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LACUNARY SERIES AND Q_K SPACES ON THE UNIT BALL

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Abstract. In this paper, we establish a necessary condition for a kind of lacunary series on the unit ball to be in Q_K. As a consequence, we prove a necessary and sufficient condition for that Q_K coincides with the Bloch space. In the case of Q_p spaces we show that the condition, which is similar to that obtained by Hu, is also sufficient. This is a generalization of the result of Aulaskari, Xiao and Zhao for Q_p spaces on the unit disk.

1. Introduction

Let B^m denote the unit ball of $\mathbf{C}^m$, S the boundary of B^m. For $z=(z_1,\cdots,z_m)$ and $a=(a_1,\cdots,a_m)$ in B^m, let $\langle z,a\rangle = z_1\overline{a}_1+\cdots+z_m\overline{a}_m$ and $|z|=\langle z,z\rangle^{1/2}$. The group of Möbius transformations of B^m is denoted by $\mathrm{Aut}(B^m)$. For $a\in B$, let $\phi_a\in\mathrm{Aut}(B^m)$ be the Möbius transformation which satisfies $\phi_a(0)=a$ and $\phi_a^{-1}=\phi_a$.

By $H(B^m)$ we denote the collection of all holomorphic functions on B^m. For $f\in H(B^m)$ and $z=(z_1,\cdots,z_m)\in B^m$, let

$$\nabla f(z)=\left(\frac{\partial f}{\partial z_1},\frac{\partial f}{\partial z_2},\cdots,\frac{\partial f}{\partial z_m}\right)$$

denote the complex gradient of f, and

$$\mathscr{R}f(z)=\sum_{j=1}^{m} z_j\frac{\partial f}{\partial z_j}$$

denote the radial derivative of f. The invariant gradient $\widetilde{\nabla}f(z)$ of f is defined by $\widetilde{\nabla}f(z)=\nabla(f\circ\phi_z)(0)$. $\widetilde{\nabla}f(z)$ and $\nabla f(z)$ are related by ([11])

$$|\widetilde{\nabla}f(z)|^2=(1-|z|^2)(|\nabla f(z)|^2-|\mathscr{R}f(z)|^2). \tag{1.1}$$

Let ν denote the Lebsegue measure on $\mathbf{C}^m=\mathbf{R}^{2m}$, so normalized that $\nu(B^m)=1$ and σ the normalized surface measure on S so that $\sigma(S)=1$. Let

$$d\tau(z)=\frac{d\nu(z)}{(1-|z|^2)^{m+1}},$$

which is Möbius invariant.

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The Möbius invariant Green function of B^m is defined by $G(z,a) = g(|\phi_a(z)|)$, where

$$g(r) = \frac{m+1}{2m} \int_r^1 \left(1-t^2\right)^{m-1} t^{-2m+1}\, dt.$$

For $m > 1$, we have

$$C_m^{-1} \left(1-r^2\right)^m r^{-2(m-1)} \le g(r) \le C_m \left(1-r^2\right)^m r^{-2(m-1)}, \tag{1.2}$$

where C_m is a constant depending on m only.

The notion of the spaces Q_p was first considered for holomorphic functions defined on the unit disk D of the complex plane [3, 5, 6, 16, 17] and, later, generalized to hyperbolic Riemann surfaces and the unit ball of $\mathbf{C}^m$ [2, 4, 7, 12]. More general spaces Q_K was introduced and investigated on the unit disk in [9], and on the unit ball B^m of $\mathbf{C}^m$ in [18].

Let $K(t)$, $0 < t < \infty$, be a non-negative and non-decreasing function, which is not equal to 0 identically. The Banach space Q_K is defined by

$$Q_K = \left\{ f \in H(B^m) : \sup_{a \in B^m} \int_{B^m} |\widetilde{\nabla} f(z)|^2 K(G(z,a))\, d\tau(z) < \infty \right\}.$$

When $K(t) = t^p$, $0 < p < \infty$, Q_K is denoted by Q_p. For $m > 1$, it is proved in [12] that Q_p contains non-constant functions if and only if $(m-1)/m < p < m/(m-1)$. For Q_K spaces [18], this condition becomes

$$\int_0^1 \frac{r^{2m-1}}{(1-r^2)^m} K(g(r))\, dr < \infty. \tag{1.3}$$

In this paper, $K(t)$ always denotes a function, formulated as above, and satisfies (1.3).

A function $f \in H(B^m)$ is called a Bloch function if

$$||f||_{\mathscr{B}} = \sup_{z \in B^m} |\widetilde{\nabla} f(z)| < \infty.$$

It is obvious that $(1-|z|^2)|\mathscr{R} f(z)| \le (1-|z|^2)|\nabla f(z)| \le |\widetilde{\nabla} f(z)|$ for $z \in B^m$. Further, Timoney [14] proved that

$$||f||_{\mathscr{B}} \le C \sup_{z \in B^m} (1-|z|^2)|\mathscr{R} f(z)|, \tag{1.4}$$

where C is an absolute constant. The class of all Bloch functions is called the Bloch space and denoted by $\mathscr{B}$. Q_K is always a subspace of $\mathscr{B}$. In [18], it is proved that $Q_K = \mathscr{B}$ if

$$\int_0^1 \frac{r^{2m-1}}{(1-r^2)^{m+1}} K(g(r))\, dr < \infty. \tag{1.5}$$

In the case $m = 1$ (see [9]), (1.5) is a necessary and sufficient condition for $Q_K = \mathscr{B}$.

Lacunary series have been involved in the study of Q_p spaces. For a lacunary series $f(z) = \sum_0^\infty a_n z^{2^n}$ on the unit disk, $0 < p \le 1$, to be in Q_p, Aulaskari, Xiao and Zhao [6] gave a necessary and sufficient condition: $f \in Q_p$ if and only if $\sum_{n=0}^\infty 2^{n(1-p)} |a_n|^2 < \infty$. This result was extended to Q_p spaces on the unit ball by Hu [10] and, recently, to Q_K spaces on the unit disk by Wulan and Zhu [15]. The purpose of this paper is to generalize the necessary condition to Q_K spaces on the unit ball for a kind of lacunary series. As an application, we show that the condition

(1.5) for $m > 1$ is also necessary for $Q_K = \mathscr{B}$. By the way, we show that the condition is also sufficient for Q_p spaces on the unit ball. The condition for Q_p spaces is similar to that of Hu.

Throughout this paper, $c, c_1, c_2, \cdots$ denote constants which depend on m only and may have different values at different places.

2. A kind of lacunary series on the unit ball

In order to extend the result of Aulaskari and others to lacunary series on the unit ball, the key point is to construct homogeneous polynomials which play the role as z^{n_k} do in the case of the unit disk. For this purpose we need the distance d (see [1]) defined by

$$d(\zeta,\xi) = \left(1 - |\langle \zeta, \xi\rangle|^2\right)^{1/2} \quad \text{for } \ \zeta, \xi \in S.$$

The d-ball $E_\delta(\zeta)$ with radius δ and center $\zeta \in S$ is defined by

$$E_\delta(\zeta) = \{\xi \in S : d(\zeta,\xi) < \delta\}.$$

A set $\Gamma \subset S$ is said to be d-separated by $\delta > 0$, if d-balls with radius δ and centers at points of Γ are pairwise disjoint. The following lemma was proved in [8].

Lemma 1. *If $\Gamma \subset S$ is d-separated by $\delta > 0$ and n is a positive integer, then*

$$\sum_{\zeta\in\Gamma} |\langle \xi, \zeta\rangle|^n \le 1 + \sum_{k=1}^{\infty} (k+2)^{2m-2} e^{-k^2\delta^2 n/2} \quad \textit{for } \ \xi \in S.$$

Let N be a positive integer and $\Gamma_N \subset S$ be the set of all points $\zeta = (\zeta_1, \cdots, \zeta_m)$ defined by

$$\zeta_1 = \sin\frac{(N+n_1)\pi}{6N}, \quad \zeta_m = e^{il_{m-1}\pi/(2N)} \cos\frac{(N+n_1)\pi}{6N} \cdots \cos\frac{(N+n_{m-1})\pi}{6N},$$

$$\zeta_k = e^{il_{k-1}\pi/(2N)} \cos\frac{(N+n_1)\pi}{6N} \cdots \cos\frac{(N+n_{k-1})\pi}{6N} \sin\frac{(N+n_k)\pi}{6N}, \quad 2 \le k \le m-1,$$

where $n_1, l_1, \cdots, n_{m-1}, l_{m-1}$ are integers between 1 and N. Note that Γ_N contains $N^{2(m-1)}$ different points.

Lemma 2. *$d(\zeta,\xi) \ge 2^{-m}/N$, if $\zeta, \xi \in \Gamma_N$ and $\zeta \ne \xi$.*

Proof. Let ζ and ξ be two distinct points in $\Gamma_N \subset S$, which are defined by $n_1, l_1, \cdots, n_{m-1}, l_{m-1}$, and $n_1', l_1', \cdots, n_{m-1}', l_{m-1}'$, respectively. For $k = 1, \cdots, m-1$, denote

$$\alpha_k = \frac{(N+n_k)\pi}{6N}, \quad \alpha_k' = \frac{(N+n_k')\pi}{6N}, \quad \beta_k = \frac{l_k\pi}{2N}, \quad \beta_k' = \frac{l_k'\pi}{2N}.$$

We have

$$\begin{aligned} \langle \zeta, \xi\rangle &= \sin\alpha_1 \sin\alpha_1' + e^{i(\beta_{m-1}-\beta_{m-1}')} \cos\alpha_1\cos\alpha_1' \cdots \cos\alpha_{m-1}\cos\alpha_{m-1}' \\ &\quad + \sum_{k=2}^{m-1} e^{i(\beta_{k-1}-\beta_{k-1}')} \cos\alpha_1 \cos\alpha_1' \cdots \cos\alpha_{k-1}\cos\alpha_{k-1}' \sin\alpha_k \sin\alpha_k'. \end{aligned}$$

We distinguish two cases.

(i) There exists a j such that $1 \le j \le m-1$ and $n_j \ne n'_j$. Then,

$$\begin{aligned}
|\langle \zeta, \xi\rangle| &\le \sin\alpha_1 \sin\alpha'_1 + \cos\alpha_1\cos\alpha'_1 \cdots \cos\alpha_{m-1}\cos\alpha'_{m-1} \\
&\quad + \sum_{k=2}^{m-1} \cos\alpha_1\cos\alpha'_1\cdots\cos\alpha_{k-1}\cos\alpha'_{k-1}\sin\alpha_k\sin\alpha'_k \\
&= 1 - \sum_{k=1}^{m-1}\sigma_k(1-\cos(\alpha_k-\alpha'_k)) \le 1-\sigma_j(1-\cos(\alpha_j-\alpha'_j)),
\end{aligned}$$

where $\sigma_k = \cos\alpha_1\cos\alpha'_1\cdots\cos\alpha_{k-1}\cos\alpha'_{k-1}$. Note that $\sigma_1 = 1$ and $4^{-m+2} \le \sigma_j < 1$. Thus, since $\pi/(6N) \le |\alpha_j - \alpha'_j| \le \pi/6$,

$$d(\zeta,\xi) \ge (1-|\langle\zeta,\xi\rangle|)^{1/2} \ge \sqrt{2}\sigma_j^{1/2}\sin\frac{|\alpha_j-\alpha'_j|}{2} \ge \sqrt{2}\sigma_j^{1/2}\sin\frac{\pi}{12N} \ge \frac{2^{-m}}{N}.$$

(ii) $n_k = n'_k$ for $k = 1, \cdots, m-1$ and there exists a j such that $1 \le j \le m-1$ and $l_j \ne l'_j$. Let $\sigma_k = \beta_k - \beta'_k$ for $k = 1,\cdots, m-1$, and let $A_1 = \sin^2\alpha_1$, $A_k = \cos^2\alpha_1\cdots\cos^2\alpha_{k-1}\sin^2\alpha_k$ for $k = 2,\cdots,m-1$, and $A_m = \cos^2\alpha_1\cdots\cos^2\alpha_{m-1}$. Then,

$$\langle\zeta,\xi\rangle = A_1 + \sum_{k=2}^{m} e^{i\sigma_{k-1}}A_k,$$

and

$$\begin{aligned}
|\langle\zeta,\xi\rangle|^2 &= \left(A_1 + \sum_{k=2}^{m} A_k\cos\sigma_{k-1}\right)^2 + \left(\sum_{k=2}^{m} A_k \sin\sigma_{k-1}\right)^2 \\
&= \sum_{k=1}^{m} A_k^2 + 2A_1\sum_{k=2}^{m}A_k\cos\sigma_{k-1} + 2\sum_{2\le k<l\le m} A_kA_l\cos(\sigma_{k-1}-\sigma_{l-1}) \\
&\le (A_1+\cdots+A_m)^2 - 2A_1A_{j+1}(1-\cos(\beta_j-\beta'_j)).
\end{aligned}$$

Since $A_1 + \cdots + A_m = 1$, $A_1 \ge 1/4$ and $A_{j+1} \ge 1/4^{m-1}$, we have

$$|\langle\zeta,\xi\rangle|^2 \le 1 - 4^{-(m-1)}\sin^2\frac{|\beta_j-\beta'_j|}{2},$$

and

$$d(\zeta,\xi) \ge 2^{-(m-1)}\sin\frac{\pi}{4N} \ge \frac{2^{-m}}{N}.$$

The lemma is proved. □

By Lemma 2, Γ_N is d-separated by $2^{-m}/(2N)$. Thus, using Lemma 1, we have

Lemma 3. *If n is a positive integer, then*

$$\sum_{\zeta\in\Gamma_N} |\langle\xi,\zeta\rangle|^n \le 1 + \sum_{k=1}^{\infty}(k+2)^{2m-2}e^{-k^2 4^{-m-3/2}n/N^2} \quad \text{for } \xi \in S.$$

The following lemma is a direct consequence of Lemma 3.

Lemma 4. *If n is a positive integer, then*

$$\sum_{\zeta\in\Gamma_N,\ \zeta\ne\xi} |\langle\xi,\zeta\rangle|^n \le \sum_{k=1}^{\infty}(k+2)^{2m-2}e^{-k^2 4^{-m-3/2}n/N^2} \quad \text{for } \xi \in \Gamma_N.$$

It is easy to see that there exists a positive integer n_0, depending on m only, such that

$$\sum_{k=1}^{\infty}(k+2)^{2m-2}e^{-4^{-m-3/2}n_0k^2}<\frac{1}{2}. \tag{2.1}$$

For an integer $n \geq n_0$, let N_n be the largest positive integer such that $n/N_n^2 \geq n_0$. Note that

$$\frac{n^{1/2}}{2n_0^{1/2}} \leq N_n \leq \frac{n^{1/2}}{n_0^{1/2}}. \tag{2.2}$$

By Lemma 4,

$$\sum_{\xi\in\Gamma_{N_n},\ \xi\neq\zeta}|\langle\zeta,\xi\rangle|^n<\frac{1}{2}\quad \text{for}\quad \zeta\in\Gamma_{N_n},$$

and

$$\begin{aligned}\sum_{\zeta,\ \xi\in\Gamma_{N_n}}\langle\zeta,\xi\rangle^n &\geq \sum_{\zeta\in\Gamma_{N_n}}1-\sum_{\zeta\in\Gamma_{N_n}}\sum_{\xi\in\Gamma_{N_n},\ \xi\neq\zeta}|\langle\zeta,\xi\rangle|^n\\ &>\sum_{\zeta\in\Gamma_{N_n}}\frac{1}{2}=\frac{1}{2}N_n^{2(m-1)}\geq 2^{1-2m}n_0^{1-m}n^{m-1}.\end{aligned} \tag{2.3}$$

Now, for $n \geq n_0$, define

$$f_n(z)=\sum_{\zeta\in\Gamma_{N_n}}\langle z,\zeta\rangle^n. \tag{2.4}$$

There is an estimate for $|f_n(z)|$. In fact, by Lemma 3 and the definition of N_n,

$$\begin{aligned}|f_n(z)| &\leq |z|^n\sum_{\zeta\in\Gamma_{N_n}}|\langle z/|z|,\zeta\rangle|^n\leq|z|^n\left(1+\sum_{k=1}^{\infty}(k+2)^{2m-2}e^{-k^24^{-m-3/2}n/N_n^2}\right)\\ &\leq|z|^n\left(1+\sum_{k=1}^{\infty}(k+2)^{2m-2}e^{-k^24^{-m-3/2}n_0}\right)=c|z|^n.\end{aligned} \tag{2.5}$$

Let $\Lambda_n \subset S$ for $n = n_0, n_0+1, \cdots$. The sequence of homogeneous polynomials

$$f_n(z)=\sum_{\zeta\in\Lambda_n}\langle z,\zeta\rangle^n$$

is called a normal sequence if it possesses the following property: there exists a positive constant C such that

(i) $|f_n(z)| \leq C|z|^n$ for $z \in B^m$ and $n = n_0, n_0+1, \cdots$,
(ii) $\sum_{\zeta,\ \xi\in\Lambda_n}\langle\zeta,\xi\rangle^n \geq C^{-1}n^{m-1}$ for $n = n_0, n_0+1, \cdots$.

Because of (2.3) and (2.5), the sequence $f_n(z)$ defined by (2.4) is a normal sequence. In what following, we will consider all lacunary series defined by normal sequences of homogeneous polynomials.

3. A necessary condition for a lacunary series to be in Q_K spaces

In this section we prove a necessary condition for a lacunary series defined by a normal sequence to belong to a Q_K space on the unit ball.

Theorem 1. *Let $f_n(z)$, $n = n_0, n_0+1, \cdots$, be a normal sequence and*

$$f(z) = \sum_{k=1}^{\infty} a_k f_{n_k}(z),$$

where $n_0 \le n_1 < n_2 < \cdots < n_k < \cdots$ is a sequence of positive integers such that $\liminf\limits_{k\to\infty} n_{k+1}/n_k > 1$. If $f \in Q_K$, then

$$\sum_{k=1}^{\infty} n_k^m K(n_k^{-m}) |a_k|^2 < \infty. \tag{3.1}$$

Proof. Let $f \in Q_K$. We have, by (1.1),

$$\left|\widetilde{\nabla} f(z)\right|^2 = (1-|z|^2)\left(\left|\sum_{k=1}^{\infty} n_k a_k \sum_{\xi\in\Lambda_{n_k}} \langle z,\xi\rangle^{n_k-1}\overline{\xi}\right|^2 - \left|\sum_{k=1}^{\infty} n_k a_k \sum_{\xi\in\Lambda_{n_k}} \langle z,\xi\rangle^{n_k}\right|^2\right),$$

and

$$\int_{B^m} |\widetilde{\nabla} f(z)|^2 |K(g(|z|))\, d\tau(z) = 2m\int_0^1 \frac{r^{2m-1}}{(1-r^2)^m}\left(\int_S \left(|A|^2 - |B|^2\right) d\sigma(\zeta)\right) K(g(r))\, dr,$$

where

$$A = \sum_{k=1}^{\infty} n_k a_k r^{n_k-1} \sum_{\xi\in\Lambda_{n_k}} \langle \zeta,\xi\rangle^{n_k-1}\overline{\xi}, \quad B = \sum_{k=1}^{\infty} n_k a_k r^{n_k} \sum_{\xi\in\Lambda_{n_k}} \langle \zeta,\xi\rangle^{n_k}.$$

If $k \ne k'$, integrating by slices gives

$$\int_S \sum_{\xi\in\Lambda_{n_k}} \langle \zeta,\xi\rangle^{n_k} \sum_{\xi\in\Lambda_{n_{k'}}} \overline{\langle \zeta,\xi\rangle}^{n_{k'}} d\sigma(\zeta) = 0$$

and

$$\int_S \sum_{\xi\in\Lambda_{n_k}} \langle \zeta,\xi\rangle^{n_k-1} \sum_{\xi\in\Lambda_{n_{k'}}} \overline{\langle \zeta,\xi\rangle}^{n_{k'}-1} d\sigma(\zeta) = 0.$$

Thus,

$$\begin{aligned}\int_{B^m} |\widetilde{\nabla} f(z)|^2 |K(g(|z|))\, d\tau(z) &= \sum_{k=1}^{\infty} 2m n_k^2 |a_k|^2 \int_0^1 \frac{r^{2n_k+2m-3}}{(1-r^2)^m} \\ &\quad \cdot \left(\int_S A_k(\zeta) d\sigma(\zeta) - r^2 \int_S B_k(\zeta)\, d\sigma(\zeta)\right) K(g(r)) dr,\end{aligned} \tag{3.2}$$

where

$$A_k(\zeta) = \sum_{\xi,\xi'\in\Lambda_{n_k}} \langle \zeta,\xi\rangle^{n_k-1}\overline{\langle \zeta,\xi'\rangle}^{n_k-1}\langle \overline{\xi},\overline{\xi'}\rangle, \quad B_k(\zeta) = \sum_{\xi,\xi'\in\Lambda_{n_k}} \langle \zeta,\xi\rangle^{n_k}\overline{\langle \zeta,\xi'\rangle}^{n_k}. \tag{3.3}$$

For $\xi, \xi' \in \Lambda_{n_k}$, let T be a unitary transformation such that $T\xi = (1, 0, \cdots, 0)$. Denote $\xi'' = T\xi' = (\xi''_1, \cdots, \xi''_m)$. Then, $\xi''_1 = \langle T\xi', T\xi\rangle = \langle \xi', \xi\rangle = \overline{\langle \xi, \xi'\rangle}$, and

$$\langle \overline{\xi}, \overline{\xi'}\rangle \int_S \langle \zeta, \xi\rangle^{n_k-1}\overline{\langle \zeta, \xi'\rangle}^{n_k-1}\, d\sigma(\zeta) = \langle \overline{\xi}, \overline{\xi'}\rangle \int_S \langle \zeta, T\xi\rangle^{n_k-1}\overline{\langle \zeta, T\xi'\rangle}^{n_k-1}\, d\sigma(\zeta)$$
$$= \langle \overline{\xi}, \overline{\xi'}\rangle \int_S \zeta_1^{n_k-1}\left(\overline{\zeta_1}\overline{\langle \xi, \xi'\rangle} + \overline{\zeta_2}\xi''_2 + \cdots + \overline{\zeta_m}\xi''_m\right)^{n_k-1} d\sigma(\zeta).$$

Integrating by slices, we obtain

$$\begin{aligned}\langle \overline{\xi}, \overline{\xi'}\rangle \int_S \langle \zeta, \xi\rangle^{n_k-1}\overline{\langle \zeta, \xi'\rangle}^{n_k-1}\, d\sigma(\zeta) &= \langle \overline{\xi}, \overline{\xi'}\rangle^{n_k} \int_S |\zeta_1|^{2(n_k-1)}\, d\sigma(\zeta) \\ &= \langle \overline{\xi}, \overline{\xi'}\rangle^{n_k} \cdot \frac{(m-1)!(n_k-1)!}{(m-1+n_k-1)!},\end{aligned} \tag{3.4}$$

where the known formula [13]

$$\int_S |\zeta_1^{\beta_1}\cdots\zeta_m^{\beta_m}|^2\, d\sigma(\zeta) = \frac{(m-1)!\beta_1!\cdots\beta_m!}{(m-1+\beta_1+\cdots+\beta_m)!}$$

is used. By the same reason,

$$\int_S \langle \zeta, \xi\rangle^{n_k}\overline{\langle \zeta, \xi'\rangle}^{n_k} d\sigma(\zeta) = \langle \overline{\xi}, \overline{\xi'}\rangle^{n_k} \cdot \frac{(m-1)!n_k!}{(m-1+n_k)!}. \tag{3.5}$$

Thus, by condition (ii) of the normal sequence, (3.3), (3.4) and (3.5),

$$\begin{aligned}&\int_S A_k(\zeta)\, d\sigma(\zeta) - r^2\int_S B_k(\zeta)\, d\sigma(\zeta) \\ &= \left(\frac{(m-1)!(n_k-1)!}{(m-1+n_k-1)!} - \frac{r^2(m-1)!n_k!}{(m-1+n_k)!}\right) \sum_{\xi,\ \xi'\in\Lambda_{n_k}} \langle \overline{\xi}, \overline{\xi'}\rangle^{n_k} \\ &\geq C^{-1}n_k^{m-1} \cdot \frac{(m-1)!(n_k-1)!\,(m-1+n_k(1-r^2))}{(m-1+n_k)!} \\ &\geq \frac{C^{-1}(m-1)!n_k^{m-2}n_k!}{(m-1+n_k)!}.\end{aligned} \tag{3.6}$$

Now, it follows from (3.2) and (3.6) that

$$\int_{B^m} |\widetilde{\nabla} f(z)|^2 |K(g(|z|))\, d\tau(z) \geq c^{-1}C^{-1}\sum_{k=1}^{\infty} |a_k|^2 n_k \int_0^1 \frac{r^{2n_k+2m-3}}{(1-r^2)^m} \cdot K(g(r))\, dr, \tag{3.7}$$

where c is a positive number depending on m only. By (1.2),

$$K(g(r)) \geq K(c_2^{-1}(1-r)^m) \quad \text{for} \quad 1/2 \leq r \leq 1.$$

Consequently,

$$\begin{aligned}&\int_0^1 \frac{r^{2n_k+2m-3}}{(1-r^2)^m} \cdot K(g(r))\, dr \geq c_1^{-1}\int_{1/2}^1 \frac{r^{2n_k-1}}{(1-r)^m} \cdot K\left(c_2^{-1}(1-r)^m\right) dr \\ &\geq c_1^{-1}\int_0^{\log 2} t^{-m}e^{-2n_k t}K(c_3^{-1}t^m)\, dt \geq c_1^{-1}K(n_k^{-m})\int_{c_3^{1/m}n_k^{-1}}^{\log 2} t^{-m}e^{-2n_k t}\, dt \\ &= c_1^{-1}n_k^{m-1}K(n_k^{-m})\int_{c_3^{1/m}}^{n_k \log 2} t^{-m}e^{-2t}\, dt.\end{aligned}$$

Let k' be sufficiently large such that $n_{k'} \log 2 \geq c_3^{1/m} + 1$. Then, for $k \geq k'$,

$$\int_0^1 \frac{r^{2n_k+2m-3}}{(1-r^2)^m} \cdot K(g(r))\, dr \geq C^{-1} c^{-1} n_k^{m-1} K(n_k^{-m}),$$

and, by (3.7),

$$\infty > \int_{B^m} |\widetilde{\nabla} f(z)|^2 |K(g(|z|))\, d\tau(z) \geq C^{-1} c^{-1} \sum_{k=k'}^{\infty} n_k^m K(n_k^{-m}) |a_k|^2.$$

This shows (3.1) and the theorem is proved. □

It can be seen from the proof that the condition (i) for the normal sequence $f_n(z)$ and the lacunary condition for the sequence n_k are not necessary for Theorem 1. As an implication of Theorem 1, we prove that (1.5) is also necessary for $Q_K = \mathscr{B}$ on the unit ball B^m.

Theorem 2. *If $Q_K = \mathscr{B}$ on the unit ball B^m, then (1.5) holds.*

Proof. Assume that $Q_K = \mathscr{B}$. Among lacunary series defined by normal sequences, we consider

$$f(z) = \sum_{k=k_0}^{\infty} f_{2^k}(z),$$

where $f_{2^k}(z)$ are constructed by (2.4)and $2^{k_0} \geq n_0$. By (2.5), $|f_{2^k}(z)| \leq c|z|^{2^k}$ for $k \geq k_0$ and $z \in B^m$. Thus,

$$\begin{aligned}(1-|z|^2)|\mathscr{R} f(z)| &\leq (1-|z|^2) \sum_{k=k_0}^{\infty} |\mathscr{R} f_{2^k}(z)| \leq (1-|z|^2) \sum_{k=k_0}^{\infty} 2^k |f_{2^k}(z)| \\ &\leq c(1-|z|^2) \sum_{k=k_0}^{\infty} 2^k |z|^{2^k} \leq 4c(1-|z|) \sum_{n=1}^{\infty} |z|^n \leq 4c.\end{aligned}$$

By (1.4), this shows that $f \in \mathscr{B}$ and, consequently, $f \in Q_K$ since $Q_K = \mathscr{B}$. Using Theorem 1 gives

$$\sum_{k=1}^{\infty} 2^{mk} K(2^{-mk}) < \infty.$$

By (1.2), we have

$$\int_{1/2}^1 \frac{r^{2m-1}}{(1-r^2)^{m+1}} K(g(r))\, dr \leq \int_{1/2}^1 \frac{K\left(c(1-r)^m\right)}{(1-r)^{m+1}}\, dr \leq c_1 \int_0^{c^{1/m} \log 2} t^{-(m+1)} K(t^m)\, dt.$$

On the other hand,

$$\begin{aligned}\int_0^{1/2} t^{-(m+1)} K(t^m)\, dt &= \sum_{k=1}^{\infty} \int_{2^{-(k+1)}}^{2^{-k}} t^{-(m+1)} K(t^m)\, dt \\ &\leq \sum_{k=1}^{\infty} 2^{-(k+1)} 2^{(m+1)(k+1)} K(2^{-mk}) \\ &= 2^m \sum_{k=1}^{\infty} 2^{mk} K(2^{-mk}),\end{aligned}$$

since K is non-decreasing. Thus,

$$\int_{1/2}^{1} \frac{r^{2m-1}}{(1-r^2)^{m+1}} K(g(r))\, dr < \infty.$$

Combining this with (1.3), we obtain (1.5). The theorem is proved. $\square$

4. The necessary and sufficient condition for a lacunary series to be in Q_p spaces

The condition (3.1) in Theorem 1 is not sufficient if one does not put any extra restriction on K. We don't know what is a better restriction. Now, we can prove the sufficiency only for Q_p spaces, $(m-1)/m < p \le 1$.

In [10], the following equivalent characterization for Q_p spaces on the unit ball B^m with $m > 1$ was proved.

Lemma 5. *Let $(m-1)/m < p \le 1$. Then, for $f \in H(B^m)$, $f \in Q_p$ if and only if*

$$\sup_{a \in B^m} \int_{B^m} (1-|z|^2)^2 |\mathscr{R} f(z)|^2 (1-|\phi_a(z)|^2)^{mp}\, d\tau(z) < \infty. \tag{4.1}$$

The following lemma can be found in [13].

Lemma 6. *If $\lambda > 0$ and $z \in B^m$, then*

$$\int_S \frac{d\sigma(\zeta)}{|1-\langle \zeta, z\rangle|^{m+\lambda}} \le \frac{c}{(1-|z|^2)^{\lambda}}. \tag{4.2}$$

Theorem 3. *Let $(m-1)/m < p \le 1$ and f be defined as in Theorem 1. Then, $f \in Q_p$ if and only if*

$$\sum_{k=1}^{\infty} |a_k|^2 n_k^{m(1-p)} < \infty. \tag{4.3}$$

Proof. The necessity of the condition (4.3) follows from Theorem 1. Now, assume that (4.3) holds. By condition (i) of the normal sequence, for $z \in B^m$,

$$|\mathscr{R} f(z)| \le \sum_{k=1}^{\infty} |a_k| n_k |f_{n_k}(z)| \le C \sum_{k=1}^{\infty} |a_k| n_k |z|^{n_k}.$$

Let $0 < \eta < 1$. For $0 < r < 1$, using Schwarz's inequality gives

$$\left(\sum_{k=1}^{\infty} |a_k| n_k r^{n_k}\right)^2 \le \left(\sum_{k=1}^{\infty} n_k^{\eta} r^{n_k}\right)\left(\sum_{k=1}^{\infty} n_k^{2-\eta} |a_k|^2 r^{n_k}\right).$$

There exists an $M > 1$ such that $n_k/(n_k - n_{k-1}) \le M$ for $k = 1, 2, \cdots$, since $\liminf_{k\to\infty} n_{k+1}/n_k > 1$. Let $\mu_0 = 0$ and $n_k |\log r| = \mu_k$ for $k = 1, 2, \cdots$. Then,

$$\begin{aligned}
\sum_{k=1}^{\infty} n_k^{\eta} r^{n_k} &= |\log r|^{-\eta} \sum_{k=1}^{\infty} \mu_k^{\eta} e^{-\mu_k} \\
&= |\log r|^{-\eta} \sum_{k=1}^{\infty} \frac{n_k}{n_k - n_{k-1}} (\mu_k - \mu_{k-1}) \mu_k^{\eta-1} e^{-\mu_k} \\
&\le M |\log r|^{-\eta} \Gamma(\eta).
\end{aligned}$$

By (1.2), the concavity of t^p and (4.2), for $a \in B^m$, we have

$$\begin{aligned}\int_S \left(1-|\phi_a(r\zeta)|^2\right)^{mp} d\sigma(\zeta) &= (1-|a|^2)^{mp}(1-r^2)^{mp}\int_S \frac{d\sigma(\zeta)}{|1-\langle ra,\zeta\rangle|^{2mp}}\\ &\le (1-|a|^2)^{mp}(1-r^2)^{mp}\left(\int_S \frac{d\sigma(\zeta)}{|1-\langle ra,\zeta\rangle|^{2m}}\right)^p\\ &\le c\cdot\frac{(1-|a|^2)^{mp}(1-r^2)^{mp}}{(1-|a|^2r^2)^{mp}} \le c(1-r^2)^{mp}.\end{aligned}$$

Combining the above estimates, we obtain

$$\begin{aligned}I(a) &= \int_{B^m}(1-|z|^2)^2|\mathscr{R}f(z)|^2(1-|\phi_a(z)|^2)^{mp}\,d\tau(z)\\ &= \int_{B^m}(1-|z|^2)^{-m+1}|\mathscr{R}f(z)|^2(1-|\phi_a(z)|^2)^{mp}\,d\nu(z)\\ &\le c_1M\Gamma(\eta)\int_0^1\frac{r^{2m-1}|\log r|^{-\eta}}{(1-r^2)^{m-1}}\sum_{k=1}^\infty|a_k|^2n_k^{2-\eta}r^{n_k}\,dr\int_S\left(1-|\phi_a(r\zeta)|^2\right)^{mp}d\sigma(\zeta)\\ &\le c_2M\Gamma(\eta)\sum_{k=1}^\infty|a_k|^2n_k^{2-\eta}\int_0^1 r^{2m-1+n_k}|\log r|^{-\eta}(1-r^2)^{1-m(1-p)}\,dr.\end{aligned}$$

It is easy to calculate that

$$\begin{aligned}\int_0^1 r^{2m-1+n_k}|\log r|^{-\eta}(1-r^2)^{1-m(1-p)}\,dr &\le 2\int_0^1 r^{n_k}|\log r|^{-\eta}(1-r)^{1-m(1-p)}\,dr\\ &= 2\int_0^\infty t^{-\eta}(1-e^{-t})^{1-m(1-p)}e^{-n_kt}\,dt\\ &\le 2\int_0^\infty t^{1-\eta-m(1-p)}e^{-n_kt}\,dt\\ &= 2n_k^{\eta+m(1-p)-2}\Gamma(2-\eta-m(1-p)).\end{aligned}$$

Thus,

$$I(a) \le cM\Gamma(\eta)\Gamma(2-\eta-m(1-p))\sum_{k=1}^\infty|a_k|^2n_k^{m(1-p)}$$

holds for $a \in B^m$. Because of (4.3), the right side of the above inequality is a finite number. Using Lemma 5, we see that $f \in Q_p$. The theorem is proved. □

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