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APPROXIMABLE QUASIDISKS

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Abstract. In this paper, we study a question posed by Anderson and Hinkkanen [AH]: what quasidisks are approximable? We show that a quasidisk bounded by an analytic curve is approximable.

0. Introduction

In 1962, Ahlfors and Weill [AW] introduced the method of quasiconformal extension to prove the univalence of a meromorphic function f satisfying the Nehari condition

$$\sup_{z\in\Delta} \frac{|S(f,z)|}{\lambda_\Delta(z)^2} < 2 \tag{1}$$

in the unit disk $\Delta = \{z \in \mathbf{C} : |z| < 1\}$, where

$$S(f,z) = \left(\frac{f''}{f'}\right)'(z) - \frac{1}{2}\left(\frac{f''}{f'}\right)^2(z)$$

is the Schwarzian derivative of f,

$$\lambda_\Delta(z) = \frac{1}{1-|z|^2}$$

is the Poincaré density at the point $z \in \Delta$, and $\mathbf{C}$ denotes the complex plane. In 1973, Becker [B] established the related univalence criterion

$$\sup_{z\in\Delta} \left|\frac{zf''(z)}{f'(z)\lambda_\Delta(z)}\right| < 1 \tag{2}$$

via the methods of Loewner chains and quasiconformal extension.

The univalence conditions (1) and (2) in the unit disk have since been generalized by several authors including, among others, Ahlfors [A], Epstein [E], Anderson and Hinkkanen [AH], Osgood and Stowe [OS]. The result of Anderson and Hinkkanen also gave univalence and quasiconformal extensibility criteria in the more general domain setting of approximable quasidisks, while Osgood and Stowe's result provided univalence conditions in an n-dimensional Riemannian manifold.

In [AH], the authors asked whether all quasidisks were approximable, and further suggested that a "sufficiently smooth" quasidisk would be approximable. The aim of this paper is to study this problem, and we will show that a quasidisk bounded by an analytic curve in $\mathbf{C}$ is indeed approximable.

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1. Preliminaries

If D is a domain in the extended complex plane $\widehat{\mathbf{C}} = \mathbf{C} \cup \infty$, we use ∂D and D^* to denote the boundary and exterior of D in $\widehat{\mathbf{C}}$ respectively. For an open disk $B(a,r)$ of radius r centered at a, $\overline{B}(a,r)$ denotes its closure. If S is a subset of the complex plane and f is a complex-valued function whose domain contains S, then $f(S)$ denotes the image of S under f. We also use f_z and $f_{\bar z}$ to denote the partial derivatives $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \bar z}$ respectively. We define quasiconformality in terms of maximal dilatation as in [LV]. A K-quasidisk is the image of the unit disk Δ under a K-quasiconformal self-mapping of $\widehat{\mathbf{C}}$ for some $K \geq 1$.

Suppose a Jordan curve C is the common boundary of the domains D and D^*. A K-quasiconformal reflection in C is a sense-reversing K-quasiconformal involution of $\widehat{\mathbf{C}}$ that preserves every point of C. We now recall the definition of an approximable quasidisk, first introduced in [AH].

Suppose D is a K-quasidisk in the finite plane $\mathbf{C}$ and g_1 is a conformal mapping of Δ onto D. If $f_1 = g_1^{-1}$, then the density of the Poincaré metric at a point $z \in D$ is given by $\lambda_D(z) = \frac{|f_1'(z)|}{1-|f_1(z)|^2}$. Let D_n be a sequence of K_1-quasidisks so that $\overline{D_n} \subseteq D_{n+1} \subseteq D$ for all n and $D = \bigcup_{n=1}^{\infty} D_n$. Suppose for some $K_1 \geq 1$ and for all n, there exist K_1-quasiconformal reflections ζ in ∂D and ζ_n in ∂D_n, such that $\zeta \in C^1(D)$ and $\zeta_n \in C^1(D_n)$, and let J and J_n be the Jacobian determinants of ζ and ζ_n respectively. Given k, k_1 satisfying $0 \leq k \leq k_1 < 1$, we write $E = |\zeta_{\bar z}|^2 - k^2|\zeta_z|^2 > 0$ and $E_n = |(\zeta_n)_{\bar z}|^2 - k_1^2|(\zeta_n)_z|^2 > 0$. The next definition and theorem are taken from [AH, pp. 837–838].

Definition 1. A domain D is an approximable K-quasidisk if the following conditions hold:

(a) there exists a constant $K_1 \geq 1$ and an exhaustion of D by K_1-quasidisks D_n of the type indicated above,

(b) each D_n has a K_1-quasiconformal reflection $\zeta_n \in C^1(D_n)$ of order 2 (i.e., $\zeta_n(\zeta_n(z)) = z$ for all z) and for each $z \in D$, we have $\lim_{n\to\infty} \zeta_n(z) = \zeta(z)$,

(c) for each $\delta > 0$ there exists $\rho \in (0,1)$ such that if $f_1(D_n) \supseteq \{w : |w| \leq \rho\}$ and $z \in D_n$ with $|f_1(z)| > \rho$, then, for such a point z,

$$\frac{|\zeta_{\bar z} J|}{E|z-\zeta(z)|^2} \leq (1+\delta)\frac{|(\zeta_n)_{\bar z} J_n|}{E_n|z-\zeta_n(z)|^2}, \tag{3}$$

$$\left|\frac{\zeta_z J k^2}{E(z-\zeta(z))^2} - \frac{(\zeta_n)_z J_n k_1^2}{E_n(z-\zeta_n(z))^2}\right| \leq \delta \frac{|(\zeta_n)_{\bar z} J_n|}{E_n|z-\zeta_n(z)|^2}, \tag{4}$$

$$|\zeta_{\bar z}| \geq c > 0 \quad \text{and} \quad c \leq |(\zeta_n)_{\bar z}| \leq \frac{1}{c}, \tag{5}$$

$$\frac{|(\zeta_n)_{\bar z} J_n|}{E_n|z-\zeta_n(z)|} \geq c\lambda_D(z), \tag{6}$$

where c depends on K and K_1 only.

Theorem 2. *Let D be an approximable K-quasidisk contained in $\mathbf{C}$ and suppose that K_1, ζ and λ_D are defined as above. Let f be meromorphic and locally univalent* *in D and suppose that g is a complex-valued function in $C^1(D)$ satisfying*

$$\left| \frac{2g\zeta_{\overline{z}}(z-\zeta) - (z-\zeta)^2\{g_{\overline{z}}\zeta_z + \zeta_{\overline{z}}[g^2 - g_z + \frac{1}{2}S(f)]\}}{J - 2g\overline{\zeta}_{\overline{z}}(z-\zeta) + (z-\zeta)^2\{g_{\overline{z}}\overline{\zeta}_z + \overline{\zeta}_{\overline{z}}[g^2 - g_z + \frac{1}{2}S(f)]\}} \right| \le k, \tag{7}$$

for some $k \in [0,1)$ and all $z \in D$. Then there is a positive number ε_1 depending only on K and K_1 such that, if

$$\limsup_{z\to\partial D} |g(z)|(\lambda_D(z))^{-1} < \varepsilon_1$$

and

$$\limsup_{z\to\partial D} |g_{\overline{z}}(z)|(\lambda_D(z))^{-2} < \varepsilon_1,$$

then f is univalent in D and has a $\frac{1+k}{1-k}$-quasiconformal extension h to $\widehat{\mathbf{C}}$ given by

$$h(z) = f(\zeta(z)) + \frac{(z-\zeta)f'(\zeta(z))}{1 + (z-\zeta(z))\left(g(\zeta(z)) - \frac{1}{2}\frac{f''}{f'}(\zeta(z))\right)} \tag{8}$$

for $z \in D^*$.

2. Quasidisks bounded by an analytic curve

An analytic Jordan curve is the image of the unit circle S^1 under a conformal map on a neighbourhood in $\widehat{\mathbf{C}}$ of S^1 to $\widehat{\mathbf{C}}$.

Remark. Any Jordan domain bounded by an analytic curve is a quasidisk (see, for example, [LV, p. 96, Theorem 8.1]).

If D is a bounded K-quasidisk having an analytic curve as its boundary and g_1 is a conformal mapping of Δ onto D, then g_1 may be analytically continued to be analytic and univalent in the closed disk $\overline{B}(0,R)$ for some $R > 1$. Let g_2 be a conformal map defined in $B(0,R)^*$ so that

$$g_2(B(0,R)^*) = g_1(B(0,R))^* \quad \text{and} \quad g_2(\infty) = \infty.$$

The quasisymmetric function $h(z) = \frac{1}{R}g_2^{-1}(g_1(Rz))$ on S^1 can be extended to a quasiconformal self-mapping $\tilde{H}$ of Δ, with maximal dilatation K_1 in Δ for some $K_1 \ge 1$. For $z \in B(0,R)$, let $H(z) = R\tilde{H}(\frac{z}{R})$. The map

$$G_1(z) = \begin{cases} g_1(z), & z \in \overline{B}(0,R), \\ g_2\left(\dfrac{R^2}{\overline{H\left(\frac{R^2}{\overline{z}}\right)}}\right), & z \in B(0,R)^*, \end{cases}$$

is a K_1-quasiconformal self-mapping of $\widehat{\mathbf{C}}$. Usually $G_1(\infty) \ne \infty$, since $G_1(\infty) = \infty$ if, and only if, $H(0) = 0$. Let r_n be a strictly increasing sequence in $(0,1)$ so that $\lim_{n\to\infty} r_n = 1$ and define $D_n = g_1(B(0,r_n))$. Let $F_n(z) = G_1(r_n z)$. Since $F_n(\Delta) = D_n$, the domain D_n is a K_1-quasidisk for each n. The maps

$$\zeta = G_1 \circ \frac{1}{\overline{G_1^{-1}}} \quad \text{and} \quad \zeta_n = G_1 \circ \frac{r_n^2}{\overline{G_1^{-1}}} \tag{9}$$

are K_1-quasiconformal reflections in ∂D and ∂D_n respectively. For each $w \in D$, let z be the point in Δ such that $w = g_1(z)$. For $w \in D \setminus g_1\left(\overline{B}\left(0, \frac{1}{R}\right)\right)$, we have

$$\zeta(w) = g_1\left(\frac{1}{\overline{g_1^{-1}(w)}}\right)$$

and, for $w \in D_n \setminus g_1\left(\overline{B}\left(0, \frac{1}{R}\right)\right)$, we have

$$\zeta_n(w) = g_1\left(\frac{r_n^2}{\overline{g_1^{-1}(w)}}\right).$$

By the chain rule, for $w \in D \setminus g_1\left(\overline{B}\left(0, \frac{1}{R}\right)\right)$, we have

$$\zeta_w(w) = 0 \quad \text{and} \quad \zeta_{\bar{w}}(w) = -\frac{g_1'\left(\frac{1}{\bar{z}}\right)}{\overline{z^2 g_1'(z)}},$$

and, for $w \in D_n \setminus g_1\left(\overline{B}\left(0, \frac{1}{R}\right)\right)$, we have

$$(\zeta_n)_w(w) = 0 \quad \text{and} \quad (\zeta_n)_{\bar{w}}(w) = -\frac{r_n^2 g_1'\left(\frac{r_n^2}{\bar{z}}\right)}{\overline{z^2 g_1'(z)}}.$$

Our next result shows that the absolute values of the partial derivative $\zeta_{\bar{w}}$ approach 1 uniformly near the boundary of D.

Proposition 3. *For each $c \in (0,1)$, there exists $\rho \in \left(\frac{1}{R}, 1\right)$ such that*

$$c \leq |\zeta_{\bar{w}}(w)| \leq \frac{1}{c}$$

for each $w \in D \setminus g_1\left(\overline{B}\left(0, \rho\right)\right)$.

Proof. Since g_1' and g_1'' are continuous on the compact set $\overline{B}(0,R) \setminus B\left(0, \frac{1}{R}\right)$ and $g_1' \neq 0$, there exist $m > 0$ and $M > 0$ such that

$$|g_1'(z)| \geq m \quad \text{and} \quad |g_1''(z)| \leq M \tag{10}$$

for all $z \in \overline{B}(0,R) \setminus B\left(0, \frac{1}{R}\right)$. Take $\rho \in \left(\frac{1}{R}, 1\right)$ sufficiently close to 1 so that

$$\frac{M}{\rho^3 m}(1-\rho^2) \leq \min\left\{1-c, \frac{1}{c} - \frac{1}{\rho^2}\right\}.$$

For $z \in \Delta \setminus \overline{B}(0,\rho)$, by integrating along the line segment connecting z and $\frac{1}{\bar{z}}$, we have

$$\left|g_1'\left(\frac{1}{\bar{z}}\right) - g'(z)\right| = \left|\int_z^{\frac{1}{\bar{z}}} g_1''(t)\,dt\right| \leq M\left|\frac{1}{\bar{z}} - z\right| \leq M\left(\frac{1}{\rho} - \rho\right).$$

Since

$$|\zeta_{\bar{w}}(w)| = \left|\frac{g_1'\left(\frac{1}{\bar{z}}\right)}{z^2 g_1'(z)}\right| = \left|\frac{1}{z^2} + \frac{g_1'\left(\frac{1}{\bar{z}}\right) - g_1'(z)}{z^2 g_1'(z)}\right|,$$

it follows from the triangle inequality that, for $z \in \Delta \setminus \overline{B}\left(0, \rho\right)$,

$$c \leq 1 - \frac{M}{\rho^3 m}(1-\rho^2) \leq |\zeta_{\bar{w}}(w)| \leq \frac{1}{\rho^2} + \frac{M}{\rho^3 m}(1-\rho^2) \leq \frac{1}{c}$$

as desired. $\square$

Proposition 4. *The functions ζ_n converge to ζ uniformly in $\widehat{\mathbf{C}}$ with respect to the chordal metric.*

Proof. Since the chordal metric is bounded, $\frac{r_n^2}{G_1^{-1}}$ converges uniformly to $\frac{1}{G_1^{-1}}$ in $\widehat{\mathbf{C}}$. It then follows from the uniform continuity of G_1 on $\widehat{\mathbf{C}}$ and from (9) that ζ_n converges to ζ uniformly in $\widehat{\mathbf{C}}$. □

Proposition 5. *For every $\varepsilon > 0$, there exists a positive integer N such that if $n \geq N$, $r_N > \frac{1}{R}$ and $w \in D_n \setminus G_1\left(\overline{B}\left(0, \frac{1}{R}\right)\right)$, then*

$$|(\zeta_n)_{\bar{w}}(w) - \zeta_{\bar{w}}(w)| < \varepsilon.$$

Proof. Since $zg_1'(z) \neq 0$ for all $z \in \overline{\Delta} \setminus B\left(0, \frac{1}{R}\right)$, there exists $M > 0$ such that

$$\frac{1}{|zg_1'(z)|} \leq M$$

for all $z \in \overline{\Delta} \setminus B\left(0, \frac{1}{R}\right)$. Since $z \neq 0$ on the compact set $\overline{\Delta} \setminus B\left(0, \frac{1}{R}\right)$, the map

$$\xi \colon z \longmapsto \frac{g_1'\left(\frac{1}{\bar{z}}\right)}{\bar{z}}$$

is uniformly continuous there and so $\xi\left(\frac{z}{r_n^2}\right)$ converges uniformly to $\xi(z)$. For every $\varepsilon > 0$, there exists a positive integer N such that if $n \geq N$, $r_N > \frac{1}{R}$ and $z \in \overline{\Delta} \setminus B\left(0, \frac{1}{R}\right)$, then

$$\left|\xi\left(\frac{z}{r_n^2}\right) - \xi(z)\right| < \frac{\varepsilon}{M}.$$

Our result then follows from the inequality

$$|(\zeta_n)_{\bar{w}}(w) - \zeta_{\bar{w}}(w)| \leq M\left|\xi\left(\frac{z}{r_n^2}\right) - \xi(z)\right|$$

for all $w \in D_n \setminus G_1\left(\overline{B}\left(0, \frac{1}{R}\right)\right)$. □

Proposition 6. *There exists $\rho \in \left(\frac{1}{R}, 1\right)$ such that*

$$\lambda_D(w) < \frac{4}{|w - \zeta(w)|}$$

for all $w \in D \setminus G_1\left(\overline{B}\left(0, \rho\right)\right)$.

Proof. From (10), there exist $m > 0$ and $M > 0$ such that $|g_1'(z)| \geq m$ and $|g_1''(z)| \leq M$ for all $z \in \overline{B}(0, R) \setminus B\left(0, \frac{1}{R}\right)$. Take $\rho \in \left(\frac{1}{R}, 1\right) \cap \left(\frac{1}{2}, 1\right)$ such that

$$\frac{1}{\rho} - \rho \leq \frac{2m}{M}.$$

Then, for $w = g_1(z) \in D \setminus g_1(\overline{B}(0, \rho))$, we have (all integrals below are taken along line segments)

$$\begin{aligned}
|w-\zeta(w)| &= \left|g_1\left(\frac{1}{\bar{z}}\right)-g_1(z)\right| = \left|\int_z^{\frac{1}{\bar{z}}} g_1'(s)\,ds\right| \\
&\le \left|\int_z^{\frac{1}{\bar{z}}} g_1'(z)\,ds\right| + \left|\int_z^{\frac{1}{\bar{z}}} (g_1'(s)-g_1'(z))\,ds\right| \\
&= |g_1'(z)|\left|\frac{1}{\bar{z}}-z\right| + \left|\int_z^{\frac{1}{\bar{z}}}\int_z^s g_1''(t)\,dt\,ds\right| \\
&\le |g_1'(z)|\left|\frac{1}{\bar{z}}-z\right| + \int_z^{\frac{1}{\bar{z}}} M|s-z||ds| = |g_1'(z)|\left|\frac{1}{\bar{z}}-z\right| + \frac{M}{2}\left|\frac{1}{\bar{z}}-z\right|^2 \\
&\le |g_1'(z)|\left|\frac{1}{\bar{z}}-z\right| + m\left|\frac{1}{\bar{z}}-z\right| \quad \text{by our choice of } \rho \\
&\le 2\,|g_1'(z)|\left|\frac{1}{\bar{z}}-z\right| < \frac{2(1-|g_1^{-1}(w)|^2)}{\rho|(g_1^{-1})'(w)|} < \frac{4}{\lambda_D(w)}
\end{aligned}$$

as desired. □

Proposition 7. *For each $\delta > 0$, there exists $\rho \in \left(\frac{1}{R}, 1\right)$ such that*

$$|w-\zeta_n(w)| < (1+\delta)|w-\zeta(w)|$$

for all $w \in D_n \setminus g_1(\overline{B}(0,\rho))$.

Proof. From (10), there exist $m > 0$ and $M > 0$ such that $|g_1'(z)| \ge m$ and $|g_1''(z)| \le M$ for all $z \in \overline{B}(0,R) \setminus B\left(0,\frac{1}{R}\right)$. Take $\rho \in \left(\frac{1}{R},1\right)$ such that

$$\frac{1}{\rho} - \rho \le \frac{m\delta}{M(1+\frac{\delta}{2})}. \tag{11}$$

Then, for $w = g_1(z) \in D_n \setminus g_1(\overline{B}(0,\rho))$, we have (all integrals below are taken along line segments)

$$\begin{aligned}
|w-\zeta_n(w)| - |w-\zeta(w)| &= \left|g_1\left(\frac{r_n^2}{\bar{z}}\right)-g_1(z)\right| - \left|g_1\left(\frac{1}{\bar{z}}\right)-g_1(z)\right| \\
&= \left|\int_z^{\frac{r_n^2}{\bar{z}}} g_1'(s)\,ds\right| - \left|\int_z^{\frac{1}{\bar{z}}} g_1'(s)\,ds\right| \\
&= \left|g_1'(z)\left(\frac{r_n^2}{\bar{z}}-z\right) + \int_z^{\frac{r_n^2}{\bar{z}}} (g_1'(s)-g_1'(z))\,ds\right| \\
&\quad - \left|g_1'(z)\left(\frac{1}{\bar{z}}-z\right) + \int_z^{\frac{1}{\bar{z}}} (g_1'(s)-g_1'(z))\,ds\right|.
\end{aligned} \tag{12}$$

For $z \in B(0, r_n) \setminus \overline{B}(0, \rho)$,

$$\left| \int_z^{\frac{1}{\bar{z}}} (g_1'(s) - g_1'(z))\, ds \right| = \left| \int_z^{\frac{1}{\bar{z}}} \int_z^s g_1''(t)\, dt\, ds \right| \le \int_z^{\frac{1}{\bar{z}}} M|s - z||ds| = \frac{M}{2} \left| \frac{1}{\bar{z}} - z \right|^2$$

$$\overset{\text{by (11)}}{<} m \left| \frac{1}{\bar{z}} - z \right| \le \left| g_1'(z) \left(\frac{1}{\bar{z}} - z \right) \right|$$

and thus, from (12), we have

$$\begin{aligned} |w - \zeta_n(w)| - |w - \zeta(w)| &\le \left| g_1'(z) \left(\frac{r_n^2}{\bar{z}} - z \right) \right| + \left| \int_z^{\frac{r_n^2}{\bar{z}}} (g_1'(s) - g_1'(z))\, ds \right| \\ &\quad - \left| g_1'(z) \left(\frac{1}{\bar{z}} - z \right) \right| + \left| \int_z^{\frac{1}{\bar{z}}} (g_1'(s) - g_1'(z))\, ds \right| \\ &< \left| \int_z^{\frac{r_n^2}{\bar{z}}} (g_1'(s) - g_1'(z))\, ds \right| + \left| \int_z^{\frac{1}{\bar{z}}} (g_1'(s) - g_1'(z))\, ds \right| \\ &= \left| \int_z^{\frac{r_n^2}{\bar{z}}} \int_z^s g_1''(t)\, dt\, ds \right| + \left| \int_z^{\frac{1}{\bar{z}}} \int_z^s g_1''(t)\, dt\, ds \right| \\ &\le \int_z^{\frac{r_n^2}{\bar{z}}} M|s - z||ds| + \int_z^{\frac{1}{\bar{z}}} M|s - z||ds| < M \left| \frac{1}{\bar{z}} - z \right|^2 \end{aligned}$$

and hence

$$\begin{aligned} \delta|w - \zeta(w)| &= \delta \left| g_1'(z) \left(\frac{1}{\bar{z}} - z \right) + \int_z^{\frac{1}{\bar{z}}} (g_1'(s) - g_1'(z))\, ds \right| \\ &\ge m\delta \left| \frac{1}{\bar{z}} - z \right| - \frac{M\delta}{2} \left| \frac{1}{\bar{z}} - z \right|^2 \\ &\overset{\text{by (11)}}{\ge} M \left| \frac{1}{\bar{z}} - z \right|^2 > |w - \zeta_n(w)| - |w - \zeta(w)| \end{aligned}$$

as desired. □

Let J and J_n be the Jacobian determinants of ζ and ζ_n respectively. Given k, k_1 satisfying $0 \le k \le k_1 < 1$, we now define, for $z \in D$,

$$E = |\zeta_{\bar{z}}|^2 - k^2|\zeta_z|^2, \qquad X = \frac{\zeta_{\bar{z}} J}{E(z - \zeta)^2},$$

$$E_n = |(\zeta_n)_{\bar{z}}|^2 - k_1^2|(\zeta_n)_z|^2, \qquad X_n = \frac{(\zeta_n)_{\bar{z}} J_n}{E_n(z - \zeta_n)^2}.$$

Lemma 8. *For each $\delta > 0$, there exists $\rho \in \left(\frac{1}{R}, 1\right)$ such that, for all n satisfying the condition $\rho < r_n$,*

$$(1 - \delta)|\zeta_{\bar{z}}| \le |(\zeta_n)_{\bar{z}}| \le (1 + \delta)|\zeta_{\bar{z}}|$$

on $D_n \setminus g_1(\overline{B}(0, \rho))$.

Proof. By Proposition 3, there exists $c > 0$ such that

$$|\zeta_{\bar{z}}| \geq c$$

on $D \setminus g_1\left(\overline{B}\left(0, \frac{1}{R}\right)\right)$. Taking $\varepsilon = c\delta$, it follows from Proposition 5 that there exists an integer $N > 0$ with $r_N > \frac{1}{R}$ such that, for all $n \geq N$,

$$|(\zeta_n)_{\bar{z}} - \zeta_{\bar{z}}| < c\delta \leq \delta|\zeta_{\bar{z}}|$$

on $D_n \setminus g_1\left(\overline{B}\left(0, \frac{1}{R}\right)\right)$. Applying the triangle inequality and taking $\rho \geq r_N$ then gives the desired result. □

Lemma 9. *For each $\delta > 0$, there exists $\rho \in \left(\frac{1}{R}, 1\right)$ such that, for all n satisfying the condition $\rho < r_n$,*

$$|X| \leq (1 + \delta)|X_n|$$

on $D_n \setminus g_1(\overline{B}(0, \rho))$.

Proof. Take $\delta' > 0$ such that $\frac{(1+\delta')^2}{1-\delta'} \leq 1 + \delta$. Applying Proposition 7 and Lemma 8 with δ' in place of δ yields the existence of $\rho \in \left(\frac{1}{R}, 1\right)$ such that

$$|X| = \frac{|\zeta_{\bar{z}}|}{|z - \zeta|^2} \leq \frac{(1 + \delta')^2}{1 - \delta'} \frac{|(\zeta_n)_{\bar{z}}|}{|z - \zeta_n|^2} \leq (1 + \delta)|X_n|,$$

as desired. □

Theorem 10. *A bounded quasidisk having an analytic curve as its boundary is an approximable quasidisk.*

Proof. If the boundary of a bounded K-quasidisk D is an analytic curve, we define D_n, ζ and ζ_n as in the beginning of this chapter. Then (a) of Definition 1 is satisfied and $\zeta_n(\zeta_n(z)) = z$. Since D is bounded, it follows from Proposition 4 that (b) of Definition 1 also holds. For each $\delta > 0$, take $\rho \in \left(\frac{1}{R}, 1\right)$ so that Propositions 3, 6, 7 and Lemmas 8, 9 all hold. Then (3) is precisely Lemma 9; (4) is trivially true since its left-hand side is zero; (5) follows from Proposition 3 and Lemma 8; and (6) is a consequence of Propositions 3, 6, 7 and Lemma 8 since $\frac{|J_n|}{E_n} = \frac{|(\zeta_n)_{\bar{z}}|^2}{|(\zeta_n)_{\bar{z}}|^2} = 1$. This fulfills (c) of Definition 1 and gives the desired conclusion. □

Theorem 2 then yields the following corollary.

Corollary 11. *Let D be a bounded K-quasidisk having an analytic curve as its boundary, let ζ be defined as in (9), let f be meromorphic and locally univalent in D and suppose that g is a complex-valued function in $C^1(D)$ satisfying (7) for some $k \in [0, 1)$ and all $z \in D$. Then there is a positive number ε_1 depending on K such that, if*

$$\limsup_{z \to \partial D} |g(z)|(\lambda_D(z))^{-1} < \varepsilon_1$$

and

$$\limsup_{z \to \partial D} |g_{\bar{z}}(z)|(\lambda_D(z))^{-2} < \varepsilon_1$$

then f is univalent in D and has a $\frac{1+k}{1-k}$-quasiconformal extension h to $\widehat{\mathbf{C}}$ given by (8) for $z \in D^*$.

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