Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 34, 2009, 615–636

MEROMORPHIC COMPOSITIONS AND TARGET FUNCTIONS

Alastair N. Fletcher and James K. Langley

University of Glasgow, Department of Mathematics
University Gardens, Glasgow G12 8QW, United Kingdom; a.fletcher@maths.gla.ac.uk

University of Nottingham, School of Mathematical Sciences
University Park, Nottingham NG7 2RD, United Kingdom; jkl@maths.nott.ac.uk

Abstract. We prove a result on the frequency of zeros of $f \circ g - Q$, where g is a transcendental entire function of finite lower order, and f and Q are meromorphic functions in the plane such that f has finite order and the growth of the target function Q is controlled by that of g. The particular case $f = Q$ is then investigated further.

1. Introduction

This paper is concerned with the zeros of functions of the form

$$F = f \circ g - Q, \tag{1}$$

where f and Q are meromorphic in the plane and g is a non-linear entire function. For convenience we will on occasions write $f[g] = f \circ g$ to denote composition, and we will use the standard notation of Nevanlinna theory [12], including the abbreviation "n.e. on E" (nearly everywhere on E) to mean as $r \to \infty$ in $E \setminus E_1$, where E_1 has finite measure.

The study of the zeros of the composition (1) has a long history. Bergweiler [1] proved a conjecture of Gross [10], to the effect that if f and g are transcendental entire functions and Q is a non-constant polynomial, then $f \circ g - Q$ has infinitely many zeros. Extensions to the case of meromorphic functions f, and further generalisations including to non-real fixpoints of compositions, as well as to quasiregular mappings, may be found in [2, 3, 25] and elsewhere.

The first result of the present paper is motivated by two papers of Katajamäki, Kinnunen and Laine [19, 20], which focus on the frequency of zeros of the composition (1). Results related to [19, 20] include those of [5, 7, 31, 32]. The main result of [20] states that if g is a transcendental entire function of finite lower order $\mu(g)$, and f is a transcendental meromorphic function in the plane of finite order, while Q is non-constant and meromorphic in the plane of order less than $\mu(g)$, then the exponent of convergence of the zeros of $f \circ g - Q$ is at least $\mu(g)$. The methods of [20] are complicated, but with a simpler proof we will establish the following stronger theorem, which in particular allows the growth of the target function Q to match that of g.

2000 Mathematics Subject Classification: Primary 30D35.

Key words: Meromorphic functions, compositions, Nevanlinna theory.

Both authors were supported by Engineering and Physical Sciences Research Council grant EP/D065321/1.

Theorem 1.1. *Let the functions f, g and Q be meromorphic in the plane with the following properties.*

(i) *f is transcendental of finite order.*
(ii) *g is transcendental entire of finite lower order.*
(iii) *There exists a set $E \subseteq [1, \infty)$ of positive lower logarithmic density such that the functions Q and $F = f \circ g - Q$ satisfy*

$$T(r, Q) = O(T(r, g)) \quad \text{on } E \tag{2}$$

and

$$\overline{N}(r, 1/F) = O(T(r, g)) \quad \text{on } E. \tag{3}$$

Then at least one of the following two conclusions is satisfied.

(a) *There exists a rational function R such that $f - R$ has finitely many zeros and $Q = R \circ g$, and this conclusion always holds if f has finitely many poles.*
(b) *There exist rational functions A, B, C such that f solves the Riccati equation*

$$y' = A + By + Cy^2, \tag{4}$$

and

$$Q' = g'(A[g] + B[g]Q + C[g]Q^2), \tag{5}$$

so that locally we may write $Q = w \circ g$ for some solution w of (4).

If (2) is replaced by

$$T(r, Q) = o(T(r, g)) \quad \text{on } E \tag{6}$$

then Q must be constant.

It is obvious that $f \circ g - Q$ may fail to have zeros if Q is a rational function of g, and in particular if Q is constant. We will give an example in §4 to show that when f has infinitely many poles case (b) can occur with the local solution w not meromorphic in the plane. Of course if $Q = w \circ g$ with w meromorphic in the plane then (2) and a well known result of Clunie (see Lemma 2.1 and [12, p. 54]) imply that w must be a rational function. We remark further that in case (b) the order and sectorial behaviour of f may be determined asymptotically from (4) [30].

The remainder of this paper is mainly concerned with the case where $Q = f$ in (1), and follows a line of investigation which was prompted by the study of the value distribution of differences $f(z + c) - f(z)$. It was conjectured in [4] that if f is transcendental and meromorphic in the plane of order less than 1 then $\Delta f(z) = f(z + 1) - f(z)$ has infinitely many zeros: such a result would represent a discrete analogue of a sharp theorem on the zeros of the derivative f' [8]. For the case where $\rho(f) < 1/6$ it was proved in [4, 22] that either Δf or $(\Delta f)/f$ has infinitely many zeros. The q-difference $f(qz) - f(z)$ was treated next in [9], in which it was shown that if f is transcendental and meromorphic in the plane with

$$\liminf_{r \to \infty} \frac{T(r, f)}{(\log r)^2} = 0,$$

and if

$$h(z) = f(az + b) - f(z), \quad a, b \in \mathbf{C}, \quad |a| \neq 0, 1,$$

then either h or h/f has infinitely many zeros: this result is sharp.

The above investigations suggest the natural question of whether $f \circ g - f$ must have zeros, when f is transcendental and meromorphic in the plane and g is a non-linear entire function. Suppose first that g is a transcendental entire function with no fixpoints and let $f = R \circ g^{\circ n}$ for some $n \in \mathbf{N}$, where R is a Möbius transformation and $g^{\circ 0} = \mathrm{id}$, $g^{\circ 1} = g$, $g^{\circ(k+1)} = g \circ g^{\circ k}$ denote the iterates of g. Then

$$F = f \circ g - f = R \circ g^{\circ(n+1)} - R \circ g^{\circ n}$$

has no zeros, since if z is a zero of F then $g^{\circ n}(z)$ is a fixpoint of g. We will deduce the following result from Theorem 1.1.

Theorem 1.2. *Let f and g be transcendental meromorphic functions in the plane such that g is entire of finite lower order while f has finite order. Assume that there exists a set $E \subseteq [1, \infty)$ of positive lower logarithmic density such that $F = f \circ g - f$ and f satisfy*

$$\overline{N}(r, 1/F) + T(r, f) = O(T(r, g)) \quad \text{on } E. \tag{7}$$

Then there exist a Möbius transformation R and polynomials P and S such that $f = R \circ g$ and

$$g(z) = z + S(z)e^{P(z)}. \tag{8}$$

In particular, if f has finitely many poles then $f = ag + b$ with $a, b \in \mathbf{C}$.

We turn next to the case where $F = f \circ g - f$ with g a non-linear polynomial.

Theorem 1.3. *Let the function f be transcendental and meromorphic of finite order ρ in the plane, with finitely many poles, and let g be a polynomial of degree $m \geq 2$. Let $F = f \circ g - f$. Then F has infinitely many zeros and if $\rho > 0$ then the exponent of convergence of the zeros of F is $\rho(F) = m\rho$.*

Finally for f with infinitely many poles we have a somewhat less complete result.

Theorem 1.4. *Let the function f be transcendental and meromorphic of order ρ in the plane, and let g be a polynomial of degree $m \geq 2$. Let $F = f \circ g - f$. If $0 < \rho < 1/m$, or if $\rho = 0$ and $m \geq 4$, then F has infinitely many zeros. If $\rho = 0$ then the equation*

$$f(g(z)) = f(z) \tag{9}$$

has infinitely many solutions z in the plane.

The authors thank the referee for several helpful suggestions to improve the readability of the paper, and for proposing a potentially very fruitful alternative approach to Theorems 1.3 and 1.4.

2. Results of Clunie, Steinmetz, Hayman and Wittich

This paper will make frequent use of the following result of Clunie [12, p. 54].

Lemma 2.1. ([12]) *Let g be a transcendental entire function and let f be a transcendental meromorphic function in the plane. Then $T(r, g) = o(T(r, f \circ g))$ as $r \to \infty$.*

The following theorem of Steinmetz [26] (see also [11]) plays a role in the present paper similar to that in [20].

Theorem 2.1. ([26]) *Suppose that g is a non-constant entire function and that $F_0, F_1, \dots, F_m$ and $h_0, h_1, \dots, h_m$ are functions meromorphic in the plane, none of which vanishes identically, such that*

$$\sum_{j=0}^{m} T(r, h_j) = O(T(r, g))$$

as $r \to \infty$ in a set of infinite measure, and

$$F_0[g]h_0 + F_1[g]h_1 + \ldots + F_m[g]h_m \equiv 0.$$

Then there exist polynomials $P_0, P_1, \dots, P_m$, not all identically zero, as well as polynomials $Q_0, Q_1, \dots, Q_m$, again not all identically zero, such that

$$P_0[g]h_0 + P_1[g]h_1 + \ldots + P_m[g]h_m \equiv 0, \quad Q_0F_0 + Q_1F_1 + \ldots + Q_mF_m \equiv 0.$$

We need next a result of Hayman.

Theorem 2.2. ([13, 16]) *Let the function g be transcendental and meromorphic of finite lower order in the plane, and let $\delta > 0$. Then there exist a positive real number C_0 and a set $E' \subseteq [1, \infty)$, of upper logarithmic density at least $1 - \delta$, such that*

$$T(2r, g) \le C_0 T(r, g) \quad \text{and} \quad T(r, g) \le C_0 T(r, g') \quad \text{for all } r \in E'.$$

Theorem 2.2 follows from [13, Lemma 4] combined with either [13, Lemma 5] or the Hayman–Miles theorem [16]. In the present paper the result will only be applied when g is a transcendental entire function of finite lower order, in which case [13, Lemma 4] gives a set E' of upper logarithmic density at least $1 - \delta$ and a positive constant C_1 such that for $r \in E'$ we have $T(2r, g) \le C_1 T(r/2, g)$, and hence

$$T(r, g) \le C_1 T(r/2, g) \le C_1 \log M(r/2, g') + O(\log r) \le (3C_1 + o(1))T(r, g').$$

We require three fairly standard lemmas concerning Riccati equations: we sketch the proofs for completeness. For a discussion of the Riccati equation see [21, Chapter 9].

Lemma 2.2. *Let the functions A, B, C and $1/C$ be analytic on the simply connected plane domain U, and let u be a meromorphic solution of the Riccati equation (4) on a non-empty domain $U' \subseteq U$. Then u extends to a meromorphic solution of (4) on U.*

Proof. Choose $z_1 \in U'$ with $u(z_1) \in \mathbf{C}$ and near z_1 write

$$\begin{aligned} &v = -Cu, \quad \frac{V'}{V} = v, \quad v' = -AC + \left(B + \frac{C'}{C}\right) v - v^2, \\ &V'' = \left(B + \frac{C'}{C}\right) V' - ACV. \end{aligned} \tag{10}$$

The coefficients of the linear equation for V in (10) are analytic on U and so V extends to be analytic on U. □

Lemma 2.3. *Let the functions A, B and C be analytic on the plane domain U, and let u and v be meromorphic solutions of (4) on U. Assume that there exists $z_0 \in U$ with $u(z_0) = v(z_0)$. Then $u \equiv v$ on U.*

Proof. Assume that $u \not\equiv v$ on U and suppose first that $u(z_0) = v(z_0) \in \mathbf{C}$. Then (4) gives

$$\frac{u' - v'}{u - v} = B + C(u + v)$$

and at z_0 the left-hand side has a pole, while the right-hand side is regular. On the other hand if u and v both have a pole at z_0 then the same argument may be applied to $1/u$ and $1/v$, which solve

$$-Y' = C + BY + AY^2. \qquad \square$$

The last of these lemmas is essentially due to Wittich [30, p. 283].

Lemma 2.4. *Let A, B and C be rational functions vanishing at infinity. Then (4) cannot have a solution which is transcendental and meromorphic in the plane.*

Proof. Let A, B and C be as in the hypotheses and assume that u is a transcendental meromorphic solution of (4) in the plane. Then $C \not\equiv 0$: if this is not the case then u has finitely many poles and cannot be transcendental by the Wiman-Valiron theory [29]. We now apply the transformations (10) and deduce that all but finitely many poles of v are simple, and that there exists a rational function R which vanishes at infinity such that all poles of the transcendental meromorphic function $w = v - R$ are simple with residue 1. Hence there exists a transcendental entire function W with $W'/W = w$. But w and W satisfy

$$w' + w^2 = -AC - R' - R^2 + \left(B + \frac{C'}{C}\right) R + \left(B - 2R + \frac{C'}{C}\right) w,$$
$$W'' = \left(-AC - R' - R^2 + \left(B + \frac{C'}{C}\right) R\right) W + \left(B - 2R + \frac{C'}{C}\right) W',$$

and the linear equation for W has a regular singular point at infinity, which contradicts the fact that W is transcendental. $\square$

3. Proof of Theorem 1.1

The proof of Theorem 1.1 will be accomplished in three main steps.

3.1. Proof of Theorem 1.1: the first part. To prove Theorem 1.1 let the functions f, g, Q, F and the set E be as in the hypotheses. If $Q = R \circ g$ is a rational function of g and if $\alpha_1, \dots, \alpha_m$ are distinct zeros of $f - R$ then the second fundamental theorem and (3) give

$$(m - 1 - o(1))T(r, g) \le \sum_{k=1}^{m} \overline{N}(r, 1/(g - \alpha_k)) \le \overline{N}(r, 1/F) = O(T(r, g)) \quad \text{n.e. on } E,$$

and so $f - R$ has finitely many zeros. Assume henceforth that Q has no representation as a rational function of g. Then in particular Q is non-constant.

Choose entire functions f_1, f_2 of finite order with no common zeros such that $f = f_1/f_2$ and define K by

$$K = f_1[g] - Q \cdot f_2[g]. \tag{11}$$

Then (3) and (11) give

$$f_1 = f \cdot f_2, \quad K = F \cdot f_2[g] \not\equiv 0. \tag{12}$$

It then follows from (2), (3), (12) and the fact that f_1 and f_2 have no common zeros that

$$\overline{N}(r,K)+\overline{N}(r,1/K)\le\overline{N}(r,1/F)+\overline{N}(r,Q)=O(T(r,g))\quad\text{on } E. \tag{13}$$

Denote positive constants by C_j. Since g has finite lower order and E has positive lower logarithmic density, Theorem 2.2 gives a set $E_1\subseteq E\subseteq[0,\infty)$ of infinite linear measure such that

$$T(2r,g)\le C_1T(r,g)\quad\text{for all } r\in E_1. \tag{14}$$

Now set

$$\gamma=\frac{K'}{K}. \tag{15}$$

Then (13), (14), (15), the lemma of the logarithmic derivative and the fact that f_1 and f_2 have finite order imply that, n.e. on E_1,

$$\begin{aligned}T(r,\gamma)&\le C_2\log^+T(r,K)+O(T(r,g))\\&\le C_2\sum_{j=1}^{2}\log^+T(r,f_j[g])+O(T(r,g))\\&\le C_2\sum_{j=1}^{2}\log^+\log^+M(r,f_j[g])+O(T(r,g))\\&\le C_2\sum_{j=1}^{2}\log^+\log^+M(M(r,g),f_j)+O(T(r,g))\\&\le C_3\log M(r,g)+O(T(r,g))\le C_4T(2r,g)\le C_5T(r,g).\end{aligned} \tag{16}$$

Differentiating (11) gives

$$g'\cdot f_1'[g]-Q'\cdot f_2[g]-Qg'\cdot f_2'[g]=K'=\gamma(f_1[g]-Q\cdot f_2[g])$$

and so

$$g'\cdot f_1'[g]-\gamma\cdot f_1[g]-Qg'\cdot f_2'[g]+(\gamma Q-Q')f_2[g]=0. \tag{17}$$

By (2) and (16) the coefficients in (17) satisfy, n.e. on E_1,

$$T(r,g')+T(r,\gamma)+T(r,Qg')+T(r,\gamma Q-Q')=O(T(r,g))$$

and so it follows from Theorem 2.1 that there exist polynomials ϕ_j, not all the zero polynomial, such that

$$\phi_1f_1'+\phi_2f_1+\phi_3f_2'+\phi_4f_2=0. \tag{18}$$

Here ϕ_1 and ϕ_3 cannot both vanish identically, since $f=f_1/f_2$ is not a rational function. Obviously (18) gives

$$\phi_1[g]f_1'[g]+\phi_2[g]f_1[g]+\phi_3[g]f_2'[g]+\phi_4[g]f_2[g]=0. \tag{19}$$

Lemma 3.1. *There exist rational functions A,B,C such that f solves the Riccati equation (4).*

Proof. Suppose first that ϕ_1 does not vanish identically in (18). Multiplying (19) by g' and (17) by $\phi_1[g]$ and subtracting we obtain

$$\begin{aligned}(g'\cdot\phi_2[g]+\gamma\cdot\phi_1[g])f_1[g]+(g'\cdot\phi_3[g]+Qg'\cdot\phi_1[g])f_2'[g]\\ +(g'\cdot\phi_4[g]+(Q'-\gamma Q)\phi_1[g])f_2[g]=0.\end{aligned}\tag{20}$$

In this equation the coefficient of $f_2'[g]$ does not vanish identically since Q is not a rational function of g. Hence using (16) again we may apply Theorem 2.1 to (20) to obtain polynomials ψ_j, not all zero, such that

$$\psi_1 f_1+\psi_2 f_2'+\psi_3 f_2=0.\tag{21}$$

Here ψ_2 cannot be the zero polynomial since f is not a rational function. Using (21) and the assumption that ϕ_1 is not the zero polynomial in (18) we therefore obtain rational functions R_j, S_j such that

$$f_1'=R_1f_1+S_1f_2,\quad f_2'=R_2f_1+S_2f_2,\quad \frac{f'}{f}=\frac{f_1'}{f_1}-\frac{f_2'}{f_2}=R_1+\frac{S_1}{f}-R_2f-S_2,\tag{22}$$

from which a Riccati equation (4) for f follows at once.

Suppose now that ϕ_1 is the zero polynomial. Then ϕ_3 does not vanish identically. We multiply (19) by Qg' and (17) by $\phi_3[g]$ and add, to obtain

$$\begin{aligned}(Qg'\cdot\phi_1[g]+g'\cdot\phi_3[g])f_1'[g]+(Qg'\cdot\phi_2[g]-\gamma\cdot\phi_3[g])f_1[g]\\ +(Qg'\cdot\phi_4[g]+(\gamma Q-Q')\phi_3[g])f_2[g]=0,\end{aligned}$$

in which the coefficient of $f_1'[g]$ cannot vanish identically since ϕ_1 is the zero polynomial but ϕ_3 is not. This time we obtain

$$\psi_1 f_1'+\psi_2 f_1+\psi_3 f_2=0$$

with polynomials ψ_j and $\psi_1\not\equiv 0$, and since $\phi_3\not\equiv 0$ in (18) this leads to (22) again. □

Since f satisfies (4) we now have

$$g'\cdot f'[g]=g'(A[g]+B[g]f[g]+C[g]f[g]^2)$$

and so

$$F'+Q'=g'\cdot f'[g]=g'(A[g]+B[g](F+Q)+C[g](F+Q)^2).\tag{23}$$

Lemma 3.2. *The function*

$$L=Q'-g'(A[g]+B[g]Q+C[g]Q^2)\tag{24}$$

vanishes identically, and we may write Q in the form $Q=w[g]$ for some local solution w of (4).

Proof. Assume that L does not vanish identically and using (24) write (23) in the form

$$L=-F'+Fg'(B[g]+C[g](2Q+F)),$$

which leads at once to

$$\frac{1}{F}=\frac{1}{L}\left(-\frac{F'}{F}+L_1+FL_2\right),\quad L_1=g'(B[g]+2C[g]Q),\quad L_2=g'\cdot C[g].\tag{25}$$

Then (2), (3) and (25) give, n.e. on E_1,

$$\begin{aligned} T(r,f[g]) &\le T(r,F)+O(T(r,g)) = m(r,1/F)+N(r,1/F)+O(T(r,g)) \\ &\le m(r,1/L)+o(T(r,F))+m(r,L_1)+m(r,L_2) \\ &\quad + N(r,1/L)+\overline{N}(r,1/F)+N(r,L_1)+N(r,L_2)+O(T(r,g)) \\ &\le o(T(r,F))+O(T(r,g)) \le o(T(r,f[g]))+O(T(r,g)), \end{aligned}$$

which contradicts Lemma 2.1. Thus L vanishes identically, which gives (5), and it follows at once that we may write Q in the form $Q = w[g]$ for some local solution w of (4). □

This completes the main part of the proof of Theorem 1.1. It remains only to deal with the case where f has finitely many poles, and that in which (2) is replaced by (6).

3.2. Proof of Theorem 1.1: the case of finitely many poles. Still with the hypotheses of Theorem 1.1, we suppose that f has finitely many poles, and continue to assume that conclusion (a) does not hold, that is, that Q is not a rational function of g. Since f is transcendental with finitely many poles the Riccati equation (4) must take the form

$$f' = A + Bf, \tag{26}$$

with $C \equiv 0$. Using (5) we now obtain

$$Q' = g'(A[g]+B[g]Q), \quad g'\cdot f'[g] = g'(A[g]+B[g]f[g]),$$

and subtraction gives

$$F' = g'\cdot f'[g] - Q' = g'\cdot B[g]F,$$

so that

$$g'\cdot B[g] = \frac{F'}{F}. \tag{27}$$

The proof of the following lemma is immediate.

Lemma 3.3. *If B has a pole at $\alpha \in \mathbf{C}$ of multiplicity β and if $z_0 \in \mathbf{C}$ is a zero of $g-\alpha$ of multiplicity γ then $g'\cdot B[g]$ has a pole at z_0 of multiplicity*

$$\beta\gamma - (\gamma-1) = (\beta-1)\gamma + 1 \ge 1.$$ □

We now consider two cases.

Case I: Suppose that B has a pole $\alpha \in \mathbf{C}$ which either is multiple or has non-rational residue.

Then (27) and Lemma 3.3 show that α is an omitted value of g and so is unique. By writing $f(w) = f_1(w-\alpha)$ we may assume that $\alpha = 0$. Hence $g = e^P$, where P is a non-constant polynomial since g has finite lower order. Moreover Q has finite lower order by (2).

If $\beta \in \mathbf{C}\setminus\{0\}$ is a pole of B then $g-\beta$ has infinitely many simple zeros, and so (27) shows that β is a simple pole of B with integer residue. Integration of (27) then gives

$$f[g] - Q = F = e^{cP}e^{P_1[e^P]+P_2[e^{-P}]}\prod_{k=1}^{m}(e^P-\alpha_k)^{q_k}, \tag{28}$$

where c is the residue of B at 0, P_1 and P_2 are polynomials, the α_k are the poles of B in $\mathbf{C} \setminus \{0\}$ (if there are none then the product is just unity) and the q_k are integers. Here at least one of P_1 and P_2 is non-constant since otherwise we obtain

$$T(r, f[g]) = O(T(r, g)) \quad \text{on } E,$$

using (2), which contradicts Lemma 2.1 since f is transcendental.

It follows at once from (28) that we may now write $Q(z) = Q_1(P(z))$ with Q_1 meromorphic of finite lower order in the plane, and (28) now leads to

$$f(e^w) - Q_1(w) = e^{cw} e^{P_1(e^w)+P_2(e^{-w})} \prod_{k=1}^{m} (e^w - \alpha_k)^{q_k}. \tag{29}$$

Thus we obtain

$$Q_1(w) - Q_1(w + 2\pi i) = (e^{c2\pi i} - 1) e^{cw} e^{P_1(e^w)+P_2(e^{-w})} \prod_{k=1}^{m} (e^w - \alpha_k)^{q_k},$$

from which it follows that c must be an integer, since otherwise the left-hand side has finite lower order while the right-hand side has infinite lower order. But then (29) shows that $Q_1(w) = Q_2(e^w)$ and $Q = Q_2(g)$ for some function Q_2 which is meromorphic in the plane, and recalling (2) and Lemma 2.1 we see that Q_2 must be a rational function, contrary to hypothesis. This contradiction disposes of Case I. We are left with:

Case II: Suppose that all poles of B are simple and have rational residues.

Let the residue of B at a pole $\alpha \in \mathbf{C}$ be p/q, where p and q are integers with no non-trivial common factor, and $q > 0$. Then (27) shows that all zeros of $g - \alpha$ have multiplicity divisible by q, and so we may write

$$\frac{pg'}{q(g - \alpha)} = \frac{h'}{h},$$

where h is a meromorphic function in the plane with $T(r, h) = O(T(r, g))$. Thus integration of (27) gives

$$f[g] - Q = F = S_1 e^{S_2[g]},$$

with S_1 a meromorphic function satisfying $T(r, S_1) = O(T(r, g))$ and S_2 a polynomial. By (2) and Theorem 2.1 there exist rational functions T_1 and T_2 with

$$f = T_1 e^{S_2} + T_2.$$

On substitution into (26) this gives

$$0 = A + Bf - f' = (BT_1 - T_1' - S_2' T_1) e^{S_2} + (A + BT_2 - T_2'),$$

and so the rational function T_2 solves the same equation (26) as f. But then the general solution of the equation $y' = A + By$ is $f + c(T_2 - f)$ with c constant, and so is meromorphic in the plane. By (5) and the fact that $C \equiv 0$ there exists a meromorphic function Q_1 in the plane with $Q = Q_1[g]$. Here Q_1 must be a rational function by (2) and Lemma 2.1 again, which contradicts the assumption that Q is not a rational function of g. This completes our discussion of the case where f has finitely many poles. □

3.3. Proof of Theorem 1.1: the case where Q is a small function. Suppose again that f, g, Q, F and E are as in the hypotheses of Theorem 1.1, but with (2) replaced by (6). If conclusion (a) holds then the rational function R must be constant, and so must Q. Assume henceforth that conclusion (b) is satisfied, and that Q is non-constant. Then (5) holds, and leads to

$$(30) \qquad A = \frac{P_1}{S}, \quad B = \frac{P_2}{S}, \quad C = \frac{P_3}{S}, \quad S[g]Q' = g'(P_1[g] + P_2[g]Q + P_3[g]Q^2),$$

where S and the P_j are polynomials, and no root α of S is such that $P_j(\alpha) = 0$ for all j. Let

$$(31) \qquad M = \max\{\deg P_j : j = 1, 2, 3\}, \quad s = \deg S.$$

Suppose that $s \leq M$ in (31). Since Q is non-constant we have

$$P_1[g] + P_2[g]Q + P_3[g]Q^2 = \sum_{k=0}^{M} b_k g^k$$

where $T(r, b_k) = o(T(r, g))$ on E and $b_M \not\equiv 0$. This implies that we may write the last equation of (30) in the form

$$g' b_M g^M = \widetilde{P}[g],$$

where $\widetilde{P}[g]$ is a polynomial of total degree at most M in g and g', with coefficients a_j which satisfy $T(r, a_j) = o(T(r, g))$ n.e. on E. Since b_M is not identically zero, Clunie's lemma [12, p. 68] gives

$$(32) \qquad T(r, g') = m(r, g') = o(T(r, g)) \quad \text{n.e. on E}.$$

But E has positive lower logarithmic density and g has finite lower order, and so (32) is impossible by Theorem 2.2. Thus $s > M$ and so A, B and C all vanish at infinity. Since f is transcendental and satisfies (4), this contradicts Lemma 2.4. The proof of Theorem 1.1 is complete. □

4. An example for Theorem 1.1

Write

$$(33) \qquad w = z^{1/2} - \frac{1}{4z}, \quad w' + w^2 = z + \frac{5}{16z^2}.$$

Consider the linear differential equation

$$(34) \qquad 16z^2 y'' = (16z^3 + 5)y,$$

which has a regular singular point at 0. By [17, Chapter VII] or direct computation, there exists a non-trivial solution y of (34) of the form $y(z) = z^c h(z)$, with $c \in \mathbf{C}$ and h an entire function. We then write

$$f(z) = \frac{y'(z)}{y(z)} = \frac{c}{z} + \frac{h'(z)}{h(z)}$$

and a simple calculation gives

$$(35) \qquad f'(z) + f(z)^2 = \frac{y''(z)}{y(z)} = \frac{c^2 - c}{z^2} + \frac{2ch'(z)}{zh(z)} + \frac{h''(z)}{h(z)} = z + \frac{5}{16z^2}.$$

Thus f solves the same Riccati equation as w. It follows from (35) and the Wiman–Valiron theory [29] that the order of h is $3/2$, so that h has infinitely many zeros and f is transcendental of finite order. Now set

$$g(z) = e^z, \quad Q(z) = w(g(z)) = e^{z/2} - \frac{1}{4e^z}, \quad G(z) = f(g(z)).$$

Then (33) and (35) imply that Q and G are both meromorphic solutions of the Riccati equation

$$Y'(z) = e^z\left(e^z + \frac{5}{16e^{2z}} - Y(z)^2\right) = A(z) + C(z)Y^2,$$

where A and C are entire and C has no zeros. Since G has infinite order and Q has finite order, the function $F = G - Q = f \circ g - Q$ does not vanish identically, and F has no zeros by Lemma 2.3.

5. Proof of Theorem 1.2

Assume that f, g and F are as in the hypotheses and suppose first that f has finitely many poles. Since f has finite order we may apply Theorem 1.1 with $Q = f$, to obtain a rational function R such that $f - R$ has finitely many zeros and $f = R[g]$. Moreover there exist a rational function R_1 and a polynomial P_1 such that

$$f(z) = R_1(z)e^{P_1(z)} + R(z) = R(g(z)). \tag{36}$$

Since R_1 has finitely many zeros it follows that g has finitely many fixpoints and so satisfies (8) with S and P polynomials. Now (8) implies that g has no finite Picard values and so since f has finitely many poles we deduce from (36) that R must be a polynomial. Furthermore, substitution of (8) into (36) shows that R must be linear, which completes the proof in this case.

Assume henceforth that f has infinitely many poles. Then by Theorem 1.1 there exist rational functions A, B and C such that f satisfies the Riccati equation (4), as well as the relation

$$f' = g'(A[g] + B[g]f + C[g]f^2). \tag{37}$$

Clearly C does not vanish identically, since f has infinitely many poles.

Lemma 5.1. *Let D be the set of all poles of A, B, C and $1/C$ in $\mathbf{C}$. Then g has no fixpoints in $\mathbf{C} \setminus D$, and g satisfies (8) with S and P polynomials.*

Proof. Suppose that $z_0 \in \mathbf{C} \setminus D$ is a fixpoint of g. Choose an open disc $U \subseteq \mathbf{C} \setminus D$ of centre z_0, and an open disc $U_1 \subseteq U$, again centred at z_0, such that $U_2 = g(U_1) \subseteq U$. Choose $z_1 \in U_1$ with $g'(z_1) \neq 0$, and choose an open disc $U_3 \subseteq U_1$, centred at z_1, on which g is univalent. Then the branch $g^{-1} \colon U_4 = g(U_3) \to U_3$ of the inverse function gives a meromorphic solution $u = f \circ g^{-1}$ of (4) on $U_4 \subseteq U_2 \subseteq U$, by (37). Now Lemma 2.2 shows that u extends to a meromorphic solution of (4) on U. But $f = u \circ g$ on $U_3 \subseteq U_1$, and so we have $f = u \circ g$ on U_1, which contains z_0. This gives

$$f(z_0) = u(g(z_0)) = u(z_0),$$

and so $f \equiv u$ on U, by Lemma 2.3. Hence $f = f \circ g$ on U_1 and so on $\mathbf{C}$, which contradicts (7). This proves the first assertion of the lemma, and the second follows at once since g has finite lower order. $\square$

Lemma 5.2. *Let (ζ_j) denote the distinct zeros of g', ordered so that $|\zeta_1| \leq |\zeta_2| \leq \ldots$. Then we have*

$$\lim_{n\to\infty} g(\zeta_n) = \infty. \tag{38}$$

Moreover, the function g has no finite asymptotic values. Next, let M be a positive real number. Then there exists an integer $N > 0$ such that if Ω is a component of the set $g^{-1}(B(0,M))$ then no value $w \in B(0,M)$ is taken more than N times in Ω, counting multiplicity. Finally, all but finitely many components of $g^{-1}(B(0,M))$ are mapped univalently onto $B(0,M)$ by g.

Here $B(A,R)$ denotes as usual the open disc of centre A and radius R.

Proof. The assertion (38) is an immediate consequence of (8) and elementary computation. The fact that g has no finite asymptotic values follows from (8) and the Denjoy–Carleman–Ahlfors theorem applied to $g_1(z) = g(z)/z$, since g_1^{-1} has $\rho(g_1)$ direct transcendental singularities over 1, and $\rho(g_1)$ direct transcendental singularities over ∞.

Hence g has finitely many critical values in $B(0,M)$ and there exists a piecewise-linear Jordan arc γ such that $G_M = B(0,M) \setminus \gamma$ is simply connected and contains no singular value of g^{-1}, and so all components of $g^{-1}(G_M)$ are mapped univalently onto G_M by g. Moreover, g has at most $N_1 < \infty$ critical points over $B(0,M)$ and so there exists $N \in \mathbf{N}$ such that each component of $g^{-1}(B(0,M))$ contains at most N components of $g^{-1}(G_M)$. It follows finally that all but finitely many components of $g^{-1}(B(0,M))$ contain no critical points of g and are mapped conformally onto $B(0,M)$ by g. □

Choose M in Lemma 5.2, so large that $D \subseteq B(0,M/2)$. Choose a component Ω of $g^{-1}(B(0,M))$ which is mapped univalently onto $B(0,M)$ by g. Then taking the branch of g^{-1} mapping $B(0,M)$ onto Ω gives a meromorphic solution $u = f \circ g^{-1}$ of (4) on $B(0,M)$, by (37). Applying Lemma 2.2 twice now shows that u extends to a meromorphic solution of (4) on $\mathbf{C}$. Since $f = u \circ g$ on Ω we have $f = u \circ g$ on $\mathbf{C}$, and because (7) implies that

$$T(r, u \circ g) = O(T(r,g)) \quad \text{on } E,$$

it follows from Lemma 2.1 that $u = R$ is a rational function.

Suppose that R is not a Möbius transformation. Then R has a multiple point $\alpha \in \mathbf{C}$: this follows from a standard argument involving continuation of R^{-1} on the extended plane punctured at $R(\infty)$. Since g takes the value α infinitely often in $\mathbf{C}$ by (8), the function $f = R \circ g$ must have infinitely many multiple points $z \in \mathbf{C}$ with $f(z) = R(\alpha) = \beta$, and β must be finite since f solves (4). Hence there are infinitely many points $z \in \mathbf{C}$ with

$$0 = f'(z) = A(z) + B(z)\beta + C(z)\beta^2,$$

and so the rational function $A(z) + B(z)\beta + C(z)\beta^2$ must vanish identically. Thus $y = \beta$ is a constant solution of (4), and $f - \beta$ has finitely many zeros by Lemma 2.3. Now set

$$H = \frac{1}{f-\beta}, \quad G = H[g] - H = \frac{f - f[g]}{(f[g]-\beta)(f-\beta)}.$$

Then H has finitely many poles. Moreover if z is a pole of $f \circ g$ but not of f, then $G(z) \neq 0$. We deduce from (7) that

$$\overline{N}(r, 1/G) + T(r, H) = O(T(r, g)) \quad \text{on } E,$$

and so applying the first part of the proof to H shows that H is a linear function of g. This is a contradiction, and so R must indeed be a Möbius transformation. The proof of Theorem 1.2 is complete. □

We remark that if we strengthen the hypothesis on the zeros of F in Theorem 1.2 by assuming that the counting function of the distinct points at which $f(g(z)) = f(z)$ is $o(T(r, g))$ on E then since R is injective we obtain

$$\overline{N}(r, 1/(g[g] - g)) = o(T(r, g)) \quad \text{on } E,$$

and by (8) the polynomial S has no zeros and must be constant.

6. Proof of Theorem 1.3: preliminaries

If g is a non-linear polynomial then ∞ is a superattracting fixpoint of g and the following lemma summarises some standard results concerning the behaviour of $g(z)$ near ∞.

Lemma 6.1 ([23, 27])**.** *(Böttcher coordinates) Let $g(z) = az^m + \ldots$ be a polynomial of degree $m \geq 2$. Then there exist a neighbourhood U of ∞ and a function ϕ analytic and univalent on U such that*

$$\phi(z) = z + O(1) \quad \text{and} \quad \phi(g(z)) = a\phi(z)^m \quad \text{for all } z \in U. \tag{39}$$

Moreover, for $j = 0, \ldots, m-1$ define u_j and $w_j(z)$ by

$$u_j = e^{2\pi ij/m}, \quad w_j(z) = \phi^{-1}(u_j\phi(z)). \tag{40}$$

Then w_j is analytic and univalent on a neighbourhood $V \subseteq U$ of ∞ and

$$w_j(z) = u_j z + O(1) \quad \text{and} \quad g(w_j(z)) = g(z) \quad \text{for all } z \in V. \tag{41}$$

□

We deduce the following simple lemma, in which $g^{\circ 0} = \text{id}$, $g^{\circ 1} = g$, $g^{\circ(k+1)} = g \circ g^{\circ k}$ as before denote the iterates of g.

Lemma 6.2. *Let $R > 0$ and let the function f be non-constant and meromorphic on the region $R < |z| < \infty$. Let g be a polynomial of degree $m \geq 2$. Then there exists a greatest non-negative integer N such that we may write $f = h_N \circ g^{\circ N}$ with h_N meromorphic on the region $R_N < |z| < \infty$ for some $R_N > 0$.*

Proof. Obviously $f = f \circ g^{\circ 0}$ so assume that there exist arbitrarily large n such that we have $f = h_n \circ g^{\circ n}$ with h_n meromorphic on a punctured neighbourhood of ∞. Let ϕ and a be as in Lemma 6.1. Then it follows easily from (39) that the iterates $g^{\circ k}$ satisfy

$$\phi \circ g^{\circ k} = b_k \phi^{m^k}, \quad b_k \in \mathbf{C} \setminus \{0\}.$$

For large z we may then write

$$v_k = \phi^{-1}\left(e^{2\pi i/m^k}\phi(z)\right), \quad \phi(g^{\circ k}(v_k)) = b_k\phi(v_k)^{m^k} = b_k\phi(z)^{m^k} = \phi(g^{\circ k}(z)),$$

and we deduce that, for arbitrarily large n,

$$f(v_n) = h_n(g^{\circ n}(v_n)) = h_n(g^{\circ n}(z)) = f(z).$$

Since $v_n \to z$ but $v_n \neq z$ this contradicts the identity theorem and proves the lemma. □

Lemma 6.3. *Let f be a function meromorphic in the plane and g a polynomial of degree $m \geq 2$. Define the u_j and $w_j(z)$ by (40), and assume that there exists $R_1 > 0$ such that*

(42) $$f(w_j(z)) = f(z) \quad \text{for } R_1 < |z| < \infty \text{ and for } j = 0, \ldots, m-1.$$

Then there exists a meromorphic function h_1 on the plane such that $f = h_1 \circ g$.

Proof. Let D be the complex plane punctured at the finitely many critical values of g. Choose v^* with $|v^*|$ large and a branch of g^{-1} mapping $w^* = g(v^*)$ to v^*. Then $h_1 = f \circ g^{-1}$ admits continuation along any path in D starting at w^*. Let σ be a path in D starting and finishing at w^*. If g_1 denotes the result obtained on continuing g^{-1} once around σ then we must have $g(g_1(w)) = w = g(g^{-1}(w))$ near w^* and so $g_1(w) = w_j(g^{-1}(w))$ for some j. Since $|g^{-1}(w)|$ is large for w near w^* we deduce from (42) that $f \circ g_1 = h_1$ near w^*. It follows that $h_1 = f \circ g^{-1}$ defines a single-valued meromorphic function on D. Since g takes each of its critical values only finitely often, the singularities of h_1 are at worst poles, and h_1 extends to a meromorphic function satisfying $f = h_1 \circ g$ near v^* and so throughout the plane. □

The next lemma is [4, Lemma 3.3].

Lemma 6.4. ([4]) *Let H be a function transcendental and meromorphic in the plane of order less than 1. Let $t_0 > 0$. Then there exists an ε-set E_1 such that*

$$\frac{H(z+c)}{H(z)} \to 1 \quad \text{as } z \to \infty \text{ in } \mathbf{C} \setminus E_1,$$

uniformly in c for $|c| \leq t_0$.

Here an ε-set is defined, following Hayman [15], to be a countable union of discs

$$E_1 = \bigcup_{j=1}^{\infty} B(b_j, r_j) \quad \text{such that} \quad \lim_{j\to\infty} |b_j| = \infty \quad \text{and} \quad \sum_{j=1}^{\infty} \frac{r_j}{|b_j|} < \infty.$$

The set of $r \geq 1$ such that the circle $S(0,r)$ of centre 0 and radius r meets the ε-set E_1 then has finite logarithmic measure [15].

The next lemma requires the Nevanlinna characteristic for a function h which is meromorphic and non-constant on a domain containing the set $\{z \in \mathbf{C} : R \leq |z| < \infty\}$, for some real $R > 0$ [6, pp. 88–98]. Such a function h has a Valiron representation [29, p. 15] of form

$$h(z) = z^n \psi(z) H(z)$$

where H is meromorphic in the plane, and the zeros and poles of H are the zeros and poles of h in $R \leq |z| < \infty$, with due count of multiplicity. Furthermore, n is an integer and ψ is analytic near ∞ with $\psi(\infty) = 1$. The Nevanlinna characteristic is then given by

$$T_R(r,h) = m_R(r,h) + N_R(r,h) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |h(re^{i\theta})| \, d\theta + N_R(r,h),$$

where

$$N_R(r,h) = \int_R^r \frac{n(t)\, dt}{t} = N(r,H)$$

and $n(t)$ is the number of poles of h, counting multiplicity, in $R \leq |z| \leq t$.

Lemma 6.5. *Let the function f be transcendental and meromorphic of order ρ in the plane, let g be a polynomial of degree $m \geq 2$, let $F = f \circ g - f = f[g] - f$, and let $w_j(z)$ be defined as in (40). Then there exist positive constants c_1, c_2 with the following properties. First, for each $a \in \mathbf{C} \cup \{\infty\}$ we have*

$$N(c_1 r^m, a, f) - O(1) \leq N(r, a, f[g]) \leq N(c_2 r^m, a, f) + O(1) \tag{43}$$

and

$$(1 - o(1))T(c_1 r^m, f) \leq T(r, f[g]) \leq (1 + o(1))T(c_2 r^m, f) \tag{44}$$

as $r \to \infty$. Moreover, if $\delta > 0$ then we have

$$T(r, F) \geq (m - 1 - \delta)T(r, f), \tag{45}$$

as $r \to \infty$, and F is transcendental of order $\rho(F) = m\rho$. If, in addition, we have $\rho < 1$, then

$$T_R(r, f \circ w_j) \leq (1 + o(1))T((1 + o(1))r, f) \tag{46}$$

as $r \to \infty$, for some appropriate choice of R.

Here we write $N(r, \infty, f) = N(r, f)$ and $N(r, a, f) = N(r, 1/(f - a))$ when $a \in \mathbf{C}$.

Proof. As observed by the referee, inequalities (43) and (44) may be found in [18, Section 14] but the proof is included here for completeness. There exist positive constants c_1, c_2 such that

$$\overline{B}(0, c_1 r^m) \subseteq g(\overline{B}(0, r)) \subseteq \overline{B}(0, c_2 r^m), \tag{47}$$

for large r, in which $\overline{B}(0, T)$ denotes the closed disc of centre 0 and radius T. To prove (43) assume that $a = \infty$. Then (47) and the fact that g has degree m give

$$m \cdot n(c_1 r^m, f) \leq n(r, f[g]) \leq m \cdot n(c_2 r^m, f)$$

for $r \geq r_0$, say. This leads in turn to

$$\begin{aligned} N(r, f[g]) &\geq \int_{r_0}^{r} \frac{n(t, f[g])\, dt}{t} - O(1) \geq m \int_{r_0}^{r} \frac{n(c_1 t^m, f)\, dt}{t} - O(1) \\ &= \int_{c_1 r_0^m}^{c_1 r^m} \frac{n(s, f)\, ds}{s} - O(1) \geq N(c_1 r^m, f) - O(1). \end{aligned}$$

This proves the first inequality of (43) and the second is established similarly. Now (44) follows at once from (43) and the first fundamental theorem, since we may choose $a, b \in \mathbf{C}$ such that $T(r, f) \sim N(r, a, f)$ and $T(r, f[g]) \sim N(r, b, f[g])$ as $r \to \infty$ [24, pp. 280–281]. In particular, the order of $f[g]$ is $m\rho$.

Next we choose a such that $T(r,f) \sim N(r,a,f)$ and a small positive τ. Then (43) gives

$$
\begin{aligned}
(m-\tau)N(r,a,f) &\le (m-1-\tau)n(r,a,f)\log r + N(r,a,f) + O(1) \\
&= \int_r^{r^{m-\tau}} \frac{n(r,a,f)\,dt}{t} + N(r,a,f) + O(1) \\
&\le \int_r^{r^{m-\tau}} \frac{n(t,a,f)\,dt}{t} + N(r,a,f) + O(1) \qquad (48) \\
&= N(r^{m-\tau},a,f) + O(1) \le N(c_1 r^m, a, f) + O(1) \\
&\le N(r,a,f[g]) + O(1).
\end{aligned}
$$

Hence we obtain

$$(m-\tau-o(1))T(r,f) \le T(r,f[g]) + O(1),$$

from which (45) follows at once. In particular F is transcendental. If f has order 0 then evidently so has F by (44), and if ρ is finite but positive then F has the same order $m\rho$ as $f[g]$. Finally if $\rho = \infty$ then F has infinite order by (45).

To prove (46) assume that $\rho < 1$. Then (41) and the same argument which established (43) give

$$N_R(r, f \circ w_j) \le (1+o(1))N((1+o(1))r, f)$$

as $r \to \infty$, and Lemma 6.4 implies that there exists an ε-set E_1 such that

$$m_R(r, f \circ w_j) \le m(r,f) + o(1)$$

for all r such that the circle $S(0,r)$ does not meet E_1, and hence for all r outside a set E_2 of finite logarithmic measure. This gives (46), initially for $r \notin E_2$, and hence without exceptional set by the Valiron representation of $f \circ w_j$ and the monotonicity of the Nevanlinna characteristic of a function meromorphic in the plane. □

7. Proof of Theorem 1.3

To prove Theorem 1.3 let f, g and F be as in the hypotheses. If f has order 0 then F is transcendental of order 0 by Lemma 6.5, and so F has infinitely many zeros.

Suppose now that $\rho > 0$. Then Lemma 6.5 gives $\rho(F) = m\rho$. Recall next from Lemma 6.2 that there exists a greatest integer $N \ge 0$ such that f has a representation $f = h_N \circ g^{\circ N}$ where $g^{\circ N}$ is the Nth iterate of g and h_N is meromorphic. Then $F = F_N \circ g^{\circ N}$ where $F_N = h_N \circ g - h_N$. Since $g^{\circ N}$ has degree m^N it follows from Lemma 6.5 that $\rho(h_N) = m^{-N}\rho$ and $\rho(F_N) = m^{1-N}\rho$. Moreover h_N has finitely many poles. Hence if it can be shown that the exponent of convergence of the zeros of F_N is $m^{1-N}\rho$ then it follows from Lemma 6.5 again that the zeros of F have exponent of convergence $m\rho$ as required.

In order to prove Theorem 1.3 it therefore suffices to consider the case where this maximal integer N is 0, and so in particular f has no representation $f = h_1[g]$ with h_1 meromorphic in the plane. By Lemma 6.3, there exists an integer $j \in \{1, \dots, m-1\}$ such that the function

$$f_j(z) = f(w_j(z)) - f(z) \qquad (49)$$

does not vanish identically near infinity, where $w_j(z)$ is defined by (40). Since w_j is the jth iterate of w_1 by (40), we may assume that f_1 does not vanish identically near infinity.

Assume that the exponent of convergence of the zeros of $F = f[g] - f$ is less than $\rho(F) = m\rho$. Then $n = m\rho$ is a positive integer by the Hadamard factorisation theorem, and there exist a polynomial P of degree n and a meromorphic function Π of order less than n, with finitely many poles, such that

$$F = f \circ g - f = \Pi e^P. \tag{50}$$

The following lemma is a standard consequence of the Poisson-Jensen formula and the fact that $\rho(\Pi) < n$.

Lemma 7.1. *Let (u_k) denote the sequence of zeros of Π with repetition according to multiplicity. Then*

$$\sum_k |u_k|^{-n} < \infty \tag{51}$$

and there exists $R_1 > 1$ with

$$\log|\Pi(z)| = o(|z|^n) \quad \textit{for } |z| > R_1, \quad z \notin H_1 = \bigcup_k B(u_k, |u_k|^{-n}). \tag{52}$$ □

On combination with (50) this leads at once to the following estimates for F.

Lemma 7.2. *There exists $d_1 \in \mathbf{R}$ with the following property. If ε is small and positive then there exists $d_2 > 0$ such that the following holds for all large z and for all $k \in \mathbf{Z}$. We have*

$$\log|F(z)| < -d_2|z|^n \quad \textit{for } d_1 + \frac{2k\pi}{n} + \varepsilon < \arg z < d_1 + \frac{(2k+1)\pi}{n} - \varepsilon \tag{53}$$

and

$$\log|F(z)| > d_2|z|^n \quad \textit{for } z \notin H_1, \tag{54}$$

$$d_1 + \frac{(2k+1)\pi}{n} + \varepsilon < \arg z < d_1 + \frac{(2k+2)\pi}{n} - \varepsilon.$$ □

Lemma 7.3. *The integers m and n are such that m divides n, and we have $\rho \geq 1$.*

Proof. Let R_2 be large and positive such that the circle $S(0, R_2)$ does not meet the exceptional set H_1 of (52); the fact that such an R_2 exists follows from (51). Let Γ be the arc given by

$$|z| = R_2, \quad d_1 - \frac{2\pi}{m} + 2\varepsilon \leq \arg z \leq d_1 - \frac{2\pi}{m} + \frac{\pi}{n} - 2\varepsilon,$$

where ε is small and positive. It follows from (41) that $w = z_1 = w_1(z)$ maps the arc Γ into the region

$$d_1 + \varepsilon < \arg w < d_1 + \frac{\pi}{n} - \varepsilon,$$

on which $F(w)$ is small by (53). This gives, for $z \in \Gamma$, using (41),

$$F(z) = f(g(z)) - f(z) = f(g(z_1)) - f(z) = F(z_1) + f(z_1) - f(z) = O\left(\left(\exp(R_2^{\rho+o(1)}\right)\right).$$

In view of (54) and the fact that $S(0, R_2)$ does not meet H_1 it follows that there must exist $k \in \mathbf{Z}$ such that

$$\left[d_1 - \frac{2\pi}{m} + 2\varepsilon, d_1 - \frac{2\pi}{m} + \frac{\pi}{n} - 2\varepsilon\right] \subseteq \left[d_1 + \frac{2k\pi}{n} - \varepsilon, d_1 + \frac{(2k+1)\pi}{n} + \varepsilon\right],$$

so that

$$\left|-\frac{2\pi}{m} - \frac{2k\pi}{n}\right| \leq 3\varepsilon.$$

Since we may assume that εmn is small, this forces $km = -n$, so that m divides n and $\rho = n/m$ is an integer. □

Next, let ε be small and positive, let R_3 be large and set

$$c = d_1 + \frac{\pi}{2n}, \quad \Omega = \{z \in \mathbf{C} : |z| > R_3, \, |\arg z - c| < \varepsilon\}.$$

Then for $z \in \Omega$ we have

$$\left|\arg w_1(z) - \frac{2\pi}{m} - c\right| < 2\varepsilon$$

and so, since $|w_1(z)| \sim |z|$ and $1/m$ is an integer multiple of $1/n$, it follows from (53) that

$$\log |F(z)| < -d_2|z|^n \quad \text{and} \quad \log |F(w_1(z))| \leq -\frac{1}{2}d_2|z|^n.$$

Using the fact that (41) and (49) give

$$F(z) = f(g(z)) - f(z) = f(g(w_1(z))) - f(z) = F(w_1(z)) + f_1(z), \tag{55}$$

we therefore obtain

$$\log |f_1(z)| \leq -\frac{1}{4}d_2|z|^n$$

for $z \in \Omega$, and hence, for some $R > 0$ and $d_3 > 0$,

$$T_R(r, f_1) \geq m_R(r, 1/f_1) - O(\log r) \geq d_3 r^n$$

as $r \to \infty$. Since Lemma 6.5 gives

$$T_R(r, f_1) \leq (2 + o(1))T((1 + o(1))r, f) \leq r^{\rho + o(1)}$$

this is a contradiction, and the proof of Theorem 1.3 is complete. □

8. Proof of Theorem 1.4

Let f, g and F be as in the hypotheses. By Lemma 6.2 again, there exists a greatest integer $N \geq 0$ such that f has a representation $f = h_N \circ g^{\circ N}$ where $g^{\circ N}$ is the Nth iterate of g and h_N is meromorphic in the plane, and $F = F_N \circ g^{\circ N}$ where $F_N = h_N \circ g - h_N$. Then the order of h_N is $m^{-N}\rho$ by Lemma 6.5, and if F has finitely many zeros so has F_N. Moreover, if the equation (9) has finitely many solutions in the plane then so has the equation

$$h_N(g(z)) = h_N(z).$$

Thus in order to prove Theorem 1.4 it suffices again to consider the case where $N = 0$ and f has no representation $f = h_1[g]$ with h_1 meromorphic in the plane. As in the proof of Theorem 1.3 we may therefore assume that the function f_1 defined by (40) and (49) does not vanish identically near ∞. Since $\rho(F) < 1$ in all cases, (40), (41) and Lemma 6.4 give an ε-set E_1 such that

$$F(w_1(z)) \sim F(u_1 z) = F(e^{2\pi i/m} z) \quad \text{for all large } z \text{ with } u_1 z \notin E_1. \tag{56}$$

Suppose first that $0 < \rho < 1/m$ but F has finitely many zeros. Then by Lemma 6.5 there exists a polynomial P such that

$$G = \frac{P}{F} \tag{57}$$

is a transcendental entire function of order $\sigma = m\rho \in (0,1)$. Moreover, $f[g]$ also has order σ. Choose a small positive ε, in particular so small that

$$0 < \sigma - \varepsilon < \sigma = m\rho < \sigma + \varepsilon < \frac{1}{1+\varepsilon} < 1, \tag{58}$$

and fix a large positive constant K. Denote by c_j positive constants which are independent of ε and K.

By the standard existence theorem for Pólya peaks [12, p. 101], there exist arbitrarily large positive s_n such that

$$\begin{aligned} \frac{T(r,f[g])}{T(s_n,f[g])} &\le \left(\frac{r}{s_n}\right)^{\sigma-\varepsilon} \qquad (1 \le r \le s_n), \\ \frac{T(r,f[g])}{T(s_n,f[g])} &\le \left(\frac{r}{s_n}\right)^{\sigma+\varepsilon} \qquad (s_n \le r < \infty). \end{aligned} \tag{59}$$

Then we have, for $s_n \le r \le 8Ks_n$, by (44), (46), (49), (57) and (59),

$$\begin{aligned} T_R(r,f_1) &\le (2+o(1))T(2r,f) \le (2+o(1))T(c_3 r^{1/m}, f[g]) \\ &\le (2+o(1))T(c_3(8K)^{1/m}s_n^{1/m}, f[g]) = o(T(s_n,f[g])) = o(T(r,f[g])), \end{aligned} \tag{60}$$

where R is chosen so that f_1 is meromorphic for $|z| \ge R$. For the same r we obtain similarly

$$T(r,f) = o(T(r,f[g])), \quad T(r,G) \sim T(r,F) \sim T(r,f[g]). \tag{61}$$

Choose z_0 with

$$|z_0| = s_n, \quad \log|G(z_0)| = \log M(s_n,G) \ge T(s_n,G), \tag{62}$$

and let C be that component of the set

$$\{z \in \mathbf{C} : \log|G(z)| \ge \varepsilon T(s_n,G)\}$$

which contains z_0. For $r \ge s_n$ let $\theta(r)$ be the angular measure of the intersection $S(0,r) \cap C$. Suppose that

$$\theta(r) \le \pi(1+\varepsilon) \quad \text{for all } r \in [2s_n, 2Ks_n]. \tag{63}$$

Then (58), (59), (61), (62), (63) and a standard application of the Carleman–Tsuji estimate for harmonic measure [28, p. 116] give

$$\begin{aligned} T(s_n,G) &\le \log|G(z_0)| \\ &\le \varepsilon T(s_n,G) + c_4 \log M(4Ks_n,G) \exp\left(-\pi \int_{2s_n}^{2Ks_n} \frac{dt}{t\theta(t)}\right) \\ &\le \varepsilon T(s_n,G) + c_5 T(8Ks_n,G) K^{-1/(1+\varepsilon)} \\ &\le T(s_n,G)\left(\varepsilon + c_5(8K)^{\sigma+\varepsilon}K^{-1/(1+\varepsilon)}\right) \le \frac{1}{2}T(s_n,G) \end{aligned}$$

since ε is small and K is large.

This contradiction shows that the assumption (63) must fail, and so there exists r_n in $[2s_n, 2Ks_n]$ such that the set

$$S_n = \{z \in S(0, r_n) : \log |G(z)| \geq \varepsilon T(s_n, G)\}$$

has angular measure greater than $\pi(1 + \varepsilon)$, and so has the set

$$T_n = \{z \in S(0, r_n) : u_1 z \in S_n\}.$$

Evidently the intersection $S_n \cap T_n$ has angular measure at least $2\pi\varepsilon$ and, for $z \in S_n \cap T_n$ such that $u_1 z$ does not belong to the ε-set E_1, we have (56) and hence $F(w_1(z)) \sim F(u_1 z)$. Thus there exists a set $U_n \subseteq S_n \cap T_n$, of angular measure at least $2\pi\varepsilon - o(1)$, such that for $z \in U_n$ we have, by (57),

$$\max\{\log |F(z)|, \log |F(w_1(z))|\} \leq -\varepsilon T(s_n, G) + O(\log r_n).$$

Using (55) and the first fundamental theorem this now gives

$$T_R(r_n, f_1) + O(\log r_n) \geq m_R(r_n, 1/f_1) \geq \frac{\varepsilon^2}{2} T(s_n, G),$$

which contradicts (60) and (61). This disposes of the case where $0 < \rho < 1/m$.

Suppose next that $\rho = 0$ and $m \geq 4$ but F has finitely many zeros. Choose small positive real numbers δ and ε and a polynomial P such that (57) again defines a transcendental entire function G, this time of order 0. Then (46), [13, Lemma 4] and the $\cos \pi\rho$ theorem [14, Ch. 6] give a set $E_2 \subseteq [1, \infty)$, of positive upper logarithmic density, such that

$$T_R(r, f_1) \leq (2 + o(1))T((1 + o(1))r, f) \leq (2 + o(1))T(r, f) \quad \text{for } r \in E_2, \tag{64}$$

and

$$\log |G(z)| \geq (1 - \varepsilon/4) \log M(r, G) \geq (1 - \varepsilon/2)T(r, F) \quad \text{for } |z| = r \in E_2. \tag{65}$$

We may assume that for all $r \in E_2$ the circle $S(0, r)$ does not meet the ε-set E_1 of (56), and so (41), (45), (56) and (65) give

$$\begin{aligned} \max\{\log |F(z)|, \log |F(w_1(z))|\} &\leq -(1 - \varepsilon)T(r, F) \\ &\leq -(1 - \varepsilon)(m - 1 - \delta)T(r, f) \end{aligned}$$

for $|z| = r \in E_2$. Using (55) this yields, for $r \in E_2$,

$$T_R(r, f_1) \geq m(r, 1/f_1) - O(\log r) \geq (1 - \varepsilon)(m - 1 - \delta)T(r, f) - O(\log r),$$

which contradicts (64) since $m \geq 4$, and completes the proof in this case.

To complete the proof of Theorem 1.4 assume that $\rho = 0$, $m \geq 2$ and that the equation (9) has finitely many solutions $z \in \mathbf{C}$. Then we may assume that

$$N(r, f) \sim T(r, f), \tag{66}$$

since if this is not the case the subsequent argument may be applied with f replaced by $A \circ f$, where A is a Möbius transformation. With these assumptions we again set $F = f[g] - f$, and F has finitely many zeros. We then obtain a stronger estimate for $T(r, F)$ than (45) as follows. With τ a small positive constant we have, by (48) and (66),

$$\begin{aligned} T(r, F) \geq N(r, F) &\geq N(r, f) + N(r, f[g]) - O(\log r) \\ &\geq (1 + m - \tau)N(r, f) - O(\log r) \geq (3 - 2\tau)T(r, f) \end{aligned}$$

as $r \to \infty$. Using the same argument as in the case $\rho = 0$, $m \geq 4$, we obtain this time

$$T_R(r, f_1) \geq m(r, 1/f_1) - O(\log r) \geq (1 - \varepsilon)(3 - 2\tau)T(r, f) - O(\log r),$$

for $r \in E_2$, which again contradicts (64). □

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Received 14 January 2009