N}$, and the identities

$$P_{j-1}y = P_{j-1}Q_k y = \begin{cases} 0 & \text{for } 1 \leq j \leq k, \\ y & \text{for } j > k, \end{cases}$$

we have that

$$\langle y, T_a^t \xi\rangle = \langle T_a y, \xi\rangle = \langle A_k y + \sum_{j=k+1}^{\infty} \langle y, \xi\rangle a_j e_j, \xi\rangle = \langle y, \xi\rangle \cdot \Big(A_k + \sum_{j=k+1}^{\infty} a_j\Big) = \langle y, \xi\rangle.$$

Hence, $\langle (I - T_a)y, \xi\rangle = 0$ for all $k \in \mathbf{N}$ and $y \in X_k$. In view of the decomposition $X = \sum_{k=1}^{\infty} X_k$ we conclude that $\langle y, (I - T_a^t)\xi\rangle = 0$ for all $y \in X$, that is, $\xi \in \ker(I - T_a^t)$. □

Proposition 5.1. *Let X be a Fréchet space. Then X is reflexive if and only if every closed subspace of X is mean ergodic.*

Proof. Suppose that X is not reflexive. By Theorem 1.1 there exists a non-reflexive, closed subspace Y with a basis. According to Theorem 1.2, the space Y must have some non-shrinking Schauder basis. In particular, Y admits a non-shrinking Schauder decomposition and hence, Theorem 1.5 implies that Y is not mean ergodic. □

Proof of Theorem 1.4. Let X be a Fréchet space with a basis. If X is non-reflexive, then Theorem 1.2 shows that X admits a non-shrinking Schauder basis (and hence, a non-shrinking Schauder decomposition). By Theorem 1.5 we can conclude that X is not mean ergodic. □

Theorem 5.2. *Let X be a Fréchet space which admits a Schauder decomposition without property (M). Then there exists a power bounded, mean ergodic operator in $\mathscr{L}(X)$ which fails to be uniformly mean ergodic.*

Proof. Let $(P_j)_j \subseteq \mathscr{L}(X)$ denote a Schauder decomposition of X as given by Lemma 4.5 and define projections $Q_j := P_j - P_{j-1}$ (with $P_0 = 0$) and closed subspaces $X_j := Q_j(X)$ of X, for $j \in \mathbf{N}$. By Lemma 4.5 there exist a seminorm $q \in \Gamma_X$ and a bounded sequence $(z_j)_j \subseteq X$ with $z_j \in X_{j+1}$ such that $q(z_j) > \frac{1}{2}$, for all $j \in \mathbf{N}$. Choose a sequence of increasing seminorms $(q_k)_k$ generating the topology of X which satisfy (5.1) and such that $q \leq q_1$.

Take any sequence of positive numbers $a = (a_j)_j$ with $\sum_{j=1}^{\infty} a_j = 1$ and set $A_n := \sum_{j=1}^{n} a_j$ for $n \in \mathbf{N}$. Define a linear map $T_a \colon X \to X$ by

$$T_a x := \sum_{k=1}^{\infty} A_k Q_k x, \quad x \in X; \tag{5.6}$$

observe that this corresponds to (5.2) for the case when $\xi = 0$. As in the proof of Theorem 1.5 given above the operator T_a is well defined, satisfies

$$q_s(T_a x) \leq 2 q_s(x), \quad x \in X,\ s \in \mathbf{N}, \tag{5.7}$$

and is power bounded.

Given $x \in \ker(I - T_a)$ we have $x = T_a x$ and so, from (5.6), it follows that $x = \sum_{k=1}^{\infty} A_k Q_k x$. Using (5.4) it follows that $Q_j x = A_j Q_j x$ and hence, since $0 < A_j < 1$, that $Q_j x = 0$ for all $j \in \mathbf{N}$. That is, $x = 0$ and so $\ker(I - T_a) = \{0\}$. To conclude that T_a is mean ergodic it suffices, by Theorem 2.12, to show that $\ker(I - T_a^t) = \{0\}$. So, suppose that $u \in X'$ satisfies $T_a^t u = u$. Let $x_j \in X_j$ be arbitrary and note that $\langle x_j, u\rangle = \langle T_a x_j, u\rangle$ for all $j \in \mathbf{N}$. It follows form (5.6) and (5.7) that

$$\langle x_j, u\rangle = \sum_{k=1}^{\infty} A_k \langle Q_k x_j, u\rangle = \sum_{k=1}^{\infty} A_k \langle Q_k Q_j x_j, u\rangle = A_j \langle x_j, u\rangle$$

for $j \in \mathbf{N}$; see also (5.4). Since $0 < A_j < 1$, we conclude that $\langle x_j, u\rangle = 0$. So, we have shown that $\langle y, u\rangle = 0$ whenever $y \in X_j$ for some $j \in \mathbf{N}$. Since $\sum_{j=1}^{\infty} Q_j = I$ (in $\mathscr{L}_s(X)$), it follows that each $x \in X$ has a decomposition $x = \sum_{j=1}^{\infty} Q_j x$ with $Q_j x \in X_j$ for all $j \in \mathbf{N}$. By continuity of u it follows that $u = 0$. So, $\ker(I - T_a^t) = \{0\}$ and hence, T_a is mean ergodic.

The proof is completed by showing, for the choice $a_j := 2^{-j}$ for $j \in \mathbf{N}$, that T_a is *not* uniformly mean ergodic. For ease of notation, set $T := T_a$. Note that $A_k = 1 - 2^{-k}$ for each $k \in \mathbf{N}$. Moreover, from (5.6), (5.7) and $\sum_{j=1}^{\infty} Q_j = I$ (in $\mathscr{L}_s(X)$), it follows that

$$T^m x = \sum_{k=1}^{\infty} A_k^m Q_k x, \quad x \in X,\ m \in \mathbf{N}.$$

Then (1.5) and direct calculation yields

$$T_{[n]} x = \frac{1}{n} \sum_{k=1}^{\infty} \frac{A_k}{(1 - A_k)} \cdot (1 - A_k^n) Q_k x, \quad x \in X,\ n \in \mathbf{N}. \tag{5.8}$$

Since T is mean ergodic, there exists $P \in \mathscr{L}(X)$ with $T_{[n]} \to P$ in $\mathscr{L}_s(X)$ as $n \to \infty$. For $j \in \mathbf{N}$ fixed and $x \in X_j$, it follows from (5.4) that $Q_k x = \delta_{kj} x$ for all $k \in \mathbf{N}$ and hence, (5.8) implies that

$$T_{[n]}x = \frac{1}{n}\frac{A_j}{(1-A_j)}\cdot(1-A_j^n)x, \quad n\in\mathbf{N}. \tag{5.9}$$

Since $0<(1-A_j^n)<1$, for each $n\in\mathbf{N}$, it follows that

$$q_k(T_{[n]}x) \le \frac{1}{n}\frac{A_j}{(1-A_j)}q_k(x), \quad k,n\in\mathbf{N}.$$

Accordingly, $q_k(T_{[n]}x)\to 0$ as $n\to\infty$ (for each $k\in\mathbf{N}$). But, $T_{[n]}\to P$ in $\mathscr{L}_s(X)$ as $n\to\infty$ and so $Px=0$. That is, $Py=0$ for every $y\in\bigcup_{j=1}^\infty X_j$. We have already seen that $\bigcup_{j=1}^\infty X_j$ is dense in X and so $P=0$, that is, $T_{[n]}\to 0$ in $\mathscr{L}_s(X)$ as $n\to\infty$.

Suppose that T is uniformly mean ergodic, in which case necessarily $T_{[n]}\to 0$ in $\mathscr{L}_b(X)$. In particular, since $(z_j)_j$ is a bounded sequence in X, we have

$$\lim_{n\to\infty}\sup_{j\in\mathbf{N}} q(T_{[n]}z_j)=0. \tag{5.10}$$

Fix $j\in\mathbf{N}$. It follows from (5.9) with $x=z_j\in X_j$ and $n=2^j$ (together with $A_j=1-2^{-j}$) that

$$q(T_{2^j}z_j)=(1-2^{-j})\cdot(1-(1-2^{-j})^{2^j})q(z_j).$$

Using the inequalities $q(z_j)>\frac12$ and $(1-2^{-j})\ge\frac12$ we conclude that

$$q(T_{2^j}z_j)>\frac14(1-(1-2^{-j})^{2^j}).$$

But, $\lim_{j\to\infty}(1-2^{-j})^{2^j}=e^{-1}$ and we have a contradiction to (5.10). Accordingly, $T=T_a$ is not uniformly mean ergodic. □

Remark 5.3. (i) As noted in Section 4, *every* Schauder decomposition in a Banach space fails to have property (M). So, for Banach spaces Theorem 5.2 reduces to [23, Theorem 2]. However, the proof of Theorem 2 given in [23] is based on Lin's criterion, namely Proposition 2.16 above which, as noted in Section 2, *fails* to hold in non-normable spaces (in general). So, the "Banach space proof" of [23] does not apply in Fréchet spaces.

(ii) For each $p\in[1,\infty)$, let $L^p_{\ell oc}(\mathbf{R})$ denote the space of all (equivalence classes of) Lebesgue measurable functions f defined on $\mathbf{R}$ which satisfy

$$q_p^{(n)}(f) := \Big(\int_{-n}^n |f(t)|^p\,dt\Big)^{1/p}<\infty,$$

for all $n\in\mathbf{N}$. Each space $L^p_{\ell oc}(\mathbf{R})$ is a separable Fréchet space (reflexive if $p\neq 1$) when equipped with the seminorms $q_p^{(1)}\le q_p^{(2)}\le\ldots$. This class of spaces has been intensively studied in [1], [3], [2]. According to Remark 4.3 (see [7, Proposition 6.2(ii)]), there exists a spectral measure in $L^p_{\ell oc}(\mathbf{R})$ which fails to be boundedly σ-additive. Hence, Proposition 4.2 implies that $L^p_{\ell oc}(\mathbf{R})$ admits a Schauder decomposition without property (M). Then Theorem 5.2 shows that $L^p_{\ell oc}(\mathbf{R})$ is *not* uniformly mean ergodic. For each $1<p<\infty$, the space $L^p_{\ell oc}(\mathbf{R})$ is mean ergodic because of its reflexivity (cf. Corollary 2.7). The case $p=1$ is different. Since the Banach space $L^1([0,1])$ is isomorphic to a complemented subspace of $L^1_{\ell oc}(\mathbf{R})$, to show that $L^1_{\ell oc}(\mathbf{R})$ is not mean ergodic it suffices to show (by an argument as in the Proof of Proposition 2.9) that $L^1([0,1])$ is not mean ergodic. But, $L^1([0,1])$ has a Schauder basis [36, p. 3] and is non-reflexive. Then [23, Corollary 1] implies that $L^1([0,1])$ is, indeed, not mean ergodic. Or, one can appeal to [20, Theorem 2].

The following result has no counterpart in Banach spaces.

Theorem 5.4. *For a Fréchet space X the following assertions are equivalent.*

(i) *X is a Montel space.*
(ii) *Every closed subspace of X is uniformly mean ergodic.*
(iii) *Every power bounded, mean ergodic operator defined on a closed subspace of X is uniformly mean ergodic.*

Proof. (i) $\Rightarrow$ (ii). This follows from Proposition 2.8 and (ii) $\Rightarrow$ (iii) is obvious.

Suppose that X is not Montel. According to Proposition 3.2, X contains a closed subspace Y which is not Montel and has a basis. Then the Schauder decomposition $(P_n)_n \subseteq \mathscr{L}(Y)$ induced by this basis has the property that each (1-dimensional) space $Q_n(Y) := (P_n - P_{n-1})(Y)$, for $n \in \mathbf{N}$, is Montel (with $P_0 := 0$). By [14, Proposition 4], the Schauder decomposition $(P_n)_n$ cannot have property (M) and hence, Theorem 5.2 guarantees the existence of a power bounded, mean ergodic operator in $\mathscr{L}(Y)$ which fails to be uniformly mean ergodic. This establishes (iii) $\Rightarrow$ (i). $\square$

Proof of Theorem 1.3. Let X be a Fréchet space with a basis. If X is Montel, then it is uniformly mean ergodic by Proposition 2.8. On the other hand, if X is not Montel, then we can choose $Y = X$ in the proof of (iii) $\Rightarrow$ (i) in Theorem 5.4 to conclude that X is not uniformly mean ergodic. $\square$

Proof of Theorem 1.6. Initially we proceed along the lines of [23, Corollary 4]. Let X be a Fréchet space which contains an isomorphic copy of c_0, say via an isomorphism $J\colon c_0 \to X$. Let $(e_n)_n$ be the standard unit basis vectors of c_0, in which case the sequence $(y_n)_n$ with $y_n := Je_n$, for $n \in \mathbf{N}$, is a Schauder basis of $Y := J(c_0)$. Let $\|\cdot\|_{c_0}$ denote the norm in c_0 and $\Gamma_X = (q_k)_k$ be increasing. Then, for each $k \in \mathbf{N}$, there exists $M_k > 0$ satisfying $q_k(Jx) \le M_k\|x\|_{c_0}$, for $x \in c_0$, and there exist $k_1 \in \mathbf{N}$ and $K > 0$ such that $\|x\|_{c_0} \le Kq_{k_1}(Jx)$, for all $x \in c_0$. By omitting q_j, for $1 \le j < k_1$, and relabelling (if necessary), we may assume that there exists $M_0 > 0$ with

$$\|x\|_{c_0} \le M_0 q_1(Jx), \quad x \in c_0, \tag{5.11}$$

and for each $k \in \mathbf{N}$ an $M_k > 0$ satisfying

$$q_k(Jx) \le M_k\|x\|_{c_0}, \quad x \in c_0. \tag{5.12}$$

Moreover, $(q_k)_k$ is still increasing and determines the topology of X. In particular, for each $u = (u_j) = \sum_{j=1}^\infty u_j e_j$ in c_0 we have

$$\sup_{j\in\mathbf{N}} |u_j| \le M_0 q_1\Big(\sum_{j=1}^\infty u_j y_j\Big) = M_0 q_1\Big(\sum_{j=1}^\infty u_j Je_j\Big)$$

and

$$q_k\Big(\sum_{j=1}^\infty u_j y_j\Big) \le M_k \sup_{j\in\mathbf{N}} |u_j|. \tag{5.13}$$

Let $(f_n) \subseteq \ell^1 = c_0'$ denote the dual basis of $(e_n)_n \subseteq c_0$. For each $n \in \mathbf{N}$, define $y_n' \in Y'$ by $y_n' := f_n \circ J^{-1}$, in which case $(y_n')_n$ is the dual basis of the Schauder basis $(y_n)_n$ of Y. Since $|\langle y, y_n'\rangle| \le M_0 q_1(y)$, for $y \in Y$ and $n \in \mathbf{N}$, we see that $(y_n')_n$ is an equicontinuous subset of Y'. By the Hahn–Banach theorem, for each $n \in \mathbf{N}$ there exists $\xi_n \in X'$ satisfying $\xi_n|_Y = y'_n$ and

$$|\langle x, \xi_n\rangle| \le M_0 q_1(x), \quad x \in X. \tag{5.14}$$

Define $x_n := \sum_{j=1}^n y_j \in Y$ and $g_n := (\xi_n - \xi_{n+1}) \in X'$ for each $n \in \mathbf{N}$. Direct calculation yields

$$\langle y_n, \xi_k\rangle = \delta_{kn} \text{ and } \langle x_n, g_k\rangle = \delta_{kn}, \quad k, n \in \mathbf{N}.$$

Define projections $P_n : X \to X$, for each $n \in \mathbf{N}$, by

$$P_n x := \sum_{k=1}^n \langle x, g_k\rangle x_k, \quad x \in X,$$

with the range of P_n equal to $\operatorname{span}(x_j)_{j=1}^n = \operatorname{span}(y_j)_{j=1}^n \subseteq Y$. Clearly $P_n P_m = P_{\min\{m,n\}}$. Set $h := \xi_1 \in X'$, in which case

$$\langle x_n, h\rangle = \langle \sum_{j=1}^n y_j, \xi_1\rangle = 1, \quad n \in \mathbf{N}.$$

Since $(P_n - P_{n-1})x_n = x_n$ (with $P_0 := 0$), we have $x_n \in (P_n - P_{n-1})(X)$ for all $n \in \mathbf{N}$. Moreover, $(x_n)_n \subseteq X$ is a bounded sequence, because $x_n = J(\sum_{j=1}^n e_j)$ with $\|\sum_{j=1}^n e_j\|_{c_0} = 1$ for all $n \in \mathbf{N}$ implies (via (5.12)) that

$$q_k(x_n) = q_k\Big(J(\sum_{j=1}^n e_j)\Big) \le M_k, \quad k, n \in \mathbf{N}.$$

Moreover, with $\varepsilon_0 := M_0^{-1}$ we see from (5.11) that

$$\varepsilon_0 \le q_1(x_n), \quad n \in \mathbf{N}.$$

On the other hand, the identities

$$P_n x = \langle x, \xi_1\rangle y_1 - \langle x, \xi_{n+1}\rangle x_n + \sum_{k=2}^n \langle x, \xi_k\rangle (x_k - x_{k-1}) = \sum_{k=1}^n (\langle x, \xi_k\rangle - \langle x, \xi_{n+1}\rangle)\, y_k,$$

valid for all $n \in \mathbf{N}$ and $x \in X$, imply (via (5.13) and (5.14)) that

$$q_k(P_n x) \le M_k \sup_{1\le k\le n} |\langle x, \xi_k\rangle - \langle x, \xi_{n+1}\rangle| \le 2M_k M_0 q_1(x), \quad x \in X;\ k, n \in \mathbf{N}. \tag{5.15}$$

Accordingly, $(P_n)_n \subseteq \mathscr{L}(X)$ is equicontinuous.

Let $a = (a_j)_j$ be any sequence of positive numbers satisfying $\sum_{j=1}^\infty a_j = 1$ and set $A_n := \sum_{j=1}^n a_j$, for $n \in \mathbf{N}$. As in the statement of Theorem 3 in [23] define

$$S_a x := x - \sum_{n=2}^\infty a_n P_{n-1} x + \sum_{n=2}^\infty a_n \langle P_{n-1} x, h\rangle x_n, \tag{5.16}$$

for each $x \in X$. To verify that $S_a \in \mathscr{L}(X)$, fix $k \in \mathbf{N}$. From the definition of S_a and the inequalities (5.13) and (5.15) we can conclude that

$$q_k(S_a x) \le q_k(x) + 2M_k M_0 q_1(x) + M_k \sup_{n\ge 2} |\langle P_{n-1} x, h\rangle|,$$

for each $x \in X$. But, (5.14) and (5.15) yield

$$|\langle P_{n-1} x, h\rangle| = |\langle P_{n-1} x, \xi_1\rangle| \le M_0 q_1(P_{n-1} x) \le 2M_0^2 M_1 q_1(x)$$

and hence,

$$q_k(S_a x) \le \big(1 + 2M_k M_0 + 2M_k M_0^2 M_1\big)\, q_k(x), \quad x \in X,$$

with the right-hand side independent of a. So, $S_a \in \mathscr{L}(X)$.

The fact that S_a is *power bounded* follows from the Claim on p. 156 of [23], stating that $S_a S_b = S_c$ for an appropriate c (expressed in terms of apriori given a and b). The argument in [23] is of a pure "algebraic computational" nature and so, also applies here.

Next we verify that S_a is *not* mean ergodic. To this end, first define $Y_n := P_n(X)$ in which case the property $P_n P_m = P_{\min\{m,n\}}$ implies that $Y_n \subseteq Y_m$ whenever $n \le m$. Let $y \in Y$. Since $(\sum_{j=1}^n e_j)_n$ is a Schauder basis of c_0, there exist scalars $(\alpha_n)_n$ such that $J^{-1}y = \sum_{n=1}^\infty \alpha_n (\sum_{j=1}^n e_j)$, that is, $y = \sum_{j=1}^\infty \alpha_j x_j$. But, $\sum_{j=1}^m \alpha_j x_j \in P_m(X) = Y_m$ for all $m \in \mathbf{N}$ and so

$$Y = \overline{\textstyle\bigcup_{m=1}^\infty Y_m}, \tag{5.17}$$

with the closure formed in X.

To see that $S_a(Y) \subseteq Y$ it suffices to verify that $S_a(Y_m) \subseteq Y$ for each $m \in \mathbf{N}$. According to (5.16), for a given $x \in X$ we have

$$S_a(P_m x) = P_m x - \sum_{n=2}^\infty a_n P_{n-1} P_m x + \sum_{n=2}^\infty a_n \langle P_{n-1} P_m x, h\rangle x_n.$$

Because of the property $P_r P_s = P_{\min\{r,s\}}$ we have

$$\sum_{n=2}^\infty a_n P_{n-1} P_m x = \sum_{n=2}^m a_n P_{n-1} x + \sum_{n>m} a_n P_m x.$$

For the same reason we also have

$$\sum_{n=2}^\infty a_n \langle P_{n-1} P_m x, h\rangle x_n = \sum_{n=2}^m a_n \langle P_{n-1} x, h\rangle x_n + \langle P_m x, h\rangle \sum_{n>m} a_n x_n.$$

Accordingly,

$$\begin{aligned} S_a(P_m x) = P_m x + \sum_{n=2}^m a_n P_{n-1} x + \Big(\sum_{n>m} a_n\Big) P_m x \\ + \sum_{n=2}^m a_n \langle P_{n-1} x, h\rangle x_n + \langle P_m x, h\rangle \sum_{n>m} a_n x_n. \end{aligned}$$

It then follows from (5.17) that $S_a(P_m x) \in Y$.

Now, $\lim_{n\to\infty} P_n y = y$ for all $y \in Y_m$ because $y = P_m y$ and so

$$\lim_{n\to\infty} P_n P_m y = \lim_{n \ge m} P_n P_m y = P_m y.$$

Since $(P_n)_n \subseteq \mathscr{L}(X)$ is equicontinuous, it follows from (5.17) that

$$y = \lim_{n\to\infty} P_n y, \quad y \in Y.$$

Define $Q_1 := P_1$ and $Q_k := (P_k - P_{k-1})$ for $k \ge 2$. Arguing as in the proof of Theorem 3 in [23], it turns out that each Q_k is a projection with $Q_k Q_j = 0$ whenever $k \ne j$ and $P_k = \sum_{j=1}^k Q_j$. Hence, $E_k := Q_k(X) = Q_k(Y)$, for $k \in \mathbf{N}$, is a Schauder decomposition of Y.

Finally, with $T_a \in \mathscr{L}(X)$ as defined by (5.2), it turns out (see the proof of Theorem 3 in [23] where the formulae (4) and (7) used there are also available here) that $S_a|_Y = T_a$. Observe that x_k belongs to E_k and satisfies $\langle x_k, h\rangle = 1$ for all $k \in \mathbf{N}$. Moreover, $(x_k)_k$ is a bounded sequence in Y. By the proof of Theorem 1.5 given above (applied to S_a in Y) there exists $y \in Y$ such that $((S_a)_{[n]}y)_n$ does *not* converge in Y. In particular, $T_a \in \mathscr{L}(X)$ is not mean ergodic. $\square$

References

[1] Albanese, A. A.: Primary products of Banach spaces. - Arch. Math. (Basel) 66, 1996, 397–405.

[2] Albanese, A. A.: On subspaces of the spaces $L^p_{\ell oc}$ and of their strong duals. - Math. Nachr. 197, 1999, 5–18.

[3] Albanese, A. A., and V. B. Moscatelli: Complemented subspaces of sums and products of copies of $L^1([0,1])$. - Rev. Mat. Univ. Complut. Madrid 9, 1996, 275–287.

[4] Altman, M.: Mean ergodic theorem in locally convex linear topological spaces. - Studia Math. 13, 1953, 190–193.

[5] Bellenot, S. F.: Basic sequences in non-Schwartz–Fréchet spaces. - Trans. Amer. Math. Soc. 258, 1980, 199–216.

[6] Bonet, J., S. Okada, and W. J. Ricker: The canonical spectral measure and Köthe function spaces. - Quaest. Math. 29, 2006, 91–116.

[7] Bonet, J., and W. J. Ricker: Spectral measures in classes of Fréchet spaces. - Bull. Soc. Roy. Sci. Liège 73, 2004, 99–117.

[8] Bonet, J., and W. J. Ricker: The canonical spectral measure in Köthe echelon spaces. - Integral Equations Operator Theory 53, 2005, 477–496.

[9] Bonet, J., and W. J. Ricker: Schauder decompositions and the Grothendieck and Dunford–Pettis properties in Köthe echelon spaces of infinite order. - Positivity 11, 2007, 77–93.

[10] Cascales, B., and J. Orihuela: On compactness in locally convex spaces. - Math. Z. 195, 1987, 365–381.

[11] Castillo, J. M. F., J. C. Díaz, and J. Motos: On the Fréchet space L_{p-}. - Manuscripta Math. 96, 1998, 219–230.

[12] Cook, T. A.: Schauder decompositions and semi-reflexive spaces. - Math. Ann. 182, 1969, 232–235.

[13] Díaz, J. C.: Montel subspaces in the countable projective limits of $L^p(\mu)$-spaces. - Canad. Math. Bull. 32, 1989, 169–176.

[14] Díaz, J. C., and M. M. Miñarro: Distinguished Fréchet spaces and projective tensor product. - Doğa-Tr. J. Math. 14, 1990, 191–208.

[15] Diestel, J.: Sequences and series in Banach spaces. - Springer Verlag, New York, 1984.

[16] Diestel, J., H. Jarchow, and A. Tonge: Absolutely summing operators. - Cambridge Univ. Press, Cambridge (U.K.), 1995.

[17] Dubinsky, E., and J. R. Retherford: Schauder basis and Köthe sequence spaces. - Trans. Amer. Math. Soc. 130, 1968, 265–280.

[18] Dunford, N.: Spectral theory I. Convergence to projections. - Trans. Amer. Math. Soc. 54, 1943, 185–217.

[19] Dunford, N., and J. T. Schwartz: Linear operators I: General theory. - Wiley-Interscience, 2nd edition, New York, 1964.

[20] Emel'yanov, E.: Banach lattices on which every power-bounded operator is mean ergodic. - Positivity 1, 1997, 291–296.

[21] Emel'yanov, E., and M. P. H. Wolff: Mean ergodicity on Banach lattices and Banach spaces. - Arch. Math. (Basel) 72, 1999, 214–218.

[22] Emilion, R.: Mean-bounded operators and mean ergodic theorems. - J. Funct. Anal. 61, 1985, 1–14.

[23] Fonf, V. P., M. Lin, and P. Wojtaszczyk: Ergodic characterizations of reflexivity in Banach spaces. - J. Funct. Anal. 187, 2001, 146–162.

[24] Hille, E.: Remarks on ergodic theorems. - Trans. Amer. Math. Soc. 57, 1945, 246–269.

[25] Hille, E., and R. S. Phillips: Functional analysis and semigroups. - Amer. Math. Soc., 4th printing of revised edition, Providence, 1981.

[26] James, R. C.: Bases and reflexivity of Banach spaces. - Ann. of Math. 52, 1950, 518–527.

[27] Jarchow, H.: Locally convex spaces. - Teubner, Stuttgart, 1981.

[28] Kalton, N. J.: Schauder bases and reflexivity. - Studia Math. 38, 1970, 255–266.

[29] Kalton, N. J.: Schauder decompositions in locally convex spaces. - Proc. Camb. Phil. Soc. 68, 1970, 377–392.

[30] Kalton, N. J.: Normalization properties of Schauder bases. - Proc. London Math. Soc. 22, 1971, 91–105.

[31] Köthe, G.: Topological vector spaces I. - Springer Verlag, 2nd revised edition, Berlin-Heidelberg-New York, 1983.

[32] Köthe, G.: Topological vector spaces II. - Springer Verlag, Berlin-Heidelberg-New York, 1979.

[33] Krengel, U.: Ergodic theorems. - de Gruyter, Berlin, 1985.

[34] Lewis, D. R.: Integration with respect to a vector measure. Pacific J. Math. 33, 1970, 157–165.

[35] Lin, M.: On the uniform ergodic theorem. - Proc. Amer. Math. Soc. 43, 1974, 337–340.

[36] Lindenstrauss, J., and L. Tzafriri: Classical Banach spaces I. - Springer Verlag, Berlin-Heidelberg-New York, 1977.

[37] Lotz, H. P.: Tauberian theorems for operators on L^∞ and similar spaces. - In: Functional Analysis: Surveys and Recent Results III, edited by K.-D. Bierstedt and B. Fuchssteiner, North Holland, Amsterdam, 1984, 117–133.

[38] Lotz, H. P.: Uniform convergence of operators on L^∞ and similar spaces. - Math. Z. 190, 1985, 207–220.

[39] Marti, J. T.: Introduction to the theory of bases. - Springer Verlag, Berlin-Heidelberg-New York, 1969.

[40] Meise, R. G., and D. Vogt: Introduction to functional analysis. - Clarendon Press, Oxford, 1997.

[41] Metafune, G., and V. B. Moscatelli: On the space $\ell^{p+} = \bigcap_{q>p} \ell^q$. - Math. Nachr. 147, 1990, 7–12.

[42] Pietsch, A.: Quasi-präkompakte Endomorphismen und ein Ergodensatz in lokalkonvexen Vektorräumen. - J. Reine Angew. Math. 207, 1961, 16–30.

[43] Radjavi, H., P.-K. Tam, and K.-K. Tan: Mean ergodicity for compact operators, - Studia Math. 158, 2003, 207–217.

[44] Retherford, J. R.: Bases, basic sequences and reflexivity of linear topological spaces. - Math. Ann. 164, 1966, 280–285.

[45] Retherford, J. R., and C. W. McArthur: Some remarks on bases in linear topological spaces. - Math. Ann. 164, 1966, 38–41.

[46] Ricker, W. J.: Spectral measures, boundedly σ-complete Boolean algebras and applications to operator theory. - Trans. Amer. Math. Soc. 304, 1987, 819–838.

[47] Ricker, W. J.: Operator algebras generated by Boolean algebras of projections in Montel spaces. - Integral Equations Operator Theory 12, 1989, 143–145.

[48] Robinson, W. B.: Extensions of basic sequences in Fréchet spaces. - Studia Math. 45, 1973, 1–14.

[49] Sine, R. C.: A mean ergodic theorem. - Proc. Amer. Math. Soc. 24, 1970, 438–439.

[50] Singer, I.: Basic sequences and reflexivity of Banach spaces. - Studia Math. 21, 1962, 351–369.

[51] Sucheston, L.: - In: Probability in Banach Spaces, Lecture Notes in Math. 526, Springer Verlag, Berlin, New York, 1976, 285–290.

[52] Walsh, B.: Structure of spectral measures on locally convex spaces. - Trans. Amer. Math. Soc. 120, 1965, 295–326.

[53] Yahdi, M.: Super-ergodic operators. - Proc. Amer. Math. Soc. 134, 2006, 2613–2620.

[54] Yosida, K.: Functional analysis. - Springer Verlag, Berlin, Heidelberg, 1965.

[55] Zippin, M.: A remark on bases and reflexivity in Banach spaces. - Israel J. Math. 6, 1968, 74–79.

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