

INITIAL-BOUNDARY VALUE PROBLEMS FOR PARABOLIC EQUATIONS

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Abstract. We prove new existence and uniqueness results for weak solutions to non-homogeneous initial-boundary value problems for parabolic equations modeled on the evolution of the p -Laplacian.

1. Introduction

In this paper we prove new existence and uniqueness results for weak solutions to non-homogeneous initial-boundary value problems for parabolic equations of the form

$$(1.1a) \quad \frac{\partial u}{\partial t} - \nabla_x \cdot A(x, t, \nabla_x u) = f \quad \text{in } \mathcal{D}'(Q_+),$$

$$(1.1b) \quad u = g \quad \text{on } (\Omega \times \{0\}) \cup (\partial\Omega \times \mathbf{R}_+).$$

Here Ω is an open and bounded set in \mathbf{R}^n and $Q_+ = \Omega \times \mathbf{R}_+$. Precise structural conditions for $A(\cdot, \cdot, \cdot)$ are given in Section 4, but the model is the following p -parabolic equation

$$(1.2a) \quad \frac{\partial u}{\partial t} - \nabla_x \cdot (|\nabla_x u|^{p-2} \nabla_x u) = f \quad \text{in } \mathcal{D}'(Q_+),$$

$$(1.2b) \quad u = g \quad \text{on } (\Omega \times \{0\}) \cup (\partial\Omega \times \mathbf{R}_+),$$

with $1 < p < \infty$.

The boundary data is prescribed on the whole parabolic boundary, $(\Omega \times \{0\}) \cup (\partial\Omega \times \mathbf{R}_+)$, and we study the problem of finding the “largest possible” classes of boundary and source data such that (1.1) has a good meaning and is uniquely solvable.

In the case of the elliptic p -laplacian:

$$(1.3a) \quad -\nabla \cdot (|\nabla u|^{p-2} \nabla u) = f \quad \text{in } \mathcal{D}'(\Omega),$$

$$(1.3b) \quad u = g \quad \text{on } \partial\Omega,$$

it is well known that $W^{1,p}(\Omega)$ is a kind of golden mean. It has the useful property that:

Given $g \in W^{1,p}(\Omega)$, there exists a unique solution $u \in W^{1,p}(\Omega)$ to the p -laplace equation (1.3) such that $u - g$ belongs to the closure of $\mathcal{D}(\Omega)$ in the $W^{1,p}(\Omega)$ -norm topology. Furthermore the source data (f in (1.3)) can then be taken as sums of first order derivatives of $L^{p/(p-1)}(\Omega)$ -functions.

2000 Mathematics Subject Classification: Primary 35K60, 35K55.

Key words: Parabolic equations, nonhomogeneous boundary conditions, nonhomogeneous initial data, Hilbert transform, anisotropic Sobolev spaces, evolution of p -Laplacian.

In this paper we construct an analogous optimal solution-space for equations of the type (1.1).

We point out that our results are new even in the linear case. In the linear case, where $p = 2$ and we denote $W^{s,2}$ by H^s , it is well known (see, e.g., [5] Vol. II) that the parabolic solution and lateral boundary value spaces, replacing the “elliptic spaces” $H^s(\Omega)$ and $H^{s-1/2}(\partial\Omega)$, are $H^{s,s/2}(\Omega \times \mathbf{R}_+)$ and $H^{s-1/2,s/2-1/4}(\partial\Omega \times \mathbf{R}_+)$. The initial data on $\Omega \times \{0\}$ should then belong to $H^{s-1}(\Omega)$ and the natural source data space is $H^{s-2,s/2-1}(\Omega \times \mathbf{R}_+)$. With additional compatibility conditions for the coupling of the data in the “corners” of the space-time cylinder we then have unique solvability for the linear case when $s > 1$ (see [5], Vol. II). When $s = 1$, the golden mean in the elliptic case, several difficulties arise in the parabolic case. One obvious difficulty is of course that we are in the borderline Sobolev imbedding case in the time direction (half-a-time derivative in $L^2(\mathbf{R}_+, L^2(\Omega))$), and are thus for instance unable to define traces on $\Omega \times \{0\}$.

In Theorem 4.10 we give optimal results in the linear limiting case ($s = 1$), and a complete description of the space of solutions (compare with the non-optimal results in, e.g., [5],[4] and [3]).

We use a similar construction of the solution space (with new technical complications) in the non-linear case when $p \neq 2$.

Our solution space for a general p , $1 < p < \infty$, (see Definition 4.6) is the sum of a Banach space carrying initial data and another Banach space carrying lateral boundary data. It is a dense subspace of the space of $L^p(Q_+)$ -functions, having half order time derivatives in $L^2(Q_+)$ and first order space derivatives in $L^p(Q_+)$.

This statement requires some explanation and the appropriate distribution theory, allowing fractional differentiation in the time direction of general L^p -functions in a space-time half cylinder, is developed. This analytic framework makes it possible to give a precise meaning to the fractional integration by parts for the time derivatives that is one of the key tools in our method. We point out that we use two different half-a-time derivatives (adjoint to each other) and that demanding these different derivatives to belong to $L^2(Q_+)$ gives rise to different function spaces. In Section 4 we investigate the relations between these different function spaces and discuss some of their basic properties. It is for instance non-trivial to show that our function spaces are well behaved when we cut off (in a smooth way) in time. This is, apart from the fact that we are in the borderline Sobolev imbedding case in the time direction, due to the fact that they have non-homogeneous summability and regularity conditions, and that they are defined as spaces of distributions.

Most of these technical problems arise already for functions defined on the real line and half-line, and for clarity we have moved most of these arguments to an auxiliary section (Section 3) dealing with this case.

The main result of this paper is Theorem 4.8 which implies, among other things, that our solution space $X^{1,1/2}(Q_+)$ really is a true analog of the space $W^{1,p}(\Omega)$ for the elliptic p -laplacian, in the sense that:

Given $g \in X^{1,1/2}(Q_+)$ there exists a unique solution $u \in X^{1,1/2}(Q_+)$ to the p -parabolic equation (1.1) such that $u - g$ belongs to the closure of $\mathcal{D}(Q_+)$ in the $X^{1,1/2}(Q_+)$ -norm topology. Furthermore the source data (f in (1.1)) can be taken as sums of first order space derivatives of $L^{p/(p-1)}(Q_+)$ -functions and half-a-time derivatives of $L^2(Q_+)$ -functions.

For simplicity we shall assume throughout the paper that the boundary of Ω is smooth, but this assumption is only used to prove that we can regularize functions near the lateral boundary so that the different spaces of test functions we use are dense in the corresponding function spaces (see Theorem 4.1).

2. Some analytical background

We will use the fractional calculus presented in [1]. Here we first give a brief review of the notation and some results. We then extend the calculus to space-time half-cylinders in order to be able to discuss initial-boundary value problems.

The Fourier transform on the Schwartz class $\mathcal{S}(\mathbf{R}^n, \mathbf{C})$ is defined by

$$(2.1) \quad \hat{u}(\xi) = \int_{\mathbf{R}^n} u(x)e^{-i2\pi x \cdot \xi} dx, \quad u \in \mathcal{S}(\mathbf{R}^n, \mathbf{C}).$$

The inverse will be denoted

$$(2.2) \quad \check{u}(\xi) = \int_{\mathbf{R}^n} u(x)e^{i2\pi x \cdot \xi} dx, \quad u \in \mathcal{S}(\mathbf{R}^n, \mathbf{C}).$$

The isotropic fractional Sobolev spaces are defined as follows.

Definition 2.1. For $s \in \mathbf{R}$ and $1 < p < \infty$ let

$$(2.3) \quad H_p^s(\mathbf{R}^n, \mathbf{C}) = \{u \in \mathcal{S}'(\mathbf{R}^n, \mathbf{C}); ((1 + |2\pi\xi|^2)^{s/2}\hat{u}(\xi))^\vee \in L^p(\mathbf{R}^n, \mathbf{C})\}.$$

They are separable and reflexive Banach spaces with the obvious norms. We will use the following multi-index notation. Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{R}^n$ be an n -tuple. We write $\alpha > 0$ if $\alpha_j > 0, j = 1, \dots, n$; $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ when $x \in \mathbf{R}^n$; $x_+^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$, (where $t_+ = \max(0, t)$ for $t \in \mathbf{R}$, with a similar definition for x_-^α) and $\Gamma(\alpha) = \Gamma(\alpha_1) \dots \Gamma(\alpha_n)$, where Γ denotes the gamma function. Furthermore we will sometimes write k for the multi-index (k, \dots, k) , the interpretation should be clear from the context. We now define the classical Riemann–Liouville convolution operators.

Definition 2.2. For a multi-index $\alpha > 0$, set

$$(2.4) \quad D_\pm^{-\alpha}u = \chi_\pm^{\alpha-1} * u, \quad u \in \mathcal{S}(\mathbf{R}^n, \mathbf{C}),$$

where the kernels $\chi_\pm^{\alpha-1}$, are given by

$$(2.5) \quad \chi_\pm^{\alpha-1} = \Gamma(\alpha)^{-1}(\cdot)_\pm^{\alpha-1}.$$

We extend the definition of D_\pm^α to general multi-indices $\alpha \in \mathbf{R}^n$ in the usual way.

Definition 2.3. For $\alpha \in \mathbf{R}^n$ set

$$(2.6) \quad D_\pm^\alpha u = D^k D_\pm^{\alpha-k} u, \quad u \in \mathcal{S}(\mathbf{R}^n, \mathbf{C}),$$

where we choose the multi-index $k \in \{0, 1, 2, \dots\}^n$ so that $k - \alpha > 0$.

The definition is independent of the choice of k .

Although it is clear in this setting how the support of a function is affected under these mappings and also for instance that the operators map real valued functions to real valued functions, other features become transparent on the Fourier transform side.

Computing in $\mathcal{S}'(\mathbf{R}^n, \mathbf{C})$, we have for all $\alpha \in \mathbf{R}^n$:

$$(2.7) \quad D_\pm^\alpha u = ((0 \pm i2\pi\xi)^\alpha \hat{u}(\xi))^\vee, \quad u \in \mathcal{S}(\mathbf{R}^n, \mathbf{C}).$$

We will use the following space of test functions.

Definition 2.4. Let

$$(2.8) \quad \mathcal{F}(\mathbf{R}^n, \mathbf{C}) = \left\{ u \in C^\infty(\mathbf{R}^n, \mathbf{C}); \|u\|_{H_p^s(\mathbf{R}^n, \mathbf{C})} < \infty, s \in \mathbf{R}, 1 < p < \infty \right\}.$$

$\mathcal{F}(\mathbf{R}^n, \mathbf{C})$ becomes a Fréchet space with the topology generated by, for instance, the following family of semi-norms $\|\cdot\|_{H_p^s(\mathbf{R}^n, \mathbf{C})}$, $s \in \{0, 1, 2, \dots\}$, $p = 1 + 2^k$, $k \in \mathbf{Z}$.

We have the following dense continuous imbeddings,

$$(2.9) \quad \mathcal{D}(\mathbf{R}^n, \mathbf{C}) \hookrightarrow \mathcal{S}(\mathbf{R}^n, \mathbf{C}) \hookrightarrow \mathcal{F}(\mathbf{R}^n, \mathbf{C}) \hookrightarrow \mathcal{E}(\mathbf{R}^n, \mathbf{C}).$$

An example of a function that belongs to $\mathcal{F}(\mathbf{R}, \mathbf{C})$ but does not belong to $\mathcal{S}(\mathbf{R}, \mathbf{C})$ is $x \mapsto 1/(1 + x^2)$.

For $\alpha \geq 0$ we now define the fractional derivatives

$$(2.10) \quad D_\pm^\alpha u = ((0 \pm i 2\pi \xi)^\alpha \hat{u})^\vee, \quad u \in \mathcal{F}(\mathbf{R}^n, \mathbf{C}).$$

The operators D_+^α and D_-^α are adjoint to each other and they are connected through the operator

$$(2.11) \quad H^\alpha = \prod_{k=1}^n (\cos(\pi\alpha_k) \text{Id} + \sin(\pi\alpha_k) H_k),$$

where Id is the identity operator and H_k is the Hilbert transform with respect to the k th variable, i.e.,

$$(2.12) \quad H_k u(t) = \pi^{-1} \lim_{\varepsilon \rightarrow +0} \int_{|s| \geq \varepsilon} \frac{u(t - se_k)}{s} ds, \quad u \in \mathcal{F}(\mathbf{R}^n, \mathbf{C}),$$

where e_k is the usual canonical k th basis vector in \mathbf{R}^n . We have the following lemma.

Lemma 2.1. For $\alpha \geq 0$, D_\pm^α are continuous linear operators on $\mathcal{F}(\mathbf{R}^n, \mathbf{C})$. For $\alpha \in \mathbf{R}^n$, H^α is an isomorphism on $\mathcal{F}(\mathbf{R}^n, \mathbf{C})$. For $\alpha, \beta \geq 0$ we have

$$(2.13) \quad D_\pm^\alpha D_\pm^\beta = D_\pm^{\alpha+\beta},$$

$$(2.14) \quad D_+^\alpha H^\alpha = D_-^\alpha.$$

Furthermore, all these operators commute on $\mathcal{F}(\mathbf{R}^n, \mathbf{C})$.

We note that for $\alpha \geq 0$

$$(2.15) \quad \int_{\mathbf{R}^n} D_+^\alpha u \Phi dx = \int_{\mathbf{R}^n} u D_-^\alpha \Phi dx, \quad u, \Phi \in \mathcal{F}(\mathbf{R}^n, \mathbf{C}),$$

and for $\alpha \in \mathbf{R}^n$

$$(2.16) \quad \int_{\mathbf{R}^n} H^\alpha u \Phi dx = \int_{\mathbf{R}^n} u H^{-\alpha} \Phi dx, \quad u, \Phi \in \mathcal{F}(\mathbf{R}^n, \mathbf{C}).$$

Now let $\mathcal{F}'(\mathbf{R}^n, \mathbf{C})$ denote the space of continuous linear functionals on $\mathcal{F}(\mathbf{R}^n, \mathbf{C})$, endowed with the weak* topology.

Inspired by (2.15) and (2.16), we extend the definition of D_\pm^α and H^α to $\mathcal{F}'(\mathbf{R}^n, \mathbf{C})$ by duality in the obvious way.

Definition 2.5. For $u \in \mathcal{F}'(\mathbf{R}^n, \mathbf{C})$ and $\alpha \geq 0$ let

$$(2.17) \quad \langle D_\pm^\alpha u, \Phi \rangle := \langle u, D_\mp^\alpha \Phi \rangle, \quad \Phi \in \mathcal{F}(\mathbf{R}^n, \mathbf{C}),$$

and for $\alpha \in \mathbf{R}^n$ let

$$(2.18) \quad \langle H^\alpha u, \Phi \rangle := \langle u, H^{-\alpha} \Phi \rangle, \quad \Phi \in \mathcal{F}(\mathbf{R}^n, \mathbf{C}).$$

The counterpart of Lemma 2.1 is valid for $\mathcal{F}'(\mathbf{R}^n, \mathbf{C})$.

Lemma 2.2. For $\alpha \geq 0$, D_\pm^α are continuous linear operators on $\mathcal{F}'(\mathbf{R}^n, \mathbf{C})$. For $\alpha \in \mathbf{R}^n$, H^α is an isomorphism on $\mathcal{F}'(\mathbf{R}^n, \mathbf{C})$. For $\alpha, \beta \geq 0$ we have

$$(2.19) \quad D_\pm^\alpha D_\pm^\beta = D_\pm^{\alpha+\beta},$$

$$(2.20) \quad D_+^\alpha H^\alpha = D_-^\alpha.$$

Furthermore, all these operators commute on $\mathcal{F}'(\mathbf{R}^n, \mathbf{C})$.

We recall that D_\pm^α and H^α all take real-valued functions (distributions) to real-valued functions (distributions), and from now on all functions and distributions will be real valued. We will denote the subspaces of real-valued functions and distributions simply by $\mathcal{F}(\mathbf{R}^n)$ and $\mathcal{F}'(\mathbf{R}^n)$.

In [1] we studied parabolic operators on a space-time cylinder $Q = \Omega \times \mathbf{R}$, where Ω was a connected and open set in \mathbf{R}^n . We then introduced the following space of test functions.

Definition 2.6. Let $\mathcal{F}_{0,\cdot}(Q)$ denote the subspace of $\mathcal{F}(\mathbf{R}^n \times \mathbf{R})$ functions with support in $K \times \mathbf{R}$ for some compact subset $K \subset \Omega$.

We put a pseudo-topology on $\mathcal{F}_{0,\cdot}(Q)$ by specifying what sequential convergence means. We say that $\Phi_i \rightarrow 0$ in $\mathcal{F}_{0,\cdot}(Q)$ if and only if the supports of all Φ_i 's are contained in a fixed set $K \times \mathbf{R}$, where $K \subset \Omega$ is a compact subset, and $\|D^\alpha \Phi_i\|_{L^p(Q)} \rightarrow 0$ as $i \rightarrow \infty$ for all multi-indices $\alpha \in \mathbf{Z}_+^{n+1}$ and $1 < p < \infty$.

The corresponding space of distributions is then defined as follows.

Definition 2.7. If u is a linear functional on $\mathcal{F}_{0,\cdot}(Q)$, then u is in $\mathcal{F}'_{\cdot,\cdot}(Q)$ if and only if for every compact set $K \subset \Omega$, there exist constants C, p_1, \dots, p_N with $1 < p_i < \infty, i = 1, \dots, N$, and multi-indices $\alpha_1, \dots, \alpha_N$ with $\alpha_i \in \mathbf{Z}_+^{n+1}, i = 1, \dots, N$, such that

$$(2.21) \quad |\langle u, \Phi \rangle| \leq C \sum_{i=1}^N \|D^{\alpha_i} \Phi\|_{L^{p_i}(Q)}$$

for all $\Phi \in \mathcal{F}_{0,\cdot}(Q)$ with support in $K \times \mathbf{R}$.

The motivation for these spaces is that they are invariant under fractional differentiation and Hilbert-transformation in the time variable, and ordinary differentiation in the space variables. In the given topologies, these operations are continuous.

For initial-boundary value problems, the parabolic operators will be defined on a space-time half-cylinder $Q_+ = \Omega \times \mathbf{R}_+$, and we shall then need the following natural spaces of test functions defined on Q_+ .

Remark. We shall use the same constructions on the real line and half-line, which can be thought of as the case $\Omega = \{0\}$ if we identify $\{0\} \times \mathbf{R}$ with \mathbf{R} and $\{0\} \times \mathbf{R}_+$ with \mathbf{R}_+ .

Definition 2.8. Let $\mathcal{F}_{0,\cdot}(Q_+)$ denote the space of those functions defined on Q_+ that can be extended to all of Q as elements in $\mathcal{F}_{0,\cdot}(Q)$.

Furthermore, let $\mathcal{F}_{0,0}(Q_+)$ denote the space of those functions defined on Q_+ that can be extended by zero to all of Q as elements in $\mathcal{F}_{0,\cdot}(Q)$.

(A zero in the first position of course corresponds to zero boundary data on the lateral boundary and a zero in the second position corresponds to zero initial data.)

By using the construction in [6] of a (total) extension operator, we see that $\mathcal{F}_{0,\cdot}(Q_+)$ can be identified with the space of all smooth functions Φ , defined on Q_+ , with support in $K \times R_+$ for some compact subset $K \subset \Omega$ (i.e., they are zero on the complement, with respect to Q_+ , of $K \times \mathbf{R}_+$), with $\|D^\alpha \Phi\|_{L^p(Q_+)} < \infty$ for all multi-indices $\alpha \in \mathbf{Z}_+^{n+1}$ and $1 < p < \infty$.

Thus, we can put an intrinsic pseudo-topology on $\mathcal{F}_{0,\cdot}(Q_+)$ by defining that $\Phi_i \rightarrow 0$ in $\mathcal{F}_{0,\cdot}(Q_+)$ if and only if the supports of all Φ_i are contained in a fixed set $K \times \mathbf{R}_+$, where $K \subset \Omega$ is a compact subset, and $\|D^\alpha \Phi_i\|_{L^p(Q_+)} \rightarrow 0$ as $i \rightarrow \infty$ for all multi-indices $\alpha \in \mathbf{Z}_+^{n+1}$ and $1 < p < \infty$. Then $\mathcal{F}_{0,0}(Q_+)$ is a closed subspace of $\mathcal{F}_{0,\cdot}(Q_+)$ with the induced topology.

We also note that $\mathcal{D}(Q_+)$ is densely continuously imbedded in $\mathcal{F}_{0,0}(Q_+)$.

Connected with these spaces of test functions are the following spaces of distributions.

Definition 2.9. If u is a linear functional on $\mathcal{F}_{0,\cdot}(Q_+)$, then u is in $\mathcal{F}'_{\cdot,0}(Q_+)$ if and only if for every compact set $K \subset \Omega$, there exist constants C, p_1, \dots, p_N with $1 < p_i < \infty, i = 1, \dots, N$, and multi-indices $\alpha_1, \dots, \alpha_N$ with $\alpha_i \in \mathbf{Z}_+^{n+1}, i = 1, \dots, N$, such that

$$(2.22) \quad |\langle u, \Phi \rangle| \leq C \sum_{i=1}^N \|D^{\alpha_i} \Phi\|_{L^{p_i}}$$

for all $\Phi \in \mathcal{F}_{0,\cdot}(Q_+)$ with support in $K \times \mathbf{R}_+$.

Furthermore, if u is a linear functional on $\mathcal{F}_{0,0}(Q_+)$, then u is in $\mathcal{F}'_{\cdot,\cdot}(Q_+)$ if and only if for every compact set $K \subset \Omega$, there exist constants C, p_1, \dots, p_N with $1 < p_i < \infty, i = 1, \dots, N$, and multi-indices $\alpha_1, \dots, \alpha_N$ with $\alpha_i \in \mathbf{Z}_+^{n+1}, i = 1, \dots, N$, such that

$$(2.23) \quad |\langle u, \Phi \rangle| \leq C \sum_{i=1}^N \|D^{\alpha_i} \Phi\|_{L^{p_i}(Q_+)}$$

for all $\Phi \in \mathcal{F}_{0,0}(Q_+)$ with support in $K \times \mathbf{R}_+$.

The importance of these spaces comes from the fact that, for a real-valued $\alpha \geq 0$, the operations

$$(2.24) \quad \frac{\partial_+^\alpha}{\partial t^\alpha} := D_+^{(0,\dots,0,\alpha)} : \mathcal{F}_{0,0}(Q_+) \longrightarrow \mathcal{F}_{0,0}(Q_+)$$

$$(2.25) \quad \frac{\partial_-^\alpha}{\partial t^\alpha} := D_-^{(0,\dots,0,\alpha)} : \mathcal{F}_{0,\cdot}(Q_+) \longrightarrow \mathcal{F}_{0,\cdot}(Q_+)$$

are continuous. Ordinary differentiations with respect to the space variables are clearly also continuous operations on these spaces. We shall also use that the Hilbert-transform in the time variable

$$(2.26) \quad h := H^{(0,\dots,0,1/2)} : \mathcal{F}_{0,0}(Q_+) \longrightarrow \mathcal{F}_{0,\cdot}(Q_+),$$

is a continuous operator.

Extending these operators by duality in the obvious way we get that

$$(2.27) \quad \frac{\partial_+^\alpha}{\partial t^\alpha} : \mathcal{F}'_{\cdot,0}(Q_+) \longrightarrow \mathcal{F}'_{\cdot,0}(Q_+),$$

$$(2.28) \quad \frac{\partial_-^\alpha}{\partial t^\alpha} : \mathcal{F}'_{\cdot,\cdot}(Q_+) \longrightarrow \mathcal{F}'_{\cdot,\cdot}(Q_+),$$

$$(2.29) \quad h : \mathcal{F}'_{\cdot,0}(Q_+) \longrightarrow \mathcal{F}'_{\cdot,\cdot}(Q_+),$$

and taking ordinary derivatives in the space variables, are continuous operations.

Using the total extension operator from [6], one can show that we can identify $\mathcal{F}'_{\cdot,0}(Q_+)$ with the space of $\mathcal{F}'_{\cdot,\cdot}(Q)$ -distributions that are zero on $\Omega \times (-\infty, 0)$.

Since $\mathcal{D}(Q_+)$ is densely continuously imbedded in $\mathcal{F}_{0,0}(Q_+)$, we get that $\mathcal{F}'_{\cdot,\cdot}(Q_+)$ is a continuously imbedded subspace of $\mathcal{D}'(Q_+)$.

We remark that the space $\mathcal{F}'_{\cdot,0}(Q_+)$ contains elements supported on $\Omega \times \{0\}$. In fact,

$$(2.30) \quad \mathcal{F}'_{\cdot,\cdot}(Q_+) \simeq \mathcal{F}'_{\cdot,0}(Q_+) / \mathcal{F}^{\circ}_{0,0}(Q_+),$$

where $\mathcal{F}^{\circ}_{0,0}(Q_+) = \{\xi \in \mathcal{F}'_{\cdot,0}(Q_+); \langle \xi, \Phi \rangle = 0, \Phi \in \mathcal{F}_{0,0}(Q_+)\}$.

Finally, since $\mathcal{F}_{0,0}(Q_+)$ is densely continuously imbedded in $L^p(Q_+)$ when $1 < p < \infty$, clearly $L^p(Q_+)$ is continuously imbedded in both $\mathcal{F}'_{\cdot,\cdot}(Q_+)$ and $\mathcal{F}'_{\cdot,0}(Q_+)$ when $1 < p < \infty$. Thus

$$(2.31) \quad \frac{\partial_+^\alpha}{\partial t^\alpha} : L^p(Q_+) \longrightarrow \mathcal{F}'_{\cdot,0}(Q_+)$$

$$(2.32) \quad \frac{\partial_-^\alpha}{\partial t^\alpha} : L^p(Q_+) \longrightarrow \mathcal{F}'_{\cdot,\cdot}(Q_+),$$

are well-defined continuous operations when $1 < p < \infty$.

3. Auxiliary spaces on the real line and half-line

We shall use the following auxiliary spaces defined on \mathbf{R} and in the definition $\frac{\partial_-^{1/2}}{\partial t^{1/2}}$ should be understood in the $\mathcal{F}'(\mathbf{R})$ distribution sense.

Definition 3.1. For $1 < p < \infty$, set

$$(3.1) \quad B^{1/2}(\mathbf{R}) = \left\{ u \in L^p(\mathbf{R}); \frac{\partial_-^{1/2} u}{\partial t^{1/2}} \in L^2(\mathbf{R}) \right\}.$$

We equip these spaces with the following norms.

$$(3.2) \quad \|u\|_{B^{1,1/2}(\mathbf{R})} := \left\| \frac{\partial_-^{1/2} u}{\partial t^{1/2}} \right\|_{L^2(\mathbf{R})} + \|u\|_{L^p(\mathbf{R})}.$$

Computing in $\mathcal{F}'(\mathbf{R})$ we see that we can represent these spaces as closed subspaces of the direct sums $L^2(\mathbf{R}) \oplus L^p(\mathbf{R})$, and thus they are reflexive and separable Banach spaces in the topologies arising from the given norms.

If $\{\psi_\varepsilon\}$ is a regularizing sequence it is clear that

$$(3.3) \quad \|\psi_\varepsilon * u\|_{B^{1/2}(\mathbf{R})} \leq \|u\|_{B^{1/2}(\mathbf{R})},$$

and thus smooth functions are dense in $B^{1/2}(\mathbf{R})$.

Due to the definition using distributions and to the inhomogeneity of our summability conditions, it is unfortunately not so easy to cut off in time and in this way show that $\mathcal{F}(\mathbf{R})$ (or $\mathcal{D}(\mathbf{R})$) is dense in $B^{1/2}(\mathbf{R})$. Nevertheless this is true.

Lemma 3.1. *The space of testfunctions $\mathcal{F}(\mathbf{R})$ is dense in $B^{1/2}(\mathbf{R})$.*

Proof. The proof is based on a non-linear version of the Riesz representation theorem.

We (temporarily) denote the closure of $\mathcal{F}(\mathbf{R})$ in $B^{1/2}(\mathbf{R})$ by $B_0^{1/2}(\mathbf{R})$, and we shall show that $B_0^{1/2}(\mathbf{R}) = B^{1/2}(\mathbf{R})$.

Set

$$(3.4) \quad T(u) = \frac{\partial u}{\partial t} + |u|^{p-2}u.$$

By fractional integration by parts

$$(3.5) \quad \langle T(u), \Phi \rangle = \int_{\mathbf{R}} \frac{\partial_+^{1/2}u}{\partial t^{1/2}} \frac{\partial_-^{1/2}\Phi}{\partial t^{1/2}} + |u|^{p-2}u\Phi dt; \quad \Phi \in \mathcal{F}(\mathbf{R}),$$

and Hölder’s inequality, it is clear that

$$(3.6) \quad T: B^{1/2}(\mathbf{R}) \longrightarrow B_0^{1/2}(\mathbf{R})^*$$

is continuous.

We notice that

$$(3.7) \quad T: B_0^{1/2}(\mathbf{R}) \longrightarrow B_0^{1/2}(\mathbf{R})^*$$

is weakly continuous and monotone (for definitions see [KS] or [1]).

By M. Riesz’ conjugate function theorem, which says that the Hilbert transform h is bounded from $L^p(\mathbf{R})$ to $L^p(\mathbf{R})$ (recall that $1 < p < \infty$), we see that the operators H^α introduced above are isomorphisms on $B_0^{1/2}(\mathbf{R})$.

Now for any $\alpha \in (0, 1/2)$ we have

$$(3.8) \quad \langle T(u), H^{-\alpha}(u) \rangle \geq \int_{\mathbf{R}} \sin(\pi\alpha) \frac{\partial_+^{1/2}u}{\partial t^{1/2}} \frac{\partial_+^{1/2}u}{\partial t^{1/2}} + (\cos(\pi\alpha) - \sin(\pi\alpha)C)|u|^p dt;$$

$u \in \mathcal{F}(\mathbf{R})$, where $C < \infty$ is a constant such that

$$(3.9) \quad \|h(u)\|_{L^p(\mathbf{R})} \leq C\|u\|_{L^p(\mathbf{R})}.$$

Choosing $\alpha \in (0, 1/2)$ small enough we see that $H^\alpha \circ T$ is coercive. It follows that T is a bijection (see [1] for this functional-analytic result and similar arguments).

Thus given $u \in B^{1/2}(\mathbf{R})$ there exists a unique $v \in B_0^{1/2}(\mathbf{R})$ such that $T(u) = T(v)$ in $\mathcal{F}'(\mathbf{R})$, i.e.,

$$(3.10) \quad \frac{\partial(u-v)}{\partial t} + (|u|^{p-2}u - |v|^{p-2}v) = 0.$$

This shows that the difference of elements with the same image has more regularity in time, namely $\frac{\partial(u-v)}{\partial t} \in L^{p/(p-1)}(\mathbf{R})$.

The class of $L^p(\mathbf{R})$ functions with derivatives in $L^{p/(p-1)}(\mathbf{R})$ is stable under regularization and thus by a continuity argument we see that we can test with $\chi(u-v)$, where χ is a cut off function in time, in equation (3.10). We get that (for a canonical continuous representative) $t \mapsto |u-v|(t)$ is decreasing. Since $u-v$ belongs to $L^p(\mathbf{R})$, we conclude that $u = v$. The lemma follows. \square

We are now in position to prove the following lemma.

Lemma 3.2. *If $u \in B^{1/2}(\mathbf{R})$ then*

$$(3.11) \quad \iint_{\mathbf{R} \times \mathbf{R}} \left| \frac{u(s) - u(t)}{s - t} \right|^2 ds dt = 2\pi \int_{\mathbf{R}} \left| \frac{\partial_-^{1/2} u}{\partial t^{1/2}} \right|^2 dt.$$

Proof. Since $\mathcal{F}(\mathbf{R})$ is dense in $B^{1/2}(\mathbf{R})$ we can compute using the Fourier transform

$$(3.12) \quad \int_{\mathbf{R}} \left| \frac{\partial_-^{1/2} u}{\partial t^{1/2}} \right|^2 dt = \int_{\mathbf{R}} 2\pi |\tau| |\hat{u}|^2 d\tau = \frac{1}{2\pi} \iint_{\mathbf{R} \times \mathbf{R}} \frac{|1 - e^{i2\pi\tau s}|^2}{s^2} |\hat{u}(\tau)|^2 d\tau ds.$$

Using Parseval's formula the lemma follows. □

We note the following scaling and translation invariance

$$(3.13) \quad \begin{aligned} & \iint_{\mathbf{R} \times \mathbf{R}} \left| \frac{u(a(s - b)) - u(a(t - b))}{s - t} \right|^2 ds dt \\ &= \iint_{\mathbf{R} \times \mathbf{R}} \left| \frac{u(s) - u(t)}{s - t} \right|^2 ds dt; \quad a, b \in \mathbf{R}. \end{aligned}$$

We also note the following fact.

Lemma 3.3. *The space $B^{1/2}(\mathbf{R})$ is continuously imbedded in the space of functions with vanishing mean oscillation, $VMO(\mathbf{R})$.*

Proof. Let $I \subset \mathbf{R}$ denote a bounded interval and let u_I denote the mean value of $u \in B^{1/2}(\mathbf{R})$ over I . Then by Jensen's inequality

$$(3.14) \quad \frac{1}{|I|} \int_I |u - u_I|^2 dt \leq \iint_{I \times I} \left| \frac{u(s) - u(t)}{s - t} \right|^2 ds dt. \quad \square$$

Using the form of the norm in Lemma 3.2, we can now show that we have good estimates in the $B^{1/2}(\mathbf{R})$ -norm for the following cut-off operation.

Lemma 3.4. *Let χ_n be the piecewise affine function that is one on $(-n, n)$, zero on $(-\infty, -2n) \cup (2n, \infty)$ and affine in between. Let $I_n = (-2n, 2n)$ and for $u \in B^{1/2}(\mathbf{R})$, denote the mean value of u over I_n by u_{I_n} . Then there exists a constant C such that*

$$(3.15) \quad \begin{aligned} & \iint_{\mathbf{R} \times \mathbf{R}} \left| \frac{\chi_n(u - u_{I_n})(s) - \chi_n(u - u_I)(t)}{s - t} \right|^2 ds dt \\ & \leq C \iint_{\mathbf{R} \times \mathbf{R}} \left| \frac{u(s) - u(t)}{s - t} \right|^2 ds dt, \end{aligned}$$

$$(3.16) \quad \|\chi_n(u - u_{I_n})\|_{L^p(\mathbf{R})}^p \leq C \|u\|_{L^p(\mathbf{R})}^p; \quad u \in B^{1/2}(\mathbf{R}).$$

Furthermore $\chi_n(u - u_{I_n}) \rightarrow u$ in $B^{1/2}(\mathbf{R})$ as $n \rightarrow \infty$.

Proof. The boundedness of the cut-off operation in the L^p -norm follows from Jensen's inequality. For the L^2 -part of the norm an elementary computation gives us

$$(3.17) \quad \begin{aligned} & \iint_{\mathbf{R} \times \mathbf{R}} \left| \frac{\chi_n(u - u_{I_n})(s) - \chi_n(u - u_I)(t)}{s - t} \right|^2 ds dt \\ & \leq C \left\{ \frac{1}{|I_n|} \int_{I_n} |u - u_{I_n}|^2 dt + \iint_{\mathbf{R} \times \mathbf{R}} \left| \frac{u(s) - u(t)}{s - t} \right|^2 ds dt \right\}, \end{aligned}$$

and thus (3.15) follows using (3.14). That $\chi_n u \rightarrow u$ in $L^p(\mathbf{R})$ is clear. If u has compact support, since $p > 1$, using Jensen’s inequality, we see that $\chi_n u_{I_n} \rightarrow 0$ in $L^p(\mathbf{R})$. Since by Jensen’s inequality $\chi_n u_{I_n}$ is uniformly bounded in $L^p(\mathbf{R})$, a density argument proves that $\chi_n(u - u_{I_n}) \rightarrow u$ in $L^p(\mathbf{R})$. That $\chi_n(u - u_{I_n}) \rightarrow u$ for the L^2 -part of the norm follows since by an elementary computation

$$(3.18) \quad \iint_{\mathbf{R} \times \mathbf{R}} \left| \frac{(1 - \chi_n)(u - u_{I_n})(s) - (1 - \chi_n)(u - u_{I_n})(t)}{s - t} \right|^2 ds dt$$

$$(3.19) \quad \leq C \left\{ \frac{1}{|I_n|} \int_{I_n} |u - u_{I_n}|^2 dt + \iint_{|t| > n} \left| \frac{u(s) - u(t)}{s - t} \right|^2 ds dt \right\}.$$

The last term clearly tends to zero as n tends to infinity. We only have to prove that also

$$(3.20) \quad \frac{1}{|I_n|} \int_{I_n} |u - u_{I_n}|^2 dt \rightarrow 0$$

as $n \rightarrow \infty$. This is true since

$$(3.21) \quad \begin{aligned} \frac{1}{|I_n|} \int_{I_n} |u - u_{I_n}|^2 dt &\leq \frac{1}{4n^2} \iint_{I_n \times I_n} |u(s) - u(t)|^2 ds dt \\ &\leq C \left\{ \frac{\log^2 n}{n^2} \iint_{|s|, |t| \leq \log n} \left| \frac{u(s) - u(t)}{s - t} \right|^2 ds dt \right. \\ &\quad \left. + \iint_{|t| \geq \log n} \left| \frac{u(s) - u(t)}{s - t} \right|^2 ds dt \right\}, \end{aligned}$$

which clearly tends to zero as n tends to infinity. □

Remark. We subtracted the mean value in the argument above in order not to have to rely on the fact that $u \in L^p(\mathbf{R})$ when proving boundedness for the half-derivatives. This is crucial when we later use the same argument on functions defined in a space-time cylinder. In preparation for this we also note that, by regularizing, the lemma gives us an explicit sequence of $\mathcal{D}(\mathbf{R})$ -functions tending to a given element in $B^{1/2}(\mathbf{R})$.

We now introduce two sets of spaces defined on the real half-line.

Definition 3.2. Let $B_0^{1/2}(\mathbf{R}_+)$ be the space of functions defined on \mathbf{R}_+ that can be extended by zero as elements in $B^{1/2}(\mathbf{R})$.

Furthermore, let $B^{1/2}(\mathbf{R}_+)$ be the space of functions defined on \mathbf{R}_+ that can be extended as elements in $B^{1/2}(\mathbf{R})$.

Remark. The space $B_0^{1/2}(\mathbf{R}_+)$ can of course be identified with the closed subspace of $B^{1/2}(\mathbf{R})$ of functions with support in \mathbf{R}_+ .

We now give two simple lemmas, giving intrinsic descriptions of $B_0^{1/2}(\mathbf{R}_+)$ and $B^{1/2}(\mathbf{R}_+)$. We omit the proofs, which are straightforward elementary computations using the form of the norm in Lemma 3.2.

Lemma 3.5. *The function space $B_0^{1/2}(\mathbf{R}_+)$ is precisely the set of $L^p(\mathbf{R}_+)$ -functions such that the following norm is bounded:*

$$(3.22) \quad \|u\|_{B_0^{1/2}(\mathbf{R}_+)} := \|u\|_{L^p(\mathbf{R}_+)} + \left\{ \int_{\mathbf{R}_+} \frac{u^2(t)}{t} dt + \iint_{\mathbf{R}_+ \times \mathbf{R}_+} \left(\frac{u(s) - u(t)}{s - t} \right)^2 ds dt \right\}^{1/2}.$$

Lemma 3.6. *The function space $B^{1/2}(\mathbf{R}_+)$ is precisely the set of $L^p(\mathbf{R}_+)$ -functions such that the following norm is bounded:*

$$(3.23) \quad \|u\|_{B^{1/2}(\mathbf{R}_+)} := \|u\|_{L^p(\mathbf{R}_+)} + \left\{ \iint_{\mathbf{R}_+ \times \mathbf{R}_+} \left(\frac{u(s) - u(t)}{s - t} \right)^2 ds dt \right\}^{1/2}.$$

Furthermore, a continuous symmetric extension operator from $B^{1/2}(\mathbf{R}_+)$ to $B^{1/2}(\mathbf{R})$ is given by $E_S(u)(t) = u(|t|)$.

We have the following density results:

Lemma 3.7. *The space $\mathcal{F}(\mathbf{R}_+)$ is dense in $B^{1/2}(\mathbf{R}_+)$ and $\mathcal{F}_0(\mathbf{R}_+)$ is dense in $B_0^{1/2}(\mathbf{R}_+)$.*

Proof. That $\mathcal{F}(\mathbf{R}_+)$ is dense in $B^{1/2}(\mathbf{R}_+)$ follows immediately from the fact that $\mathcal{F}(\mathbf{R})$ is dense in $B^{1/2}(\mathbf{R})$. The argument to prove that $\mathcal{F}_0(\mathbf{R}_+)$ is dense in $B_0^{1/2}(\mathbf{R}_+)$ is a little more delicate. Given $u \in B_0^{1/2}(\mathbf{R}_+)$, a priori we only know that there exists a sequence of testfunctions in $\mathcal{F}(\mathbf{R})$ approaching u in the $B^{1/2}(\mathbf{R})$ -norm.

Given $u \in B_0^{1/2}(\mathbf{R}_+)$ we will show that we can cut-off. Let χ_n be the piecewise affine function that is one on $(0, n)$, zero on $(2n, \infty)$ and affine in between. We will show that $\chi_n u \rightarrow u$ in $B_0^{1/2}(\mathbf{R}_+)$. Taking this for granted we can regularize with a regularizing sequence having support in \mathbf{R}_+ which gives us the lemma.

That $\chi_n u \rightarrow u$ in $L^p(\mathbf{R}_+)$ is clear. We now estimate the L^2 -part of the norm. An elementary computation gives us

$$(3.24) \quad \int_{\mathbf{R}_+} \frac{((1 - \chi_n)u)^2(t)}{t} dt + \iint_{\mathbf{R}_+ \times \mathbf{R}_+} \frac{((1 - \chi_n)u(s) - (1 - \chi_n)u(t))^2}{(s - t)^2} ds dt \leq C \left\{ \frac{1}{2n} \int_0^{2n} u^2(t) dt + \iint_{(n, \infty) \times \mathbf{R}_+} \left(\frac{u(s) - u(t)}{s - t} \right)^2 ds dt + \int_n^\infty \frac{u^2(t)}{t} dt \right\}.$$

The last two terms above clearly tend to zero as $n \rightarrow \infty$. To estimate the first term, we integrate by parts (we may assume that u is smooth, it is the decay at infinity that is the issue).

$$(3.25) \quad \begin{aligned} \frac{1}{2n} \int_0^{2n} u^2(t) dt &= \frac{1}{2n} \int_0^{2n} \left(\int_0^{2n} \frac{u^2(s)}{s} ds - \int_0^t \frac{u^2(s)}{s} ds \right) dt \\ &\leq \frac{1}{2n} \int_{\log n}^{2n} \int_t^{2n} \frac{u^2(s)}{s} ds dt + \frac{\log n}{2n} \int_0^{2n} \frac{u^2(s)}{s} ds, \end{aligned}$$

which clearly tends to zero as n tends to infinity. The lemma follows. □

We now give the following equivalent characterization of $B_0^{1/2}(\mathbf{R}_+)$.

Lemma 3.8. *A function $u \in L^p(\mathbf{R}_+)$ belongs to $B_0^{1/2}(\mathbf{R}_+)$ if and only if the $\mathcal{F}'_0(\mathbf{R}_+)$ -distribution derivative $\frac{\partial_+^{1/2}u}{\partial t^{1/2}} \in L^2(\mathbf{R}_+)$. Furthermore, an equivalent norm on $B_0^{1/2}(\mathbf{R}_+)$ is given by*

$$(3.26) \quad \|u\| = \|u\|_{L^p(\mathbf{R}_+)} + \left\| \frac{\partial_+^{1/2}u}{\partial t^{1/2}} \right\|_{L^2(\mathbf{R}_+)}.$$

Remark. We recall that the $\mathcal{F}'_0(\mathbf{R}_+)$ -distribution derivative, apart from what happens inside \mathbf{R}_+ , also controls what happens on the boundary $\{0\}$. The fact that $\frac{\partial_+^{1/2}u}{\partial t^{1/2}} \in L^2(\mathbf{R}_+)$ thus actually contains a lot of information about u 's behaviour at 0.

Proof. It is clear that a function in $B_0^{1/2}(\mathbf{R}_+)$ has the $\mathcal{F}'_0(\mathbf{R}_+)$ -distribution derivative $\frac{\partial_+^{1/2}u}{\partial t^{1/2}}$ in $L^2(\mathbf{R}_+)$.

On the other hand, let E_0 be the extension by zero operator. Then if $u \in L^p(\mathbf{R}_+)$ and the $\mathcal{F}'_0(\mathbf{R}_+)$ -distribution derivative $\frac{\partial_+^{1/2}u}{\partial t^{1/2}} \in L^2(\mathbf{R}_+)$, we have

$$(3.27) \quad \int_{\mathbf{R}} E_0(u) \frac{\partial_-^{1/2}\Phi}{\partial t^{1/2}} dt = \int_{\mathbf{R}} E_0\left(\frac{\partial_+^{1/2}u}{\partial t^{1/2}}\right) \Phi dt; \quad \Phi \in \mathcal{F}(\mathbf{R}).$$

This shows that the $\mathcal{F}'(\mathbf{R})$ -distribution derivative $\frac{\partial_+^{1/2}E_0(u)}{\partial t^{1/2}}$ belongs to $L^2(\mathbf{R})$. An easy computation shows that

$$(3.28) \quad \begin{aligned} \int_{\mathbf{R}_+} \left| \frac{\partial_+^{1/2}u}{\partial t^{1/2}} \right|^2 dt &= \int_{\mathbf{R}} \left| \frac{\partial_+^{1/2}E_0(u)(t)}{\partial t^{1/2}} \right|^2 dt \\ &\sim \iint_{\mathbf{R}_+ \times \mathbf{R}_+} \left| \frac{E_0(u)(s) - E_0(u)(t)}{s - t} \right|^2 ds dt + \int_{\mathbf{R}_+} \frac{E_0(u)^2}{t} dt. \end{aligned}$$

Since $E_0(u) = u$ on $(0, \infty)$, the lemma follows. □

We now give a corresponding equivalent norm on $B^{1/2}(\mathbf{R}_+)$.

Lemma 3.9. *If $u \in B^{1/2}(\mathbf{R}_+)$, then the $\mathcal{F}'(\mathbf{R}_+)$ -distribution derivative $\frac{\partial_-^{1/2}u}{\partial t^{1/2}}$ belongs to $L^2(\mathbf{R}_+)$.*

Furthermore, an equivalent norm on $B^{1/2}(\mathbf{R}_+)$ is given by

$$(3.29) \quad \|u\| = \|u\|_{L^p(\mathbf{R})} + \left\| \frac{\partial_-^{1/2}u}{\partial t^{1/2}} \right\|_{L^2(\mathbf{R}_+)}.$$

Remark. In contrast to the $\mathcal{F}'_0(\mathbf{R}_+)$ -distribution derivative, the $\mathcal{F}'(\mathbf{R}_+)$ -distribution derivative that we use in this definition “does not see” what happens on the boundary, $\{0\}$.

Proof. Since $\mathcal{F}(\mathbf{R}_+)$ is dense in $B^{1/2}(\mathbf{R}_+)$, it is enough to show that

$$(3.30) \quad \int_{\mathbf{R}_+} \left| \frac{\partial_-^{1/2}u}{\partial t^{1/2}} \right|^2 dt \sim \iint_{\mathbf{R}_+ \times \mathbf{R}_+} \left| \frac{u(s) - u(t)}{s - t} \right|^2 ds dt,$$

for functions in $\mathcal{F}(\mathbf{R}_+)$, where \sim means that the seminorms are equivalent.

For $p = 2$ we (temporarily) denote the closure of $\mathcal{F}(\mathbf{R}_+)$ in the norm

$$(3.31) \quad \|u\| = \left\| \frac{\partial_-^{1/2} u}{\partial t^{1/2}} \right\|_{L^2(\mathbf{R}_+)} + \|u\|_{L^2(\mathbf{R}_+)},$$

by H .

It follows directly from the definitions, and the fact that $\mathcal{F}(\mathbf{R}_+)$ is dense in $B^{1/2}(\mathbf{R}_+)$, that $B^{1/2}(\mathbf{R}_+)$ is continuously imbedded in H .

We shall now show that in fact $H = B^{1/2}(\mathbf{R}_+)$.

Let T denote the operator $T: u \mapsto \frac{\partial u}{\partial t} + u$. Then $T: B_0^{1/2}(\mathbf{R}_+) \rightarrow H^*$ is continuous. This follows from fractional integration by parts,

$$(3.32) \quad \langle Tu, \Phi \rangle = \left(\frac{\partial_+^{1/2} u}{\partial t^{1/2}}, \frac{\partial_-^{1/2} \Phi}{\partial t^{1/2}} \right)_{L^2} + (u, \Phi)_{L^2}; \quad \Phi \in \mathcal{F}(\mathbf{R}_+), \quad u \in \mathcal{F}_0(\mathbf{R}_+),$$

and the fact that $\mathcal{F}(\mathbf{R}_+)$ is dense in H and that $\mathcal{F}_0(\mathbf{R}_+)$ is dense in $B_0^{1/2}(\mathbf{R}_+)$.

Now by the Hahn–Banach theorem, given $\xi \in H^*$ there exist elements $u, v \in L^2(\mathbf{R}_+)$ such that

$$(3.33) \quad \langle \xi, \Phi \rangle = \left(u, \frac{\partial_-^{1/2} \Phi}{\partial t^{1/2}} \right)_{L^2} + (v, \Phi)_{L^2}; \quad \Phi \in \mathcal{F}(\mathbf{R}_+).$$

We can thus extend ξ by zero to an element $E_0(\xi)$ of $B^{1/2}(\mathbf{R})^*$. Since $T: B^{1/2}(\mathbf{R}) \rightarrow B^{1/2}(\mathbf{R})^*$ is an isomorphism, we can find a unique element $u \in B^{1/2}(\mathbf{R})$ such that $Tu = E_0(\xi)$ in $\mathcal{F}'(\mathbf{R})$. But this holds if and only if $u \in B_0^{1/2}(\mathbf{R}_+)$ and $Tu = \xi$ in $\mathcal{F}'_0(\mathbf{R}_+)$.

Thus $T: B_0^{1/2}(\mathbf{R}_+) \rightarrow H^*$ is an isomorphism.

Furthermore, by direct computation (or by interpolation (recall that $p = 2$)), we know that

$$(3.34) \quad T: B_0^{1/2}(\mathbf{R}_+) \rightarrow B^{1/2}(\mathbf{R}_+)^*$$

is an isomorphism.

Since $\mathcal{F}(\mathbf{R}_+)$ is densely continuously imbedded in both H and $B^{1/2}(\mathbf{R}_+)$ and thus H^* and $B^{1/2}(\mathbf{R}_+)^*$ both are well defined subspaces in $\mathcal{F}'_0(\mathbf{R}_+)$, we see that H^* and $B^{1/2}(\mathbf{R}_+)^*$ are identical as subspaces of $\mathcal{F}'_0(\mathbf{R}_+)$ and equivalent as Hilbert spaces.

Since $B^{1/2}(\mathbf{R}_+) \hookrightarrow H$, by Riesz representation theorem, this implies that H and $B^{1/2}(\mathbf{R}_+)$ have equivalent norms.

From a scaling argument it now follows that

$$(3.35) \quad \int_{\mathbf{R}_+} \left| \frac{\partial_-^{1/2} u}{\partial t^{1/2}} \right|^2 dt \sim \iint_{\mathbf{R}_+ \times \mathbf{R}_+} \left| \frac{u(s) - u(t)}{s - t} \right|^2 ds dt,$$

for functions in $B^{1/2}(\mathbf{R}_+)$. The lemma follows. □

4. Parabolic equations

We shall consider operators of the form

$$(4.1) \quad Tu = \frac{\partial u}{\partial t} - \nabla_x \cdot A(x, t, \nabla_x u),$$

on a space-time cylinder $Q_+ = \Omega \times \mathbf{R}_+$, where Ω is an open and bounded set in \mathbf{R}^n with smooth boundary.

We shall assume the following structural conditions for the function $A: \Omega \times \mathbf{R}_+ \times \mathbf{R}^n \rightarrow \mathbf{R}^n$.

- (1) $Q_+ \ni (x, t) \mapsto A(x, t, \xi)$ is Lebesgue measurable for every fixed $\xi \in \mathbf{R}^n$.
- (2) $\mathbf{R}^n \ni \xi \mapsto A(x, t, \xi)$ is continuous for almost every $(x, t) \in Q_+$.
- (3) For every $\xi, \eta \in \mathbf{R}^n, \xi \neq \eta$ and almost every $(x, t) \in Q_+$, we have

$$(4.2) \quad (A(x, t, \xi) - A(x, t, \eta), \xi - \eta) > 0.$$

- (4) There exists $p \in (1, \infty)$, a constant $\lambda > 0$ and a function $h \in L^1(Q_+)$ such that for every $\xi \in \mathbf{R}^n$ and almost every $(x, t) \in Q_+$:

$$(4.3) \quad (A(x, t, \xi), \xi) \geq \lambda|\xi|^p - h(x, t).$$

- (5) There exists a constant $\Lambda \geq \lambda > 0$ and a function $H \in L^{p/(p-1)}(Q_+)$ such that for every $\xi \in \mathbf{R}^n$ and almost every $(x, t) \in Q_+$:

$$(4.4) \quad |A(x, t, \xi)| \leq \Lambda|\xi|^{p-1} + H(x, t).$$

The Carathéodory conditions 1 and 2 above guarantee that the function $Q \ni (x, t) \mapsto A(x, t, \Phi(x, t))$ is measurable for every function $\Phi \in L^p(Q_+, \mathbf{R}^n)$. Condition 3 is a strict monotonicity condition that gives us uniqueness results. Conditions 4 (coercivity) and 5 (boundedness) give us apriori estimates that imply existence results (see [1]).

We now introduce some function spaces, and in their definitions $\partial_-^{1/2}/\partial t^{1/2}$ should be understood in the $\mathcal{F}'_{\cdot, \cdot}(Q)$ distribution-sense.

Definition 4.1. For $1 < p < \infty$, set

$$(4.5) \quad B_{\cdot, \cdot}^{1,1/2}(Q) = \left\{ u \in L^p(Q); \frac{\partial_-^{1/2} u}{\partial t^{1/2}} \in L^2(Q), \frac{\partial u}{\partial x_i} \in L^p(Q), i = 1, \dots, n \right\}.$$

We equip these spaces with the following norms.

$$(4.6) \quad \|u\|_{B_{\cdot, \cdot}^{1,1/2}(Q)} = \left\| \frac{\partial_-^{1/2} u}{\partial t^{1/2}} \right\|_{L^2(Q)} + \|u\|_{L^p(Q)} + \sum_{i=1}^n \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(Q)}.$$

Computing in $\mathcal{F}'_{\cdot, \cdot}(Q)$ we see that we can represent these spaces as closed subspaces of the direct sum $L^2(Q) \oplus L^p(Q) \oplus \dots \oplus L^p(Q)$, and thus they are reflexive and separable Banach spaces in the topologies arising from the given norms.

Since the lateral boundary is smooth (in fact Lipschitz continuous suffices), we can extend an element in $B_{\cdot, \cdot}^{1,1/2}(Q)$ to all of $\mathbf{R}^n \times \mathbf{R}$ and then cut off in the space variables. By regularizing it is clear that functions smooth up to the boundary are dense in $B_{\cdot, \cdot}^{1,1/2}(Q)$. To show that $\mathcal{F}'_{\cdot, \cdot}(Q)$ is dense in $B_{\cdot, \cdot}^{1,1/2}(Q)$ we only have to prove that we can “cut off” in time. This will follow as in Lemma 3.4 once we have the following result.

Lemma 4.1. *If $u \in B_{\cdot, \cdot}^{1,1/2}(Q)$, then*

$$(4.7) \quad \iiint_{\Omega \times \mathbf{R} \times \mathbf{R}} \left| \frac{u(x, s) - u(x, t)}{s - t} \right|^2 dx ds dt = 2\pi \iint_Q \left| \frac{\partial_-^{1/2} u}{\partial t^{1/2}} \right|^2 dx dt.$$

Proof. That $\frac{\partial_-^{1/2} u}{\partial t^{1/2}} = v$ means that

$$(4.8) \quad \iint_Q u(x, t) \frac{\partial_+^{1/2} \Phi(x, t)}{\partial t^{1/2}} dx dt = \iint_Q v(x, t) \Phi(x, t) dx dt; \quad \Phi \in \mathcal{F}_{0,\cdot}(Q).$$

Now for almost every $x \in \Omega$, $\Omega \ni x \mapsto u(x, \cdot) \in L^p(\mathbf{R})$ and $\Omega \ni x \mapsto v(x, \cdot) \in L^2(\mathbf{R})$ are well defined. Let S denote the set of common Lebesgue points. Since the Lebesgue points of a function can only increase by multiplication with a smooth function, by taking limits of mean values, we get that

$$(4.9) \quad \int_{\mathbf{R}} u(x, t) \frac{\partial_+^{1/2} \Phi(x, t)}{\partial t^{1/2}} dt = \int_{\mathbf{R}} v(x, t) \Phi(x, t) dt; \quad \Phi \in \mathcal{F}_{0,\cdot}(Q),$$

for all $x \in S$. This implies that for almost every $x \in \Omega$ the $L^p(\mathbf{R})$ function $t \mapsto u(x, t)$ has half a derivative equal to $v(x, t) \in L^2(\mathbf{R})$. So from the one-dimensional result it follows that

$$(4.10) \quad \iint_{\mathbf{R} \times \mathbf{R}} \left| \frac{u(x, s) - u(x, t)}{s - t} \right|^2 ds dt = 2\pi \int_{\mathbf{R}} \left| \frac{\partial_-^{1/2} u}{\partial t^{1/2}} \right|^2 dt,$$

for almost every $x \in \Omega$. Integrating with respect to x , the lemma follows. □

We conclude that:

Lemma 4.2. *The space of testfunctions $\mathcal{F}_{\cdot,\cdot}(Q)$ is dense in $B_{\cdot,\cdot}^{1,1/2}(Q)$.*

We now introduce the following subspace that corresponds to zero boundary data on the lateral boundary $\partial\Omega \times \mathbf{R}$ and as $|t| \rightarrow \infty$.

Definition 4.2. Let $B_{0,\cdot}^{1,1/2}(Q)$ denote the closure of $\mathcal{F}_{0,\cdot}(Q)$ in the $B_{\cdot,\cdot}^{1,1/2}(Q)$ -topology.

We shall work with the following two sets of function spaces on Q_+ .

Definition 4.3. Let $B_{*,\cdot}^{1,1/2}(Q_+)$ denote the space of functions defined on Q_+ that can be extended to elements in $B_{*,\cdot}^{1,1/2}(Q)$.

Furthermore, let $B_{*,0}^{1,1/2}(Q_+)$ denote the space of functions defined on Q_+ that can be extended by zero to elements in $B_{*,\cdot}^{1,1/2}(Q)$.

Here $*$ optionally stands for \cdot or 0 . A zero in the first position corresponds to zero boundary data on the lateral boundary and a zero in the second position corresponds to zero initial data.

Clearly $B_{*,0}^{1,1/2}(Q_+)$ can be identified with a closed subspace of $B_{*,\cdot}^{1,1/2}(Q)$.

We give the following two simple lemmas concerning these spaces and, as in the case of the real line, we omit the easy proofs.

Lemma 4.3. *The function space $B_{*,0}^{1,1/2}(Q_+)$ becomes a Banach space with the norm*

$$(4.11) \quad \begin{aligned} \|u\|_{B_{*,0}^{1,1/2}(Q_+)} = & \|u\|_{L^p(Q_+)} + \|\nabla_x u\|_{L^p(Q_+)} + \left\{ \int_{Q_+} \frac{u^2(x, t)}{t} dt dx \right. \\ & \left. + \iiint_{\Omega \times \mathbf{R}_+ \times \mathbf{R}_+} \left(\frac{u(x, s) - u(x, t)}{s - t} \right)^2 dx ds dt \right\}^{1/2}. \end{aligned}$$

Lemma 4.4. *The function space $B_{*,\cdot}^{1,1/2}(Q_+)$ becomes a Banach space with the norm*

$$(4.12) \quad \begin{aligned} \|u\|_{B_{*,\cdot}^{1,1/2}(Q_+)} &= \|u\|_{L^p(Q_+)} + \|\nabla_x u\|_{L^p(Q_+)} \\ &+ \left\{ \iiint_{\Omega \times \mathbf{R}_+ \times \mathbf{R}_+} \left(\frac{u(x, s) - u(x, t)}{s - t} \right)^2 dx ds dt \right\}^{1/2}. \end{aligned}$$

Furthermore, a continuous symmetric extension mapping from $B_{*,\cdot}^{1,1/2}(Q_+)$ to $B_{*,\cdot}^{1,1/2}(Q)$ is given by $E_S(u)(x, t) = u(x, |t|)$.

Computing in $\mathcal{F}'_{\cdot,0}(Q_+)$ we can give an equivalent characterization of $B_{\cdot,0}^{1,1/2}(Q_+)$.

Lemma 4.5. *A function $u \in L^p(Q_+)$ belongs to $B_{\cdot,0}^{1,1/2}(Q_+)$ if and only if the $\mathcal{F}'_{\cdot,0}(Q_+)$ -distribution derivative $\frac{\partial_+^{1/2} u}{\partial t^{1/2}}$ belongs to $L^2(Q_+)$, and the $\mathcal{F}'_{\cdot,0}(Q_+)$ -distribution derivatives $\nabla_x u \in L^p(Q_+)$. Furthermore, an equivalent norm on $B_{\cdot,0}^{1,1/2}(\mathbf{R}_+)$ is then given by*

$$(4.13) \quad \|u\| = \|\nabla_x u\|_{L^p(Q_+)} + \|u\|_{L^p(Q_+)} + \left\| \frac{\partial_+^{1/2} u}{\partial t^{1/2}} \right\|_{L^2(Q_+)}.$$

Proof. As on the real line. □

Using the corresponding result on the real half-line and the same type of argument as in the proof of Lemma 4.1, we see that an equivalent norm on $B_{*,\cdot}^{1,1/2}(Q_+)$ is given by

$$(4.14) \quad \|u\| = \left\| \frac{\partial_-^{1/2} u}{\partial t^{1/2}} \right\|_{L^2(Q_+)} + \|u\|_{L^p(Q_+)} + \sum_{i=1}^n \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(Q_+)},$$

where $\frac{\partial_-^{1/2}}{\partial t^{1/2}}$ is understood in the $\mathcal{F}'_{\cdot,0}(Q_+)$ -distribution sense.

We have the following density results:

Theorem 4.1. *The space of testfunctions $\mathcal{F}_{*,*}(Q_+)$ is dense in $B_{*,*}^{1,1/2}(Q_+)$. Furthermore, the space of testfunctions $\mathcal{F}_{0,*}(Q_+)$ is dense in $B_{0,*}^{1,1/2}(Q_+)$.*

Proof. Since the boundary of Ω is smooth we have good extension operators in the space variables, and we can also translate the support of functions away from the lateral boundary without spreading the support in the time direction. The result thus follows exactly as in Lemma 3.7. □

We point out the following result that follows immediately from the given norms.

Lemma 4.6. *The space $B_{*,0}^{1,1/2}(Q_+)$ is continuously imbedded in $B_{*,\cdot}^{1,1/2}(Q_+)$.*

We also remark that the (semi)norms $\left\| \frac{\partial_-^{1/2} u}{\partial t} \right\|_{L^2(Q_+)}$ and $\left\| \frac{\partial_+^{1/2} u}{\partial t} \right\|_{L^2(Q_+)}$ are not equivalent. In fact in Lemma 4.8 below we show that $B_{0,0}^{1,1/2}(Q_+)$ is a dense subspace of $B_{0,\cdot}^{1,1/2}(Q_+)$. This is of course connected with the well known fact that if $u \in L^2(Q)$ and $\frac{\partial_-^{1/2} u}{\partial t^{1/2}} \in L^2(Q)$, it is in general impossible to define a trace on $\Omega \times \{0\}$ (for instance the function $(x, t) \mapsto \log |\log |t||$ locally belongs to this space). Still a function in $B_{\cdot,0}^{1,1/2}(Q_+)$ is of course zero on $\Omega \times \{0\}$ in the sense that

$$(4.15) \quad \iint_{Q_+} \frac{u^2(x, t)}{t} dx dt < \infty.$$

We shall now discuss homogeneous data on the whole parabolic boundary.

4.1. Homogeneous data. We introduce the following space of $\mathcal{F}'_{0,\cdot}(Q)$ -distributions defined globally in time, but supported in Q_+ .

Definition 4.4. Let

$$(4.16) \quad B_{\cdot,0}^{-1,-1/2}(Q_+) := \left\{ \xi \in B_{0,\cdot}^{1,1/2}(Q)^*; \xi = 0 \text{ in } \Omega \times (-\infty, 0) \right\}.$$

From Theorem 4.3 and Theorem 4.4 in [1] follows

Theorem 4.2. For T as defined in (4.1), satisfying the structural conditions (1)–(5),

$$(4.17) \quad T: B_{0,0}^{1,1/2}(Q_+) \longrightarrow B_{\cdot,0}^{-1,-1/2}(Q_+)$$

is a bijection.

We shall now show that $B_{\cdot,0}^{-1,-1/2}(Q_+)$ can be identified with the dual space of $B_{0,\cdot}^{1,1/2}(Q_+)$.

Lemma 4.7. We can identify $B_{\cdot,0}^{-1,-1/2}(Q_+)$ with $B_{0,\cdot}^{1,1/2}(Q_+)^*$.

Remark. Note that we here identify a subspace of $\mathcal{F}'_{0,\cdot}(Q)$ with a subspace of $\mathcal{F}'_{\cdot,0}(Q_+)$.

Proof. Given $\xi \in B_{0,\cdot}^{1,1/2}(Q_+)^*$ we have (by the Hahn–Banach theorem) $u_0 \in L^2(Q_+)$ and $u_i \in L^{p'}(Q_+)$, $i = 1, \dots, n$, such that

$$(4.18) \quad \langle \xi, \Phi \rangle = \iint_{Q_+} u_0 \frac{\partial_-^{1/2} \Phi}{\partial t} + \sum_{i=1}^n u_i \frac{\partial \Phi}{\partial x_i} dx dt; \quad \Phi \in \mathcal{F}_{0,\cdot}(Q_+).$$

It is thus clear that we can extend this ξ to all of $\mathcal{F}_{0,\cdot}(Q)$ by zero. Set

$$(4.19) \quad \langle \xi_0, \Phi \rangle = \iint_Q E_0(u_0) \frac{\partial_-^{1/2} \Phi}{\partial t} + \sum_{i=1}^n E_0(u_i) \frac{\partial \Phi}{\partial x_i} dx dt; \quad \Phi \in \mathcal{F}_{0,\cdot}(Q),$$

where E_0 denotes the operator that extends a function with 0 to all of Q . The mapping $B_{0,\cdot}^{1,1/2}(Q_+)^* \ni \xi \mapsto \xi_0 \in B_{\cdot,0}^{-1,-1/2}(Q_+)$ is clearly injective, but it is also surjective. This follows since given $\xi \in B_{\cdot,0}^{-1,-1/2}(Q_+)$, by Theorem 4.2 above, there exists a (unique) $u_\xi \in B_{0,0}^{1,1/2}(Q_+)$ such that

$$(4.20) \quad \frac{\partial u_\xi}{\partial t} - \nabla_x \cdot (|\nabla_x u_\xi|^{p-2} \nabla_x u_\xi) = \xi,$$

i.e.,

$$(4.21) \quad \langle \xi, \Phi \rangle = \iint_Q \frac{\partial_+^{1/2} u_\xi}{\partial t} \frac{\partial_-^{1/2} \Phi}{\partial t} + (|\nabla_x u_\xi|^{p-2} \nabla_x u_\xi) \cdot \nabla_x \Phi dx dt; \quad \Phi \in \mathcal{F}_{0,\cdot}(Q),$$

and we see that ξ has the required form. □

Thus we can reformulate Theorem 4.2.

Theorem 4.3. For T as defined in (4.1), satisfying the structural conditions (1)–(5),

$$(4.22) \quad T: B_{0,0}^{1,1/2}(Q_+) \longrightarrow B_{0,\cdot}^{1,1/2}(Q_+)^*$$

is a bijection.

Remark. This theorem of course means that given $\xi \in B_{0,\cdot}^{1,1/2}(Q_+)^*$ there exists a unique $u \in B_{0,0}^{1,1/2}(Q_+)$ such that

$$(4.23) \quad \langle T(u), \Phi \rangle = \langle \xi, \Phi \rangle; \quad \Phi \in B_{0,\cdot}^{1,1/2}(Q_+).$$

Which means precisely that

$$(4.24) \quad \langle \xi, \Phi \rangle = \iint_{Q_+} \frac{\partial_+^{1/2} u_\xi}{\partial t} \frac{\partial_-^{1/2} \Phi}{\partial t} + A(x, t, \nabla_x u) \cdot \nabla_x \Phi \, dx \, dt; \quad \Phi \in \mathcal{F}_{0,\cdot}(Q_+),$$

since $\mathcal{F}_{0,\cdot}(Q_+)$ is dense in $B_{0,\cdot}^{1,1/2}(Q_+)$.

The following structure theorem for our source data space is an immediate consequence of the Hahn–Banach theorem.

Theorem 4.4. Given $\xi \in B_{0,\cdot}^{1,1/2}(Q_+)^*$ there exist functions $u_0 \in L^2(Q_+)$ and $u_1, \dots, u_n \in L^{p/(p-1)}(Q_+)$ such that

$$(4.25) \quad \xi = \frac{\partial_+^{1/2} u_0}{\partial t} + \sum_{i=1}^n \frac{\partial u_i}{\partial x_i}$$

in $\mathcal{F}'_{\cdot,0}(Q_+)$.

Our next result implies that in general it is actually enough to test our equations with $\mathcal{F}_{0,0}(Q_+)$ instead of $\mathcal{F}_{0,\cdot}(Q_+)$.

Lemma 4.8. *The continuous imbedding*

$$(4.26) \quad B_{0,0}^{1,1/2}(Q_+) \hookrightarrow B_{0,\cdot}^{1,1/2}(Q_+)$$

is dense.

Proof. It is enough to show that if $\xi \in B_{0,\cdot}^{1,1/2}(Q_+)^*$ and $\langle \xi, \Phi \rangle = 0$ for all $\Phi \in B_{0,0}^{1,1/2}(Q_+)$, then $\xi = 0$.

Now given $\xi \in B_{0,\cdot}^{1,1/2}(Q_+)^*$, by Theorem 4.3, there exists a unique $u_\xi \in B_{0,0}^{1,1/2}(Q_+)$ such that

$$(4.27) \quad \frac{\partial u_\xi}{\partial t} - \nabla_x \cdot (|\nabla_x u_\xi|^{p-2} \nabla_x u_\xi) = \xi.$$

Now if $\langle \xi, \Phi \rangle = 0$ for all $\Phi \in B_{0,0}^{1,1/2}(Q_+)$, then with $\Phi = u_\xi$ we get

$$(4.28) \quad \iint_{Q_+} |\nabla_x u_\xi|^p \, dx \, dt = 0.$$

By the Poincaré inequality $u_\xi = 0$, and so $\xi = 0$. □

4.2. Non-homogeneous initial data. We will first introduce the space that will carry the initial data. In the definition, all derivatives should be understood in the $\mathcal{F}'_{\cdot,\cdot}(Q_+)$ -distribution sense.

Definition 4.5. Let

$$(4.29) \quad B_I(Q_+) = \left\{ u \in B_{0,\cdot}^{1,1/2}(Q_+) \cap C_b([0, \infty), L^2(\Omega)); \frac{\partial u}{\partial t} \in L^{p'}(\mathbf{R}_+, W^{-1,p'}(\Omega)) \right\}.$$

Here $C_b([0, \infty), L^2(\Omega))$ denotes the space of bounded continuous functions from $[0, \infty)$ into $L^2(\Omega)$, and $\frac{\partial u}{\partial t} \in L^{p'}(\mathbf{R}_+, W^{-1,p'}(\Omega))$ means exactly that

$$(4.30) \quad \left| \left\langle u, \frac{\partial \Phi}{\partial t} \right\rangle \right| \leq C \|\nabla_x \Phi\|_{L^p(Q_+)}; \quad \Phi \in \mathcal{F}_{0,0}(Q_+),$$

for some constant $C > 0$. The smallest possible constant is by definition $\left\| \frac{\partial u}{\partial t} \right\|_{L^{p'}(\mathbf{R}_+, W^{-1,p'}(\Omega))}$.

We equip $B_I(Q_+)$ with the following norm

$$(4.31) \quad \|u\|_{B_I(Q_+)} := \|u\|_{B_{0,\cdot}^{1,1/2}(Q_+)} + \sup_{t \in \mathbf{R}_+} \|u(\cdot, t)\|_{L^2(\Omega)} + \left\| \frac{\partial u}{\partial t} \right\|_{L^{p'}(\mathbf{R}_+, W^{-1,p'}(\Omega))}.$$

Using Theorem 4.3 and the monotonicity of $A(x, t, \cdot)$ we shall now prove that we always have a unique solution in $B_I(Q_+)$ to the following initial value problem.

Theorem 4.5. *Given $u_0 \in L^2(\Omega)$, there exists a unique element $u \in B_I(Q_+)$ such that*

$$(4.32a) \quad \frac{\partial u}{\partial t} - \nabla_x \cdot A(x, t, \nabla_x u) = 0 \quad \text{in } \mathcal{F}'_{\cdot,\cdot}(Q_+),$$

$$(4.32b) \quad u = u_0 \quad \text{on } \Omega \times \{0\}.$$

Proof. Uniqueness follows immediately from the monotonicity of $A(x, t, \cdot)$ by pairing with a cut off function in time multiplied with the difference of two solutions. To prove existence we first note that if $u_0 \in \mathcal{D}(\Omega)$, we can extend it for instance to a smooth testfunction $U_0 \in \mathcal{D}(\Omega \times (-2, 2))$ such that $U_0(x, t) = u_0(x)$ when $-1 < t < 1$.

Since $\frac{\partial U_0}{\partial t} \in B_{0,\cdot}^{1,1/2}(Q_+)^*$, by Theorem 4.3, we know that there exists a unique $w \in B_{0,0}^{1,1/2}(Q_+)$ such that

$$(4.33) \quad \frac{\partial w}{\partial t} - \nabla_x \cdot A(x, t, \nabla_x w + \nabla_x U_0) = -\frac{\partial U_0}{\partial t} \quad \text{in } B_{0,\cdot}^{1,1/2}(Q_+)^*.$$

Then clearly $u = (w + U_0) \in B_{0,\cdot}^{1,1/2}(Q_+)$ solves (4.32), and the initial value is taken in the sense that

$$(4.34) \quad \iint_{\Omega \times (0,1)} \frac{(u(x, t) - u_0(x))^2}{t} dx dt < \infty.$$

By standard arguments it follows from (4.32) that $u \in B_I(Q_+)$ and so the initial data is actually taken in $C_b([0, \infty), L^2(\Omega))$ -sense.

Given $u_0 \in L^2(\Omega)$ we now choose a sequence $\mathcal{D}(\Omega) \ni u_0^n \rightarrow u_0$ in $L^2(\Omega)$.

Let u^n denote the solution of (4.32) with initial data u_0^n . By testing with $u^n \chi$, where χ is a standard cut off function in time, in (4.32), we get that

$$(4.35) \quad \sup_{t \in \mathbf{R}_+} \int_{\Omega} (u^n - u^m)^2(x, t) dx \leq \int_{\Omega} (u_0^n - u_0^m)^2(x) dx.$$

It is also clear that $\|\nabla_x u^n\|_{L^p(Q_+)}$ is bounded by a constant independent of n .

Finally we note that we can extend u^n symmetrically to Q and the extended function $E_S(u^n) \in B_{0,\cdot}^{1,1/2}(Q)$ will satisfy $\frac{\partial E_S(u^n)}{\partial t} \in L^{p'}(\mathbf{R}, W^{-1,p'}(\Omega))$.

We then have

$$(4.36) \quad \iint_Q \frac{\partial_-^{1/2} E_S(u^n)}{\partial t^{1/2}} \frac{\partial_-^{1/2} \Phi_k}{\partial t^{1/2}} dx dt = \int_{\mathbf{R}} \left\langle \frac{\partial E_S(u^n)}{\partial t}, h(\Phi_k) \right\rangle dt,$$

for a sequence $\mathcal{F}_{0,\cdot}(Q) \ni \Phi_k \rightarrow E_S(u^n)$ in $B_{0,\cdot}^{1,1/2}(Q)$.

This implies that $\left\| \frac{\partial_-^{1/2} E_S(u^n)}{\partial t^{1/2}} \right\|_{L^2(Q_+)}$ is bounded by a constant independent of n .

We conclude that $\|u^n\|_{B_I(Q_+)} \leq C$, where $C < \infty$ is a constant independent of n .

We can now extract a weakly convergent subsequence and in fact, as we have seen, we actually have strong convergence in $C_b([0, \infty), L^2(\Omega))$ and thus the limit function satisfies the initial conditions.

Finally a Minty argument using the monotonicity of $A(x, t, \cdot)$ shows that the limit function solves (4.32). The theorem follows. \square

4.3. Fully non-homogeneous initial-boundary values. We shall now introduce the function space that will carry both initial and lateral boundary data.

Since we have continuous imbeddings $B_{0,\cdot}^{1,1/2}(Q_+) \hookrightarrow B_{\cdot,\cdot}^{1,1/2}(Q_+)$ and $B_I(Q_+) \hookrightarrow B_{\cdot,\cdot}^{1,1/2}(Q_+)$, the following definition makes sense.

Definition 4.6. Let

$$(4.37) \quad X^{1,1/2}(Q_+) = B_{\cdot,0}^{1,1/2}(Q_+) + B_I(Q_+)$$

be equipped with the norm

$$(4.38) \quad \|u\|_{X^{1,1/2}(Q_+)} = \inf_{(u_1, u_2) \in K_u} \left(\|u_1\|_{B_{\cdot,0}^{1,1/2}(Q_+)} + \|u_2\|_{B_I(Q_+)} \right),$$

where the infimum is taken over the set

$$(4.39) \quad K_u = \left\{ (u_1, u_2); u_1 + u_2 = u, u_1 \in B_{\cdot,0}^{1,1/2}(Q_+), u_2 \in B_I(Q_+) \right\}.$$

The following imbeddings are immediate

$$(4.40) \quad \|u\|_{X^{1,1/2}(Q_+)} \leq \|u\|_{B_{\cdot,0}^{1,1/2}(Q_+)}; \quad u \in B_{\cdot,0}^{1,1/2}(Q_+),$$

$$(4.41) \quad \|u\|_{X^{1,1/2}(Q_+)} \leq \|u\|_{B_I(Q_+)}; \quad u \in B_I(Q_+),$$

$$(4.42) \quad \|u\|_{B_{\cdot,\cdot}^{1,1/2}(Q_+)} \leq C \|u\|_{X^{1,1/2}(Q_+)}; \quad u \in X^{1,1/2}(Q_+).$$

For an element in $X^{1,1/2}(Q_+)$ we can always define the trace on $\Omega \times \{0\}$.

Theorem 4.6. *There exists a continuous linear and surjective trace operator*

$$(4.43) \quad Tr_0: X^{1,1/2}(Q_+) \longrightarrow L^2(\Omega).$$

There also exists a bounded extension operator

$$(4.44) \quad E: L^2(\Omega) \longrightarrow X^{1,1/2}(Q_+)$$

such that $Tr_0 \circ E = Id_{L^2(\Omega)}$.

Proof. Given $u \in X^{1,1/2}(Q_+)$, there exist $u_1 \in B_{\cdot,0}^{1,1/2}(Q_+)$ and $u_2 \in B_I(Q_+)$ such that $u = u_1 + u_2$. Since $u_2 \in B_I(Q_+) \implies u_2 \in C_b([0, +\infty), L^2(\Omega))$, $u_2|_{\Omega \times \{0\}}$ is a well defined element of $L^2(\Omega)$. We now define $u|_{\Omega \times \{0\}} = u_2|_{\Omega \times \{0\}}$. We have to show that this is independent of the decomposition of u , but if we have two different

decompositions $u_1 + u_2 = v_1 + v_2$ as above, then $(u_2 - v_2) \in B_I(Q_+) \cap B_{\cdot,0}^{1,1/2}(Q_+)$, which implies that

$$(4.45) \quad \iint_{\Omega \times (0, +\infty)} \frac{(u_2 - v_2)^2(x, t)}{t} dx dt < +\infty,$$

and so $u_2(\cdot, 0) = v_2(\cdot, 0)$ since they both belong to $C_b([0, +\infty), L^2(\Omega))$.

Now

$$(4.46) \quad \|u(\cdot, 0)\|_{L^2(\Omega)} = \|u_2(\cdot, 0)\|_{L^2(\Omega)} \leq C \|u_2\|_{B_I(Q_+)},$$

for any decomposition $u = u_1 + u_2$ as above. Taking the infimum over all such decompositions gives:

$$(4.47) \quad \|u(\cdot, 0)\|_{L^2(\Omega)} \leq C \|u\|_{X^{1,1/2}(Q_+)}, \quad u \in X^{1,1/2}(Q_+).$$

Now given $u_0 \in L^2(\Omega)$, let $E(u_0)$ be the (unique) solution in $B_I(Q_+)$ of the initial value problem:

$$(4.48a) \quad \frac{\partial u}{\partial t} - \nabla_x \cdot (|\nabla_x u|^{p-2} \nabla_x u) = 0 \quad \text{in } Q_+ = \Omega \times \mathbf{R}_+,$$

$$(4.48b) \quad u = u_0 \quad \text{on } \Omega \times \{0\}.$$

Clearly this extension map satisfies $Tr_0 \circ E = Id_{L^2(\Omega)}$ and furthermore

$$(4.49) \quad \|E(u_0)\|_{B_I(Q_+)} \leq C \|u_0\|_{L^2(\Omega)},$$

and thus

$$(4.50) \quad \|E(u_0)\|_{X^{1,1/2}(Q_+)} \leq C \|u_0\|_{L^2(\Omega)}. \quad \square$$

Remark. Note that if $p = 2$, the extension map is linear.

Theorem 4.7. *We have the following imbedding:*

$$(4.51) \quad \|u\|_{B_{\cdot,0}^{1,1/2}(Q_+)} \leq C \|u\|_{X^{1,1/2}(Q_+)}; \quad u \in B_{\cdot,0}^{1,1/2}(Q_+).$$

Proof. If $u \in B_{\cdot,0}^{1,1/2}(Q_+)$ and $u = u_1 + u_2$ with $u_1 \in B_{\cdot,0}^{1,1/2}(Q_+)$ and $u_2 \in B_I(Q_+)$, then $u_2(\cdot, 0) = 0$ since $u_2 \in B_{\cdot,0}^{1,1/2}(Q_+) \cap B_I(Q_+)$. Thus u_2 can be extended by zero to all of Q . Since, by a continuity argument,

$$(4.52) \quad \left\| \frac{\partial_+^{1/2} u_2}{\partial t} \right\|_{L^2(Q_+)}^2 = - \int_{\mathbf{R}_+} \left\langle \frac{\partial u_2}{\partial t}, h(u_2) \right\rangle dt, \quad u_2 \in B_{\cdot,0}^{1,1/2}(Q_+) \cap B_I(Q_+).$$

We get

$$(4.53) \quad \|u_1\|_{B_{\cdot,0}^{1,1/2}(Q_+)} + \|u_2\|_{B_I(Q_+)} \geq C \left(\|u_1\|_{B_{\cdot,0}^{1,1/2}(Q_+)} + \|u_2\|_{B_{\cdot,0}^{1,1/2}(Q_+)} \right) \\ \geq C \|u_1 + u_2\|_{B_{\cdot,0}^{1,1/2}(Q_+)} = C \|u\|_{B_{\cdot,0}^{1,1/2}(Q_+)},$$

where $C > 0$. Taking the infimum concludes the proof. □

We immediately get the following

Corollary 4.1. *There exist constants $C_1, C_2 > 0$ such that*

$$(4.54) \quad C_1 \|u\|_{B_{0,0}^{1,1/2}(Q_+)} \leq \|u\|_{X^{1,1/2}(Q_+)} \leq C_2 \|u\|_{B_{0,0}^{1,1/2}(Q_+)}; \quad u \in \mathcal{F}_{0,0}(Q_+).$$

Thus $B_{0,0}^{1,1/2}(Q_+)$ is the closure of $\mathcal{F}_{0,0}(Q_+)$ in the $X^{1,1/2}(Q_+)$ -norm topology.

We are now ready to state our main theorem.

Theorem 4.8. Given $f \in B_{0,\cdot}^{1,1/2}(Q_+)^*$ and $g \in X^{1,1/2}(Q_+)$, there exists a unique element $u \in X^{1,1/2}(Q_+)$ such that

$$(4.55a) \quad \frac{\partial u}{\partial t} - \nabla_x \cdot (A(x, t, \nabla_x u)) = f \quad \text{in } \mathcal{F}'_{\cdot,\cdot}(Q_+),$$

$$(4.55b) \quad u - g \in B_{0,0}^{1,1/2}(Q_+).$$

Proof. Let $w = u - g$. Then (4.55) is equivalent to

$$(4.56a) \quad \frac{\partial w}{\partial t} - \nabla_x \cdot (A(x, t, \nabla_x (w + g))) = f - \frac{\partial g}{\partial t} \quad \text{in } \mathcal{F}'_{\cdot,\cdot}(Q_+),$$

$$(4.56b) \quad w \in B_{0,0}^{1,1/2}(Q_+).$$

Here $\frac{\partial g}{\partial t} \in \mathcal{F}'_{\cdot,\cdot}(Q_+)$ has a unique extension to an element in $B_{0,\cdot}^{1,1/2}(Q_+)^*$. In fact, if $g \in X^{1,1/2}(Q_+)$, we can write $g = g_1 + g_2$, where $g_1 \in B_{\cdot,0}^{1,1/2}(Q_+)$ and $g_2 \in B_I(Q_+)$. Thus

$$(4.57) \quad \begin{aligned} \left| \left\langle g, \frac{\partial \Phi}{\partial t} \right\rangle \right| &= \left| \left\langle g_1, \frac{\partial \Phi}{\partial t} \right\rangle + \left\langle g_2, \frac{\partial \Phi}{\partial t} \right\rangle \right| \\ &\leq C \left(\|g_1\|_{B_{\cdot,0}^{1,1/2}(Q_+)} + \|g_2\|_{B_I(Q_+)} \right) \|\Phi\|_{B_{0,\cdot}^{1,1/2}(Q_+)}; \quad \Phi \in \mathcal{F}_{0,0}(Q_+). \end{aligned}$$

Since, by Lemma 4.8 and Theorem 4.1, $\mathcal{F}_{0,0}(Q_+)$ is dense in $B_{0,\cdot}^{1,1/2}(Q_+)$, it is clear that we have a unique extension. If the function $A(\cdot, \cdot, \cdot)$ satisfies the structural conditions 1–5 given above, then also $A(\cdot, \cdot, \cdot + g)$, with $g \in X^{1,1/2}(Q_+)$, satisfies the same structural conditions (with new constants λ, Λ and functions H, h depending on g). Thus Theorem 4.3, and the remark following Theorem 4.3, tell us that (4.56) has a unique solution. This implies that $u = w + g$ is the unique solution to (4.55). \square

Remark. Note that since $\mathcal{D}(Q_+)$ is densely continuously imbedded in $\mathcal{F}_{0,0}(Q_+)$ it is equivalent to demand that (4.55) should hold in $\mathcal{D}'(Q_+)$.

We shall conclude with a comment on the linear case.

The function spaces we have introduced so far coincides with well known function spaces existing in the literature when $p = 2$. When $p = 2$ we shall follow existing notation and replace B with H for all spaces (for instance, if $p = 2$ we shall write $H_{0,\cdot}^{1,1/2}(Q_+)$ instead of $B_{0,\cdot}^{1,1/2}(Q_+)$ and so on).

The Sobolev space $H_{\cdot,\cdot}^{1/2,1/4}(\partial\Omega \times \mathbf{R}_+)$ below is defined by pull-backs in local charts on $\partial\Omega$.

Theorem 4.9. If $p = 2$ there exists a linear, continuous and surjective trace operator

$$(4.58) \quad Tr: X^{1,1/2}(Q_+) \longrightarrow H_{\cdot,\cdot}^{1/2,1/4}(\partial\Omega \times \mathbf{R}_+).$$

There also exists a continuous and linear extension operator

$$(4.59) \quad E: H_{\cdot,\cdot}^{1/2,1/4}(\partial\Omega \times \mathbf{R}_+) \longrightarrow X^{1,1/2}(Q_+),$$

such that $Tr \circ E = Id|_{H_{\cdot,\cdot}^{1/2,1/4}(\partial\Omega \times \mathbf{R}_+)}$.

Proof. Using a partition of unity argument and the Fourier multiplier operators

$$(4.60) \quad m_s(D)u = ((1 + i2\pi\tau + 4\pi^2|\xi|^2)^{-s}\hat{u})^\vee; \quad s \in \mathbf{R},$$

which preserves forward support in time, and have the property that

$$(4.61) \quad m_s(D) (L^2(\mathbf{R}^n \times \mathbf{R})) = H_{\cdot, \cdot}^{2s, s}(\mathbf{R}^n \times \mathbf{R}),$$

we can construct continuous linear operators:

$$(4.62) \quad Tr: H_{\cdot, 0}^{1, 1/2}(Q_+) \longrightarrow H_{\cdot, \cdot}^{1/2, 1/4}(\partial\Omega \times \mathbf{R}_+)$$

and

$$(4.63) \quad E: H_{\cdot, \cdot}^{1/2, 1/4}(\partial\Omega \times \mathbf{R}_+) \longrightarrow H_{\cdot, 0}^{1, 1/2}(Q_+),$$

such that $Tr \circ E = Id|_{H_{\cdot, \cdot}^{1/2, 1/4}(\partial\Omega \times \mathbf{R}_+)}$. Now given $u \in X^{1, 1/2}(Q_+)$, let $u = u_1 + u_2$ where $u_1 \in H_{\cdot, 0}^{1, 1/2}(Q_+)$ and $u_2 \in H_I(Q_+)$. We define $u|_{\partial\Omega \times \mathbf{R}_+} = u_1|_{\partial\Omega \times \mathbf{R}_+}$. This definition is independent of the decomposition of u . In fact, if $u_1 + u_2 = v_1 + v_2$ are two decompositions as above, then $u_1 - v_1 \in L^2(\mathbf{R}_+, H_0^1(\Omega))$, and so $(u_1 - v_1)|_{\partial\Omega \times \mathbf{R}} = 0$.

Now

$$(4.64) \quad \|Tr(u)\|_{H_{\cdot, \cdot}^{1/2, 1/4}(\partial\Omega \times \mathbf{R}_+)} \leq C \|u_1\|_{H_{\cdot, 0}^{1, 1/2}(Q_+)}$$

for any decomposition. Taking the infimum proves the continuity of Tr . The continuity of the extension operator E follows from the imbedding $H_{\cdot, 0}^{1, 1/2}(Q_+) \hookrightarrow X^{1, 1/2}(Q_+)$. \square

Combining our trace theorems with Theorem 4.8 gives us in the linear case:

Theorem 4.10. *If*

$$(4.65) \quad Tu = \frac{\partial u}{\partial t} - \nabla_x \cdot (A(x, t, \nabla_x u)),$$

is a linear operator, satisfying the structural conditions 1–5 above, then

$$(4.66) \quad \begin{aligned} X^{1, 1/2}(Q_+) \ni u &\mapsto (Tu, u|_{\partial\Omega \times \mathbf{R}_+}, u|_{\Omega \times \{0\}}) \\ &\in H_{0, \cdot}^{1, 1/2}(Q_+)^* \times H_{\cdot, \cdot}^{1/2, 1/4}(\partial\Omega \times \mathbf{R}_+) \times L^2(\Omega) \end{aligned}$$

is a linear isomorphism.

Acknowledgements. I thank Johan Råde for useful remarks and stimulating discussions in connection with this work, and Anders Holst, Per-Anders Ivert and Stefan Jakobsson for reading and commenting on this paper.

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Received 26 November 2008