Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 35, 2010, 179–195

LAURENT SEPARATION, THE WIENER ALGEBRA AND RANDOM WALKS

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Abstract. Let φ, f_0 belong to the algebra $\mathscr{W}$ of absolutely convergent complex Fourier series on $\mathbf{T} = \{|z| = 1\}$. We define $f_n \in \mathscr{W}$ by

$$(*) \qquad f_1(z) = \varphi(z) f_0(z) \quad \text{and} \quad f_{n+1}(z) = \varphi(z) f_n(z)^+ \quad \text{for } n \in \mathbf{N},$$

where $(\dots)^+$ denotes the analytic part of the Laurent series. We derive a number of generating functions all of which contain

$$p(z, w) = \exp\Big(\big[\log\big(1 - w\varphi(z)\big)\big]^-\Big) \quad (|z| \ge 1, |w| < 1).$$

The Laurent separation is a discrete equivalent to the Wiener–Hopf factorization of probability theory and allows us to obtain rather concrete results.

The recursion $(*)$ comes from the study of the random walk on $\mathbf{Z}$ defined by

$$S_{n+1} = S_0 + X_1 + \dots + X_n,$$

where S_0 is a random variable with generating function f_0 specifying the initial distribution, the X_ν are i.i.d. with generating function φ and the random walk stops if it hits $(-\infty, -1]$, which is a version of the ruin problem. We also consider the technical problems which arise if X is replaced by $-X$. The results will also be applied to the minimum problem for random walks.

1. Introduction

1.1. Let X be a random variable with values in $\mathbf{Z}$ and generating function

$$(1.1) \qquad \varphi(z) = \sum_{k \in \mathbf{Z}} a_k z^k, \quad a_k = \mathbf{P}(X = k)$$

and X_n $(n \in \mathbf{N})$ independent random variables that are distributed like X. Let S_0 be another random variable with values in $\mathbf{Z}$ that is independent of the X_n and has the generating function

$$(1.2) \qquad f_0(z) = \sum_{k \in \mathbf{Z}} b_{0,k} z^k, \quad b_{0,k} = \mathbf{P}(S_0 = k).$$

We do not make any assumptions about expectations or other moments. The random variables

$$(1.3) \qquad S_n := S_0 + X_1 + \dots + X_n \quad (n \in \mathbf{N})$$

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doi:10.5186/aasfm.2010.3510

2000 Mathematics Subject Classification: Primary 30B10, 30H05, 60G50.

Key words: Generating functions, Laurent separation, Wiener algebra, Wiener–Hopf, random walks, ruin, minimum.

*Supported in part by the project MTM 2006-14449-C02-01 of the Spanish Government and the European Union.

have the generating functions $F_n(z) := f_0(z)\varphi(z)^n$, $(n \in \mathbf{N})$, and we have the recursion $F_1 = \varphi f_0$, $F_{n+1} = \varphi F_n$ for $n \in \mathbf{N}$.

The stochastic process $(S_n)_{n\geq 0}$ defines a random walk in $\mathbf{Z}$, and the coefficient of z^k in $F_n(z)$ is the probability $\mathbf{P}(S_n = k)$ of being at k at time n. Note that $\mathbf{P}$ depends on S_0 and X. The *ruin problem* deals with the random variable R defined by

$$R = n \iff S_\nu \geq 0 \; (\nu > 0, \nu < n),\; S_n < 0 \quad (n \in \mathbf{N}), \tag{1.4a}$$

$$R = \infty \iff S_\nu \geq 0 \text{ for all } \nu \in \mathbf{N}. \tag{1.4b}$$

Then $\mathbf{P}(S_n = k, R \geq n)$ is the probability of being at k at time n under the restriction that $(-\infty, -1]$ was not hit before, except possibly in the initial state. It is easily seen that this is the coefficient of z^k in $f_n(z)$ where the f_n are determined recursively by

$$f_1 = \varphi f_0, \tag{1.5a}$$

$$f_{n+1} = \varphi f_n^+ \quad \text{for } n \in \mathbf{N}, \tag{1.5b}$$

and f_n^+ is the result of discarding the terms with negative exponents from the Laurent series f_n, see (1.8) below.

More generally, one could remove all terms with exponents $\geq d$ by means of the operation $f(z) \to z^{-d}\big(f(z)z^{-d}\big)^+$ instead of $f(z) \to f(z)^+$. We will only have to deal with $d = 1$ and use the hat sign, such as in $\hat{f}_n$, for quantities related to this case. It arises if the ruin problem is considered on the basis of the modified random variable

$$\hat{R} = n \iff S_\nu > 0 \; (\nu > 0, \nu < n)\,,\; S_n \leq 0 \quad (n \in \mathbf{N}), \tag{1.6a}$$

$$\hat{R} = \infty \iff S_\nu > 0 \text{ for all } \nu \in \mathbf{N}. \tag{1.6b}$$

The generating functions $\hat{f}_n(z)$ of the probabilities $\mathbf{P}(S_n = k, \hat{R} \geq n)$ of being at k at time n under the restriction that $(-\infty, 0]$ was not hit before, except possibly in the initial state, satisfy (1.5) with $f_n(z) = \hat{f}_n(z)/z$.

The anomalous form of the first step in (1.5) could be avoided by starting the recursion after the first step with the initial function f_1 and substituting $f_1 = \varphi f_0$ in the results. This, however, would be less transparent, for in important applications the initial distribution is deterministic whereas even the state after the first step depends on φ and can be complicated.

1.2. The question when and where a random walk on $\mathbf{Z}$ first hits a half-line like $(-\infty, 0]$ or $(-\infty, 0)$ is extensively discussed in [Spi76, Chap. IV]. Our Section 2 can be considered as a streamlined version of these probabilistic results put into a general complex-analytic context. We define a sequence of functions f_n by the recursion (1.5) which is considered for complex coefficients. For our purposes, the natural function space for the generating functions is the *Wiener algebra* $\mathscr{W}$ of absolutely convergent complex Fourier series on $\mathbf{T} = \{|z| = 1\}$, that is

$$h(z) = \sum_{k\in\mathbf{Z}} c_k z^k, \quad \|h\| = \sum_{k\in\mathbf{Z}} |c_k| < \infty. \tag{1.7}$$

See e.g. [Kat04] and [CC74] for information about the Wiener algebra. We shall use the beautiful *Wiener–Levy theorem* [Zyg68, p. 245][Kat04, p. 247]: Let ψ be analytic in a domain $U \subset \mathbf{C}$. If $h \in \mathscr{W}$ and $h(\mathbf{T}) \subset U$ then $\psi \circ h \in \mathscr{W}$.

Our main tool is the Laurent separation $h = h^+ + h^-$ defined by

$$h^+(z) = \sum_{k\geq 0} c_k z^k, \quad h^-(z) = \sum_{k<0} c_k z^k. \tag{1.8}$$

The subsets of all functions in $\mathscr{W}$ of these forms are subalgebras $\mathscr{W}^\pm$. They are projections of $\mathscr{W}$ onto $\mathscr{W}^\pm$. If convenient we write $h(z)^\pm$ instead of $h^\pm$, and when these operators are applied to functions of several variables then, by convention, they refer to the variable z. Let $\mathscr{W}^\pm$ denote the subalgebras of functions of this form. The functions in $\mathscr{W}^+$ are analytic in $\mathbf{D} = \{|z| < 1\}$ and continuous in $\overline{\mathbf{D}} = \{|z| \leq 1\}$ whereas the functions in $\mathscr{W}^-$ are analytic in $\{|z| > 1\}$ and continuous in $\{|z| \geq 1\}$ with $h^-(\infty) = 0$. The results hold for the most general $\varphi \in \mathscr{W}$ with $|\varphi(z)| \leq 1$ on $\mathbf{T}$.

In Section 3 the coefficients of f_0^+ and f_0^- in (2.24) and (2.25) are investigated more closely. In particular, this section contains recursion formulas which can be used for numerical computations. In Theorem 3.2 we give a structural characterization of the function

$$p(z,w) = \exp\Big(\big[\log\big(1 - w\varphi(z)\big)\big]^-\Big) \quad (|z| \geq 1, |w| < 1).$$

Moreover, in Theorem 3.3 a result is obtained by function-theoretic methods which connects the results of this paper with the special case that was considered in detail in [JP07].

In Section 4 these results are applied to probability theory. We consider two versions of the ruin problems and the minimum problem. We are interested in the size of the minimum and the time when it is first attained in a finite section of the random walk. Contrary to [JP07], the situation is now symmetric with respect to the sign of X, so the results can be applied to more general ruin problems, as presented, e.g., in the book of Asmussen [Asm00].

2. The function-theoretic problem

2.1. Throughout the paper we assume that φ is a fixed function in the Wiener algebra $\mathscr{W}$ that is bounded by 1 on $\mathbf{T}$. We always write

$$\varphi(z) = \sum_{k\in\mathbf{Z}} a_k z^k \quad (z \in \mathbf{T}) \tag{2.1}$$

with $a_k \in \mathbf{C}$. Thus we assume

$$\|\varphi\| = \sum_{k\in\mathbf{Z}} |a_k| < \infty, \tag{2.2}$$

$$|\varphi(z)| \leq 1 \quad \text{for } |z| = 1. \tag{2.3}$$

Let $f_0 \in \mathscr{W}$ be given. We recursively define $f_1, f_2, \ldots$ by (1.5). We see that

$$\|f_1\| \leq \|\varphi\|\|f_0\| \quad \text{and} \quad \|f_{n+1}\| \leq \|\varphi\|\|f_n^+\| \leq \|\varphi\|\|f_n\| \text{ for } n \geq 1.$$

It follows that

$$\|f_n\| \leq \|f_0\|\|\varphi\|^n < \infty$$

and therefore $f_n \in \mathscr{W}$.

The generating function of (f_n) is defined by

$$g(z,w) = \sum_{n=0}^{\infty} f_n(z) w^n \quad (z \in \mathbf{T}). \tag{2.4}$$

This series converges for $|w| < 1/\|\varphi\|$ and we shall show later (Theorem 2.1) that it actually converges for $|w| < 1$. The Laurent separation is

$$g^{\pm}(z,w) = \sum_{n=0}^{\infty} f_n^{\pm}(z) w^n \quad \text{for } |z| \le 1 \text{ or } |z| \ge 1. \tag{2.5}$$

It follows from (2.5) and (1.5) that

$$\begin{aligned} g^{+}(z,w) + g^{-}(z,w) - f_0(z) &= \sum_{n=0}^{\infty} f_{n+1}(z) w^{n+1} = w\varphi(z)\Big(f_0^{-}(z) + \sum_{n=0}^{\infty} f_n^{+}(z) w^n\Big) \\ &= w\varphi(z)\big(f_0^{-}(z) + g^{+}(z,w)\big). \end{aligned}$$

This implies the Wiener–Hopf type *functional equation*

$$\big(1 - w\varphi(z)\big) g^{+}(z,w) + g^{-}(z,w) = f_0^{+}(z) + \big(1 + w\varphi(z)\big) f_0^{-}(z) \quad (z \in \mathbf{T}). \tag{2.6}$$

2.2. We write $\varphi(z)^n = \sum_{k\in\mathbf{Z}} a_{n,k} z^k$. Then

$$\log\big(1 - w\varphi(z)\big) = -\sum_{n=1}^{\infty} \varphi(z)^n \frac{w^n}{n} = -\sum_{k\in\mathbf{Z}} \Big(\sum_{n=1}^{\infty} a_{n,k} \frac{w^n}{n}\Big) z^k. \tag{2.7}$$

This function is continuous in $z \in \mathbf{T}$ and analytic in $w \in \mathbf{D}$ because of (2.3). Hence the same is true for the function

$$p(z,w) := \exp\Big(-\sum_{k<0} \Big(\sum_{n=1}^{\infty} a_{n,k} \frac{w^n}{n}\Big) z^k \Big) \tag{2.8}$$

which satisfies

$$\frac{1 - w\varphi(z)}{p(z,w)} = \exp\Big(-\sum_{k\ge 0} \Big(\sum_{n=1}^{\infty} a_{n,k} \frac{w^n}{n}\Big) z^k \Big). \tag{2.9}$$

This is related to the Wiener factorization theorem [CC74, p. 494], see also [Spi76, p. 180]. With

$$r(w) := \frac{1 - w\varphi(z)}{p(z,w)}\bigg|_{z=0} = \exp\Big(-\sum_{n=1}^{\infty} a_{n,0} \frac{w^n}{n}\Big) \tag{2.10}$$

we have, in view of (2.8) and (2.9),

$$p(z,w) = \sum_{k\le 0} p_k(w) z^k, \qquad \frac{1}{p(z,w)} = \sum_{k\le 0} q_k(w) z^k \quad \text{for } |z| \ge 1, \tag{2.11}$$

$$\frac{1 - w\varphi(z)}{r(w)p(z,w)} = \sum_{k\ge 0} p_k(w) z^k, \quad \frac{r(w)p(z,w)}{1 - w\varphi(z)} = \sum_{k\ge 0} q_k(w) z^k \quad \text{for } |z| \le 1, \tag{2.12}$$

with coefficients

$$p_k(w) = \sum_{n=0}^{\infty} p_{n,k} w^n, \quad q_k(w) = \sum_{n=0}^{\infty} q_{n,k} w^n \quad (k \in \mathbf{Z}) \tag{2.13}$$

which are analytic in $|w| < 1$ and satisfy

$$p_0(w) = q_0(w) = 1 \quad \text{for } w \in \mathbf{D} \tag{2.14}$$

and

$$p_k(0) = q_k(0) = 0 \quad \text{for } k \neq 0. \tag{2.15}$$

2.3. Let $\tilde{\varphi}(z) = \varphi(1/z)$. The quantities derived from $\tilde{\varphi}$ are labeled by a tilde, like $\tilde{p}$ or $\tilde{p}_k$. In the probabilistic interpretation, $\tilde{\varphi}$ is the generating function for the transition probabilities of the reversed random walk, see [Spi76, p. 111]. From $\tilde{\varphi}^n(z) = \tilde{\varphi}(z)^n = \varphi(1/z)^n = \varphi^n(1/z) = \widetilde{\varphi^n}(z)$ follows that

$$\tilde{a}_{n,k} = a_{n,-k}. \tag{2.16}$$

In particular, $\tilde{a}_{n,0} = a_{n,0}$, so (2.10) shows that

$$\tilde{r}(w) = r(w). \tag{2.17}$$

We obtain from (2.9) and (2.10) that

$$\tilde{p}(z, w) = \frac{1 - w\varphi(1/z)}{r(w)p(1/z, w)}. \tag{2.18}$$

From (2.12), (2.17), (2.18) and (2.11) follows that

$$\sum_{k\geq 0} \tilde{p}_k(w) z^k = \frac{1 - w\tilde{\varphi}(z)}{r(w)\tilde{p}(z, w)} = p(1/z, w) = \sum_{k\leq 0} p_k(w) z^{-k} = \sum_{k\geq 0} p_{-k}(w) z^k,$$

hence $\tilde{p}_k(w) = p_{-k}(w)$ for $k \geq 0$. Because of $\tilde{\tilde{p}}_k = p_k$ this holds for all k, hence

$$\tilde{p}_k(w) = p_{-k}(w) \quad (k \in \mathbf{Z}), \tag{2.19}$$

Similarly it follows from (2.12), (2.17), (2.18) and (2.11) that

$$\tilde{q}_k(w) = q_{-k}(w) \quad (k \in \mathbf{Z}). \tag{2.20}$$

2.4. We derive explicit expressions for the generating function g as well as for g^+ and g^- in terms of f_0, φ and p.

Theorem 2.1. *Let $w \in \mathbf{D}$. Then $p(\cdot, w)$, $1/p(\cdot, w)$ and $g(\cdot, w)$ belong to $\mathscr{W}$ and we have*

$$g^+(z, w) = \frac{p(z, w)}{1 - w\varphi(z)} \left(\frac{f_0^+(z) + w\varphi(z) f_0^-(z)}{p(z, w)} \right)^+, \tag{2.21}$$

$$g^-(z, w) = f_0^-(z) + p(z, w) \left(\frac{f_0^+(z) + w\varphi(z) f_0^-(z)}{p(z, w)} \right)^-. \tag{2.22}$$

Proof. The function $\psi(s) = \log(1 - ws)$ is analytic in $\{|s| < 1/|w|\}$. Since $\varphi \in \mathscr{W}$ by (2.2) and furthermore $|\varphi(z)| \leq 1$ for $z \in \mathbf{T}$ by (2.3), we conclude from the Wiener-Levy theorem that $\log(1 - w\varphi) \in \mathscr{W}$. Hence $\big(\log(1 - w\varphi)\big)^{\pm} \in \mathscr{W}$ and, by the Wiener-Levy theorem with $h(s) = \exp(\mp s)$, we obtain from (2.8) that $p(\cdot, w)^{\pm} \in \mathscr{W}$. Finally it will follow from (2.21) and (2.22) that $g^+(\cdot, w)$, $g^-(\cdot, w) \in \mathscr{W}$, hence $g(\cdot, w) \in \mathscr{W}$.

The functional equation (2.6) implies that

$$\frac{f_0^+(z) + w\varphi(z) f_0^-(z)}{p(z, w)} = \frac{1 - w\varphi(z)}{p(z, w)} g^+(z, w) + \frac{g^-(z, w) - f_0^-(z)}{p(z, w)}. \tag{2.23}$$

The first term on the right is in $\mathscr{W}^+$ by (2.9) whereas the second belongs to $\mathscr{W}^-$ by (2.8). Hence (2.23) is a Laurent separation, which implies (2.21) and (2.22). □

An alternative form for (2.21) and (2.22) is

$$g^+(z,w) = \frac{p(z,w)}{1-w\varphi(z)}\left[f_0^+(z)\frac{1}{p(z,w)} - f_0^-(z)\frac{1-w\varphi(z)}{p(z,w)}\right]^+, \tag{2.24}$$

$$g^-(z,w) = 2f_0^-(z) + p(z,w)\left[f_0^+(z)\frac{1}{p(z,w)} - f_0^-(z)\frac{1-w\varphi(z)}{p(z,w)}\right]^-. \tag{2.25}$$

2.5. Now we consider the special cases where $f_0(z) = z^m$, $m \in \mathbf{Z}$. We write g_m instead of g to indicate the dependence on m. The cases $m \ge 0$ and $m < 0$ have to be treated separately. If we set $f_0(z) = z^m$ in (2.6) then it follows that for $z \in \mathbf{T}$

$$\big(1 - w\varphi(z)\big)g_m^+(z,w) + g_m^-(z,w) = z^m \qquad \text{if } m \ge 0, \tag{2.26}$$

$$\big(1 - w\varphi(z)\big)g_m^+(z,w) + g_m^-(z,w) = \big(1 + w\varphi(z)\big)z^m \qquad \text{if } m < 0. \tag{2.27}$$

Theorem 2.2. *Let $f_0(z) = z^m$ with $m \ge 0$. Then*

$$g_m^+(z,w) = \frac{p(z,w)}{1-w\varphi(z)}\sum_{k=0}^{m} q_{k-m}(w)z^k, \tag{2.28}$$

$$g_m^-(z,w) = p(z,w)\sum_{k<0} q_{k-m}(w)z^k. \tag{2.29}$$

Furthermore the threefold generating function satisfies

$$\sum_{m=0}^{\infty} g_m(z,w)\zeta^{-m} = \frac{1}{1-z\zeta^{-1}}\left(1 + \frac{w\varphi(z)p(z,w)}{1-w\varphi(z)}\frac{1}{p(\zeta,w)}\right) \quad \text{for } |z| = 1 < |\zeta|. \tag{2.30}$$

Proof. We see from (2.11) that

$$\frac{z^m}{p(z,w)} = \sum_{j\le 0} q_j(w)z^{m+j} = \sum_{k\le m} q_{k-m}(w)z^k.$$

Hence (2.28) and (2.29) follow from (2.24) and (2.25) respectively.

To prove (2.30), we use (2.28). Changing the order of summation and writing $j = k - m$ we obtain

$$\begin{aligned}\sum_{m=0}^{\infty} g_m^+(z,w)\zeta^{-m} &= \frac{p(z,w)}{1-w\varphi(z)}\sum_{k=0}^{\infty}(z\zeta^{-1})^k\sum_{j\le 0} q_j(w)\zeta^{-j} \\ &= \frac{p(z,w)}{1-w\varphi(z)}\frac{1}{1-z\zeta^{-1}}\frac{1}{p(\zeta,w)}\end{aligned} \tag{2.31}$$

because of (2.11). Now it follows from (2.26) that

$$\sum_{m=0}^{\infty} g_m^-(z,w)\zeta^{-m} = \sum_{m=0}^{\infty}(z\zeta^{-1})^m - \frac{p(z,w)}{(1-z\zeta^{-1})p(\zeta,w)}.$$

Adding up we obtain (2.30). □

For $m = 0$ and $f_0(z) = 1$ we get

$$g_0^+(z,w) = \frac{p(z,w)}{1 - w\varphi(z)}, \tag{2.32}$$

$$g_0^-(z,w) = 1 - p(z,w), \tag{2.33}$$

$$g_0(z,w) = 1 + \frac{w\varphi(z)p(z,w)}{1 - w\varphi(z)}. \tag{2.34}$$

Theorem 2.3. *If $f_0(z) = z^m$ with $m < 0$ then*

$$g_m^+(z,w) = -\frac{r(w)p(z,w)}{1 - w\varphi(z)} \sum_{k=0}^{\infty} p_{k-m}(w) z^k, \tag{2.35}$$

$$g_m^-(z,w) = 2z^m - r(w)p(z,w) \sum_{k=m}^{-1} p_{k-m}(w) z^k. \tag{2.36}$$

Furthermore, the threefold generating function satisfies

$$\sum_{m<0} g_m(z,w)\zeta^{-m} = \frac{1}{z\zeta^{-1} - 1}\left(1 + \frac{w\varphi(z)p(z,w)}{1 - w\varphi(z)} \frac{1 - w\varphi(\zeta^{-1})}{p(\zeta^{-1},w)}\right) \quad \textit{for } |z| = 1 > |\zeta|. \tag{2.37}$$

Proof. By (2.12) we have

$$\left(z^m \frac{1 - w\varphi(z)}{p(z,w)}\right)^- = z^m p(w) \sum_{k=0}^{|m|-1} p_k(w) z^k = r(w) \sum_{j=m}^{-1} p_{j-m}(w) z^j,$$

and (2.36) follows immediately if this is substituted into (2.25). To prove (2.35), we apply the relation $a^+ = a - a^-$ to (2.24), make the same substitution and obtain

$$\begin{aligned} g_m^+(z,w) &= -z^m + z^m \frac{r(w)p(z,w)}{1 - w\varphi(z)} \sum_{k=0}^{|m|-1} p_k(w) z^k \\ &= -z^m + z^m \frac{r(w)p(z,w)}{1 - w\varphi(z)} \left(\sum_{k=0}^{\infty} p_k(w) z^k - \sum_{k=|m|}^{\infty} p_k(w) z^k\right). \end{aligned}$$

Now (2.35) follows with (2.12). The relation (2.37) is proved in a similar manner as (2.30). □

For $m = -1$, $f_0(z) = 1/z$ it follows that

$$z g_{-1}^+(z,w) = -1 + \frac{p(z,w)}{1 - w\varphi(z)} r(w), \tag{2.38}$$

$$z g_{-1}^-(z,w) = 2 - p(z,w)\, r(w), \tag{2.39}$$

$$z g_{-1}(z,w) = 1 + \frac{w\varphi(z)p(z,w)}{1 - w\varphi(z)} r(w). \tag{2.40}$$

This case has a particular importance because of its relation to the alternative ruin definition in terms of (1.6). If the recursion starts with $\hat{f}_0(z) = 1$ then

$$\hat{g}_0(z,w) = \sum_{n \geq 0} \hat{f}_n(z) w^n = z g_{-1}(z,w). \tag{2.41}$$

From (2.40) and (2.34) follows that $\hat{g}_0(z,w) - 1 = r(w)(g_0(z,w) - 1)$ or

$$\hat{b}_{n,k} = r(w) b_{n,k} \quad \text{for } n \geq 1 \text{ and } k \in \mathbf{Z}.$$

Formula (2.39) implies that

$$z\left(\frac{\hat{g}_0(z,w)}{z}\right)^{-} = 2 - p(z,w) r(w).$$

For $z \to \infty$ the right hand side tends to $2 - r(w)$ and the left hand side to $\sum_{n=0}^{\infty} \hat{b}_{n,0} w^n$. Because of $\hat{b}_{0,0} = 1$ it follows that

$$\sum_{n=1}^{\infty} \hat{b}_{n,0} w^n = 1 - r(w) = 1 - \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} a_{n,0} w^n\right). \tag{2.42}$$

For $w \to 1$ one obtains

$$\sum_{n=1}^{\infty} \hat{b}_{n,0} = 1 - \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} a_{n,0}\right). \tag{2.43}$$

In the probabilistic interpretation $\hat{b}_{n,0}$ is the probability that a random walk, starting in 0 and restricted to $\mathbf{N}_0$, returns to 0 *the first time* at n, whereas $a_{n,0}$ is the probability that an arbitrary random walk starting in 0 returns to 0 at n.

By means of (2.18), the quantities $\tilde{g}_0$, $\hat{\tilde{g}}_0$ etc. based on $\tilde{\varphi}$ instead of φ, as introduced in Section 2.3, can be expressed through $\varphi(1/z)$ and $p(1/z,w)$. We derive one of these relations which will be used later. It follows from (2.41), (2.40), (2.18) and (2.17) that

$$\begin{aligned}\hat{\tilde{g}}_0(\zeta,w) &= 1 + \frac{w\tilde{\varphi}(\zeta)\tilde{p}(\zeta,w)}{1 - w\tilde{\varphi}(\zeta)} r(w) = 1 + \frac{w\varphi(1/\zeta)}{p(1/\zeta,w)} \\ &= 1 - r(w) + \frac{1}{p(1/\zeta,w)} + r(w) - \frac{1 - w\varphi(1/\zeta)}{p(1/\zeta,w)}.\end{aligned}$$

Since this is a Laurent separation, we obtain

$$\hat{\tilde{g}}_0^{+}(\zeta,w) = 1 - r(w) + \frac{1}{p(1/\zeta,w)}. \tag{2.44}$$

3. The Structure of $p(z,w)$ and the coefficients

3.1. Now we study the function $p(z,w)$ in more detail. Its definition (2.8) is rather formal and computationally very complicated. We first note a relation which follows from (2.33) and (2.32).

$$p(z,w) = 1 - g_0^{-}(z,w) = \big(1 - w\varphi(z)\big) g_0^{+}(z,w) \quad \text{for } w \in \mathbf{D}. \tag{3.1}$$

Theorem 3.1. *If G is the domain where φ is analytic and $w \neq 0$, then $p(\cdot,w)$ is analytic precisely in $G \cup \{|z| > 1\}$ and its zeros are the zeros of $1 - w\varphi$ in $G \cap \mathbf{D}$.*

Proof. The definition (2.8) shows that $p(z,w)$ is analytic in $\{|z| > 1\}$ and $\neq 0$ in $\{|z| \geq 1\}$. For $z \in \mathbf{D}$ we use (2.9) to conclude that $1 - w\varphi(z)$ and $p(z,w)$ have the same singularities and zeros. □

3.2. Now we turn to the coefficients. We write

$$f_n(z) = \sum_{k \in \mathbf{Z}} b_{n,k} z^k \quad \text{for } |z| = 1 \tag{3.2}$$

so that, by (2.4),

$$g(z,w)=\sum_{n=0}^{\infty}\sum_{k\in\mathbf{Z}} b_{n,k}z^k w^n \quad \text{for } |z|=1,\ |w|<1. \tag{3.3}$$

In particular, the given function is

$$f_0(z)=\sum_{k\in\mathbf{Z}} b_{0,k}z^k. \tag{3.4}$$

The recursive definition (1.5) and (2.1) lead to the recursion formula

$$b_{1,k}=\sum_{j\in\mathbf{Z}} a_{k-j}b_{0,j} \quad \text{for } k\in\mathbf{Z}, \tag{3.5a}$$

$$b_{n+1,k}=\sum_{j\geq 0} a_{k-j}b_{n,j} \quad \text{for } n\in\mathbf{N},\ k\in\mathbf{Z}. \tag{3.5b}$$

By (2.10)–(2.13) we have for $w\in\mathbf{D}$

$$p(z,w)=\sum_{n=0}^{\infty}\sum_{k\leq 0} p_{n,k}z^k w^n, \qquad \frac{1}{p(z,w)}=\sum_{n=0}^{\infty}\sum_{k\leq 0} q_{n,k}z^k w^n \quad (|z|\geq 1), \tag{3.6}$$

$$\frac{1-w\varphi(z)}{r(w)p(z,w)}=\sum_{n=0}^{\infty}\sum_{k\geq 0} p_{n,k}z^k w^n, \quad \frac{r(w)p(z,w)}{1-w\varphi(z)}=\sum_{n=0}^{\infty}\sum_{k\geq 0} q_{n,k}z^k w^n \quad (|z|\leq 1), \tag{3.7}$$

$$r(w)=\sum_{n=0}^{\infty} r_n w^n. \tag{3.8}$$

From (2.14) and (2.15) follows that $p_{0,0}=1$, $p_{0,k}=0$ $(k<0)$, $p_{n,0}=0$ $(n>0)$ and from (3.1) that

$$p_{n,k}=-b_{0;n,k} \quad \text{for } n\geq 1,\ k<0, \tag{3.9}$$

where $b_{0;n,k}$ denote the coefficients belonging to $f_0=1$, see Section 2.5.

Multiplication of the second equations in (2.12) and (2.11) with $(1-w\varphi(z))$ and comparison with the respective first equations in (2.11) and (2.12) gives

$$r(w)\sum_{k\leq 0} p_k(w)z^k=\sum_{k\geq 0} q_k(w)z^k - w\sum_{k\in\mathbf{Z}}\Big(\sum_{j\geq 0} a_{k-j}q_j(w)\Big)z^k,$$

$$r(w)\sum_{k\geq 0} p_k(w)z^k=\sum_{k\leq 0} q_k(w)z^k - w\sum_{k\in\mathbf{Z}}\Big(\sum_{j\leq 0} a_{k-j}q_j(w)\Big)z^k.$$

Comparing coefficients we obtain

$$r(w)p_k(w)=\begin{cases} -w\sum_{j\geq 0} a_{k-j}q_j(w) & \text{for } k<0,\\ -w\sum_{j\leq 0} a_{k-j}q_j(w) & \text{for } k>0,\end{cases} \tag{3.10}$$

$$r(w)=1-w\sum_{j\geq 0} a_{-j}q_j(w)=1-w\sum_{j\leq 0} a_{-j}q_j(w), \tag{3.11}$$

$$q_k(w) = \begin{cases} w\sum\limits_{j\le 0} a_{k-j}q_j(w) & \text{for } k<0, \\ w\sum\limits_{j\ge 0} a_{k-j}q_j(w) & \text{for } k>0. \end{cases} \tag{3.12}$$

It is easy to derive from these formulas that

$$p_{1,k} = -a_k, \quad q_{1,k} = a_k \quad \text{for } k \neq 0.$$

For $n > 1$ there are no explicit expressions for the $p_{n,k}$ and $q_{n,k}$. However, the numerical calculation of the $p_{n,k}$, $k < 0$, is possible via (3.9), for the $b_{0;n,k}$ can be calculated recursively by means of (3.5). Another way, which allows to calculate all the $p_{n,k}$ and $q_{n,k}$, is based on the formulas (3.10)–(3.12). Substitution of the second series in (2.13) into (3.12) gives the recursion formulas

$$q_{n+1,k} = \sum_{j\le 0} a_{k-j}q_{n,j} \quad \text{for } n \in \mathbf{N}_0,\ k<0, \tag{3.13}$$

$$q_{n+1,k} = \sum_{j\ge 0} a_{k-j}q_{n,j} \quad \text{for } n \in \mathbf{N}_0,\ k>0. \tag{3.14}$$

Starting with $q_{0,k}$, which is $= 1$ for $k = 0$ and $= 0$ for $k \neq 0$ by (2.14) and (2.15), this allows the recursive computation of the $q_{n,k}$, see the proof of Theorem 3.2 below. Thereafter, the r_n can be calculated from (3.11) by the recursion

$$r_0 = 1, \quad r_{n+1} = \sum_{j\le 0} a_{-j}q_{n,j} = \sum_{j\ge 0} a_{-j}q_{n,j} \quad \text{for } n \in \mathbf{N}_0.$$

Finally, the $p_{n,k}$ can be calculated from the $q_{n,k}$ and the r_n using (3.10).

The recursion formulas (3.5), (3.9) and (3.13) imply that $-p_{n,k}$ and $q_{n,k}$ for $n \geq 1$, $k < 0$ has the form

$$\sum_{j_1+\ldots+j_n=k} \nu(j_1,\ldots,j_n)a_{j_1}\cdots a_{j_n}, \quad \nu \in \mathbf{N}_0. \tag{3.15}$$

The same is true of the coefficients $b_{m;n,k}$ of g_m for $n \geq 1$, $k \in \mathbf{Z}$.

Now we derive a characterization of $p(z,w)$ that is more structural than (2.8).

Theorem 3.2. *For each $w \in \mathbf{D}$, the function $p(z,w)$ is uniquely characterized by the conditions*

(i) $\dfrac{1}{p(\cdot,w)} - 1 \in \mathscr{W}^-$,

(ii) $\dfrac{1-w\varphi}{p(\cdot,w)} \in \mathscr{W}^+$.

Proof. It is easy to show that the functions in (2.8) and (2.9) satisfy (i) and (ii).

Now we assume that p satisfies the conditions (i) and (ii). Using (i), we define $q_k(w)$, $k \leq 0$, for this p by (2.11) and multiply this relation with $(1-\varphi(z))$. Then (ii) implies that the first case of (3.12) holds. Next we write this in matrix form, observing that (i) implies that $q_0(w) = 1$:

$$\begin{pmatrix} q_{-1}(w) \\ q_{-2}(w) \\ q_{-3}(w) \\ \vdots \end{pmatrix} = w \begin{pmatrix} a_{-1} \\ a_{-2} \\ a_{-3} \\ \vdots \end{pmatrix} + w \begin{pmatrix} a_0 & a_1 & a_2 & \dots \\ a_{-1} & a_0 & a_1 & \dots \\ a_{-2} & a_{-1} & a_0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} q_{-1}(w) \\ q_{-2}(w) \\ q_{-3}(w) \\ \vdots \end{pmatrix}, \tag{3.16}$$

or, shorter,

$$q(w) = wa + wAq(w). \tag{3.17}$$

If we define the norm of complex infinite vectors $\omega = (\omega_i)_{i=1,2,\dots}$ and matrices $(\Omega_{ij})_{i,j=1,2,\dots}$ by

$$\|\omega\| = \sum_{i=1}^{\infty} |\omega_i| \quad \text{and} \quad \|\Omega\| = \sup_{i\geq 1} \sum_{j=1}^{\infty} |\Omega_{i,j}|,$$

the vector space becomes a Banach space and the matrix algebra a Banach algebra. Then

$$\|q(w)\| = \sum_{k<0} |q_k(w)| = \left\| \frac{1}{p(\cdot, w)} \right\| - 1,$$

$$\|a\| = \sum_{k<0} |a_k| \leq \|\varphi\|, \quad \|A\| = \sup_{i\geq 1} \sum_{j=1}^{\infty} |a_{j-i}| = \|\varphi\|,$$

the norms one the right hand sides being those in $\mathscr{W}$. Therefore the series

$$(1 - wA)^{-1} = 1 + wA + w^2A^2 + \dots$$

converges absolutely for $|w| < \|\varphi\|^{-1}$ and is an analytic function of w. It follows that

$$q(w) = w(1 - wA)^{-1}a = wa + w^2Aa + w^3A^2a + \dots \quad (|w| < \|\varphi\|^{-1}),$$

so every component $q_{-k}(w)$ $(k < 0)$ can be expanded into a power series as in (2.13) which converges absolutely at least for $|w| < \|\varphi\|^{-1}$. Then the recursion formula (3.13) holds which together with the initial condition $q_{0,0} = 1$, $q_{0,k} = 0$ for $k < 0$ uniquely determines the coefficients $q_{n,k}$ of $1/p$. □

3.3. Using Theorem 3.2 we determine $p(z, w)$ for an important special case; compare Theorem 3.1.

Theorem 3.3. *Suppose that φ^- has a meromorphic continuation to* $\mathbf{D}$ *with the poles $\zeta_1, \dots, \zeta_d$ counting multiplicity. Then*

$$p(z, w) = \prod_{k=1}^{d} \frac{z - z_k(w)}{z - \zeta_k}, \tag{3.18}$$

where the z_k are the zeros of $1 - w\varphi$ in $\mathbf{D}$.

If $a_k = 0$ for $k < -d$ and $a_{-d} \neq 0$ then $\zeta_1 = \dots = \zeta_d = 0$ and thus $p(z, w) = \prod_{k=1}^{d} \left(1 - z^{-1}z_k(w)\right)$. Only this case was considered in [JP07].

Proof. We consider the Blaschke product

$$\psi(z) = \prod_{k=1}^{d} \frac{z - \zeta_k}{1 - \overline{\zeta_k} z}. \tag{3.19}$$

Let φ be defined in $\mathbf{D}$ as the sum of φ^+ and the analytic continuation of φ^-. Then $\psi(1-w\varphi)$ is analytic in $\mathbf{D}$ and continuous in $\overline{\mathbf{D}}$. Let $|w| \le \rho < 1$. By (2.3) there exists r with $|\zeta_k| < r < 1$ such that $|\varphi(z)| < 1/\rho$ for $|z| = r$. Hence

$$|w\varphi(z)\psi(z)| < |\psi(z)| \quad \text{for } |z| = r.$$

Hence it follows from Rouché's theorem that $\psi(1-w\varphi)$ and therefore $1-w\varphi$ has precisely d zeros in $|z| < r$ as has ψ. Thus the product (3.18) is well defined.

Now we apply Theorem 3.2. It is clear from (3.18) that (i) is satisfied, and (ii) is true because the $z_k(w)$ are all the zeros of $1-w\varphi$. □

3.4. Finally we consider the *symmetric case* that $\varphi(z) = \varphi(1/z)$. Then $a_k = a_{-k}$ for all k and it follows that the coefficients of $\varphi(z)^n$ satisfy $a_{n,k} = a_{n,-k}$. Hence we obtain from (2.18) that

$$1 - w\varphi(z) = r(w)p(z^{-1},w)p(z,w) \quad \text{for } |z| = 1,\ |w| < 1. \tag{3.20}$$

For $z = 1$ we obtain

$$p(1,w) = \sqrt{(1-\varphi(1)w)/r(w)} \quad \text{for } |w| < 1. \tag{3.21}$$

Example 3.1. Let $\operatorname{Re}\alpha \ge 0$ and

$$\varphi(z) = \exp\Big[-\alpha + \frac{\alpha}{2}\Big(z + \frac{1}{z}\Big)\Big] = \mathrm{e}^{-\alpha} I_0(\alpha) + \sum_{k=1}^{\infty} \mathrm{e}^{-\alpha} I_k(\alpha)\Big(z^k + \frac{1}{z^k}\Big)$$

where I_k are the modified Bessel functions. Then

$$\big|\varphi(\mathrm{e}^{\mathrm{i}t})\big| = \exp\Big[-2(\operatorname{Re}\alpha)\sin^2\frac{t}{2}\Big] \le 1\,.$$

It follows from (3.21) that

$$p(1,w) = \sqrt{1-w}\,\exp\Big[\mathrm{e}^{-\alpha}\sum_{n=1}^{\infty} I_0(\alpha n)\frac{w^n}{n}\Big].$$

Example 3.2. Now suppose that

$$\varphi(z) = a_0 + \sum_{k=1}^{d} a_k(z^k + z^{-k}).$$

The Chebychev polynomials T_k satisfy $T_k\big(\frac{1}{2}(z+z^{-1})\big) = \frac{1}{2}(z^k+z^{-k})$ [MOS66, p. 257]. With $\zeta = \frac{1}{2}(z+z^{-1})$ we there fore can write

$$\chi(z) = a_0 + 2\sum_{k=1}^{d} a_k T_k(\zeta) = \sum_{k=1}^{d} c_k \zeta^k.$$

The polynomial $1 - w\chi(z)$ has d zeros $\zeta_k(w)$ and each of these zeros gives rise to two zeros $z_k^{\pm} = \zeta_k \pm \sqrt{\zeta_k^2 - 1}$ of $1 - w\varphi(z)$, which satisfy $|z_k^-| < 1 < |z_k^+|$. Using (3.18) we can compute $p(z,w)$ and we can obtain $r_0(w)$ by (3.21). This leads to an explicit formula if $d = 2$, which corresponds to the symmetric pentanomial distribution.

4. Application to random walks

In this section we consider the probabilistic setting described in the introduction. It is characterized by $a_k \geq 0$ for $k \in \mathbf{Z}$ and the property

$$\varphi(1) = \sum_{k\in\mathbf{Z}} a_k = 1 \tag{4.1}$$

which sharpens (2.2) and (2.3) and has not been used in the preceding sections.

4.1. First we consider the *ruin problem*. For the sake of a simpler notation we now assume that $S_0 = m \in \mathbf{Z}_0$, so $f_0(z) = z^m$. The probability measure $\mathbf{P}$ considered in the introduction and the stopping times R and $\hat{R}$ according to (1.4) and (1.6) will be denoted by $\mathbf{P}_m$, R_m and $\hat{R}_m$. In the classical language, R_m is the moment when a player with initial capital m is ruined; we allow m to be negative.

For $m \in \mathbf{Z}$, $k \in \mathbf{Z}$ and $n \in \mathbf{N}_0$ we set

$$b_{m;n,k} = \mathbf{P}_m\big(S_n = k,\ S_\nu \geq 0\ (\nu \geq 1,\ \nu < n)\big) \quad \text{for } n \geq 1. \tag{4.2}$$

For $n = 0$ and $n = 1$ the conditions on ν are not satisfied by any ν. Clearly

$$b_{m;0,k} = 1 \text{ for } m = k, = 0 \text{ else},$$
$$b_{m;1,k} = \mathbf{P}_m(S_1 = k) = \mathbf{P}_m(m + X_1 = k) = a_{k-m}.$$

An alternative notation, valid for all $n \geq 0$, is

$$b_{m;n,k} = \mathbf{P}_m(S_n = k,\ R_m \geq n). \tag{4.3}$$

As in (3.3) with $f_0(z) = z^m$, we consider the generating function

$$g_m(z, w) = z^m + \sum_{n=1}^{\infty} \sum_{k\in\mathbf{Z}} b_{m;n,k} z^k w^n \quad \text{for } |z| = 1,\ |w| = 1. \tag{4.4}$$

Let $n \geq 1$. Since $S_{n+1} = S_n + X_{n+1}$ by (1.3), it follows from (4.2) and the independence that

$$b_{m;n+1,k} = \sum_{j\geq 0} \mathbf{P}_m(S_n = j,\ S_\nu \geq 0\ (\nu < n),\ X_{n+1} = k - j) = \sum_{j=0}^{\infty} a_{k-j} b_{m;n,j}.$$

This is the recursion formula (3.5b) which is equivalent to our basic relation (1.5b). Moreover, we also have

$$b_{m;1,k} = a_{k-m} = \sum_{j\in\mathbf{Z}}^{\infty} a_{k-j} b_{m;0,j},$$

which is the same as (3.5a) and therefore equivalent to (1.5a). Hence we can apply all our previous results with $\varphi(1) = 1$ and $\|\varphi\| = 1$, see (2.2) and (2.3).

Theorem 4.1. *Let $S_0 = m$ and S_n be defined by* (1.3). *If $m \geq 0$, then*

$$\sum_{n=0}^{\infty} \sum_{k\geq 0} \mathbf{P}_m(S_n = k,\ R_m > n) z^k w^n = \frac{p(z, w)}{1 - w\varphi(z)} \sum_{k=0}^{m} q_{k-m}(w) z^k, \tag{4.5}$$

$$\sum_{n=1}^{\infty} \sum_{k<0} \mathbf{P}_m(S_n = k,\ R_m = n) z^k w^n = p(z, w) \sum_{k<0} q_{k-m}(w) z^k. \tag{4.6}$$

If $m < 0$, then

$$\sum_{n=0}^{\infty}\sum_{k\ge 0}\mathbf{P}_m(S_n = k,\ R_m > n)z^k w^n = -\frac{r(w)p(z,w)}{1-w\varphi(z)}\sum_{k=0}^{\infty} p_{k-m}(w)z^k, \tag{4.7}$$

$$\sum_{n=1}^{\infty}\sum_{k<0}\mathbf{P}_m(S_n = k,\ R_m = n)z^k w^n = z^m - r(w)p(z,w)\sum_{k=m}^{-1} p_{k-m}(w)z^k. \tag{4.8}$$

Proof. We apply the Laurent separation to (4.4) and use (4.3). Then the assertion follows from Theorem 2.2 and Theorem 2.3. □

The functions $q_k(w)$ were defined in (2.11) and can be computed from the recursion formula (3.13). The function $p(z,w)$ was formally introduced in (2.8) and was discussed throughout Section 3. Since $p = 1 - g_0^-$ by (3.1), we obtain from (4.4) the probabilistic interpretation

$$p(z,w) = 1 - \sum_{n=1}^{\infty}\sum_{k<0}\mathbf{P}_0(S_n = k,\ R_0 = n)z^k w^n. \tag{4.9}$$

If X is bounded below or if, more generally, the generating function φ is meromorphic in $\mathbf{D}$ then $p(z,w)$ is given by the analytic formula (3.18).

Now we put $z = 1$ in (4.6) and (4.8). Since $R_m = n$ implies $S_n < 0$ we obtain with (2.11)

$$\sum_{n=1}^{\infty}\mathbf{P}_m(R_m = n)w^n = p(1,w)\sum_{j<-m} q_j(w) = 1 - p(1,w)\sum_{j=0}^{m} q_{-j}(w) \quad \text{for } m \ge 0,$$

$$\sum_{n=1}^{\infty}\mathbf{P}_m(R_m = n)w^n = 1 - r(w)p(1,w)\sum_{j=0}^{|m|-1} p_j(w) \quad \text{for } m < 0.$$

If we let $w \to 1-$, we obtain

$$\mathbf{P}_m(R_m = \infty) = p(1,1)\sum_{j=0}^{m} q_{-j}(1) \quad \text{for } m \ge 0, \tag{4.10}$$

$$\mathbf{P}_m(R_m = \infty) = r(1)p(1,1)\sum_{j=0}^{|m|-1} p_j(1) \quad \text{for } m < 0. \tag{4.11}$$

4.2. We now consider the modified ruin problem on the basis of (1.6). Then $\hat{R}_m$ is equivalent to R_{m-1} in (1.4), and is related to the generating function $\hat{g}_m(z,w)$ in an analogous way as R_m is related to $g_m(z,w)$. Obviously $\hat{g}_m(z,w) = zg_{m-1}(z,w)$. For simplicity we restrict ourselves to $S_0 = 0$ and obtain from (2.38), (2.39) that

$$\hat{g}_0^+(z,w) = zg_{-1}^+(z,w) = -1 + \frac{p(z,w)}{1-w\varphi(z)}\,r(w), \tag{4.12}$$

$$\hat{g}_0^-(z,w) = zg_{-1}^-(z,w) = 2 - p(z,w)\,r(w), \tag{4.13}$$

$$\hat{g}_0(z,w) = zg_{-1}(z,w) = 1 + \frac{w\varphi(z)p(z,w)}{1-w\varphi(z)}r(w)\,. \tag{4.14}$$

4.3. Now we turn to the *minimum problem.* We start with $S_0 = 0$ and define

$$M_n = \min\{S_\nu : 0 \le \nu \le n\} \quad (n \in \mathbf{N}_0). \tag{4.15}$$

Theorem 4.2. *For $|\zeta| \geq 1$ and $w \in \mathbf{D}$ we have*

$$\sum_{n=0}^{\infty} \sum_{\mu \leq 0} \mathbf{P}_0(M_n = \mu)\zeta^\mu w^n = \frac{p(1,w)}{(1-w)p(\zeta,w)}\,. \tag{4.16}$$

Proof. For $\mu \geq 0$ we consider $S_{\mu;n} := \mu + S_n$. Then we have for $\mu \geq 0$, $n \geq 0$

$$\mathbf{P}_0(M_n = -\mu) = \mathbf{P}_0\big(S_{\mu;\nu} \geq 0 \ \ (\nu \leq n)\big) - \mathbf{P}_0\big(S_{\mu-1;\nu} \geq 0 \ \ (\nu \leq n)\big)\,. \tag{4.17}$$

This is obvious for $\mu \geq 1$, $n \geq 1$. In the other cases, (4.17) can be read off from the following table:

μ	n	$\mathbf{P}_0(M_n = -\mu)$	$\mathbf{P}_0\big(S_{\mu;\nu} \geq 0 \ (\nu \leq n)\big)$	$\mathbf{P}_0\big(S_{\mu-1;\nu} \geq 0 \ (\nu \leq n)\big)$
≥ 1	0	$\mathbf{P}_0(S_0 = -\mu) = 0$	$\mathbf{P}_0(\mu \geq 0) = 1$	$\mathbf{P}_0(\mu \geq 1) = 1$
0	≥ 1	$\mathbf{P}_0\big(S_\nu \geq 0 \ (\nu \leq n)\big)$	$\mathbf{P}_0\big(S_\nu \geq 0 \ (\nu \leq n)\big)$	$\mathbf{P}_0\big(S_\nu \geq 1 \ (\nu \leq n)\big) = 0$
0	0	$\mathbf{P}_0(S_0 = 0) = 1$	$\mathbf{P}_0(0 \geq 0) = 1$	$\mathbf{P}_0(-1 \geq 0) = 0$

Since $S_{\mu;0} = \mu$ it follows from (4.2) and (4.4) that for $\mu \in \mathbf{Z}$

$$\begin{aligned} g_\mu^+(1,w) &= \sum_{n=0}^{\infty} \sum_{k \geq 0} \mathbf{P}_0\big(S_{\mu;n} = k,\ S_{\mu;\nu} \geq 0 \ \ (\nu \geq 1,\ \nu < n)\big) w^n \\ &= \sum_{n=0}^{\infty} \mathbf{P}_0\big(S_{\mu;\nu} \geq 0 \ \ (\nu \geq 1,\ \nu \leq n)\big) w^n. \end{aligned}$$

For $\mu \geq 1$ we can write (4.17) in the form

$$\mathbf{P}_0(M_n = -\mu) = \mathbf{P}_0\big(S_{\mu;\nu} \geq 0 \ \ (\nu \geq 1,\ \nu \leq n)\big) - \mathbf{P}_0\big(S_{\mu-1;\nu} \geq 0 \ \ (\nu \geq 1,\ \nu \leq n)\big)$$

and obtain

$$\sum_{n=0}^{\infty} \mathbf{P}_0(M_n = -\mu) w^n = g_\mu^+(1,w) - g_{\mu-1}^+(1,w).$$

For $\mu = 0$, (4.17) becomes

$$\mathbf{P}_0(M_n = 0) = \mathbf{P}_0\big(S_{0;\nu} \geq 0 \ \ (\nu \geq 1,\ \nu \leq n)\big)$$

and therefore

$$\sum_{n=0}^{\infty} \mathbf{P}_0(M_n = 0) w^n = g_0^+(1,w).$$

This implies

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{\mu=0}^{\infty} \mathbf{P}_0(M_n = -\mu)\zeta^{-\mu} w^n &= \sum_{\mu=0}^{\infty} g_\mu^+(1,w)\zeta^{-\mu} - \sum_{\mu=1}^{\infty} g_{\mu-1}^+(1,w)\zeta^{-\mu} \\ &= (1-\zeta^{-1}) \sum_{\mu=0}^{\infty} g_\mu^+(1,w)\zeta^{-\mu}. \end{aligned}$$

Hence (4.16) follows from (2.31) for $z = 1$ because $\varphi(1) = 1$. □

Multiplying (4.16) by $1 - \zeta^{-1}$ we obtain

$$\sum_{\mu \leq 0} \mathbf{P}_0(M_0 = \mu)\zeta^\mu + \sum_{n=1}^{\infty} \sum_{\mu \leq 0} \big(\mathbf{P}_0(M_n = \mu) - \mathbf{P}_0(M_{n-1} = \mu)\big)\zeta^\mu w^n = \frac{p(1,w)}{p(\zeta,w)}.$$

The coefficients $p_{n,k}$ and $q_{n,k}$ were defined in (3.6) and (3.7). We write

$$p_n^* = \sum_{k\le 0} p_{n,k} = 1 - \sum_{k<0} |p_{n,k}| \quad (n \in \mathbf{N}_0),$$

see (3.15). Then $p(1,w) = \sum_{n=0}^{\infty} p_n^* w^n$. Using (2.11) we obtain

$$\mathbf{P}_0(M_n = \mu) - \mathbf{P}_0(M_{n-1} = \mu) = \sum_{\nu=0}^{\infty} p_{n-\nu}^* q_{\nu,\mu}. \tag{4.18}$$

4.4. Finally, we apply Laurent separation to another approach to the minimum problem for random walks starting with $S_0 = 0$, which supplies additional information about the terminal position S_n and the first time N_n at which the minimum is attained, cf. [Spi76, p. 205 ff]. It is defined by

$$N_n = \min\{t \in [0,n] : S_t = M_n\}. \tag{4.19}$$

Theorem 4.3. *For* $|\zeta| \ge 1$ *and* $w \in \mathbf{D}$ *we have*

$$\begin{aligned} &\sum_{\mu\le 0}\sum_{n=0}^{\infty}\sum_{\nu=0}^{n}\sum_{h=0}^{\infty} \mathbf{P}_0(M_n = \mu,\ S_n = \mu + h,\ N_n = \nu) z^h s^\nu w^n \zeta^\mu \\ &= \frac{p(z,w)}{p(\zeta, sw)(1 - w\varphi(z))}. \end{aligned} \tag{4.20}$$

For $z \to 1$ we obtain from (4.20) and (4.1) the threefold generating function

$$\sum_{\mu\le 0}\sum_{n=0}^{\infty}\sum_{\nu=0}^{n} \mathbf{P}_0(M_n = \mu,\ N_n = \nu) s^\nu w^n \zeta^\mu = \frac{p(1,w)}{p(\zeta, sw)(1-w)},$$

and then (4.16) by letting $s \to 1$.

Proof. From $S_0 = 0$ follows that $M_n \le 0$ and $M_n = 0$ if and only if $S_t \ge 0$ for $1 \le t \le n$. Let $M_n = -\mu$ and $N_n = \nu$. We first consider the case $\mu \ge 1$, $n \ge \nu \ge 1$.

(i) Let $\tilde{S}_0 = 0$ and $\tilde{S}_j = S_{\nu-j} - S_\nu = -X_\nu - X_{\nu-1} - \ldots - X_{\nu-j+1}$ for $j = 1, \ldots, \nu$. Then $\tilde{S}_1 \ge 1$, …, $\tilde{S}_{\nu-1} \ge 1$, $\tilde{S}_\nu = \mu$. This event depends only on $X_1, \ldots, X_\nu$ and is described by $\hat{\tilde{R}}_0 \ge \nu$, $\tilde{S}_\nu = \mu$, therefore has the probability $\hat{\tilde{b}}_{0;\nu,\mu}$, see Section 2.3 and Section 4.2. The tilde in $\hat{\tilde{g}}_0$, $\hat{\tilde{b}}_{0;n,k}$, $\hat{\tilde{\mathbf{P}}}_0$ and $\hat{\tilde{R}}_0$ denotes terms belonging to $\tilde{\varphi}$ and $f_0 = 1$.

(ii) Let $\check{S}_0 = 0$ and $\check{S}_j = S_{\nu+j} - S_\nu = X_{\nu+1} + \ldots + X_{\nu+j}$ for $j = 1, \ldots, n-\nu$. Then $\check{S}_1 \ge 0$, …, $\check{S}_{\nu-1} \ge 0$, $\check{S}_\nu = h$ for some $h \ge 0$. This event depends only on $X_{\nu+1}, \ldots, X_n$ and is described by $R_0 \ge n - \nu$, $S_{n-\nu} = h$, therefore has the probability $b_{0;n-\nu,h}$.

With these variables, the event $(M_n = -\mu,\ S_n = -\mu + h, N_n = \nu)$ can be described as $(\hat{\tilde{R}}_0 \ge \nu,\ \tilde{S}_\nu = \mu,\ R_0 \ge n - \nu,\ S_{n-\nu} = h)$, and it follows from the independence that its probability is $\tilde{\mathbf{P}}_0(\hat{\tilde{R}}_0 \ge \nu,\ \tilde{S}_\nu = \mu)\, \mathbf{P}_0(R_0 \ge n-\nu,\ S_{n-\nu} = h)$, hence

$$\mathbf{P}_0(M_n = -\mu, S_n = -\mu + h, N_n = \nu) = \hat{\tilde{b}}_{0;\nu,\mu} b_{0;n-\nu,h}\,. \tag{4.21}$$

Moreover, $\mathbf{P}_0(M_n = -\mu,\ S_n = -\mu + h, N_n = 0) = 0 = \hat{\tilde{b}}_{0;0,\mu}$ for $\mu \geq 1$ and all $n \geq 0$, so (4.21) holds for $\mu \geq 1$, $n \geq \nu \geq 0$. Furthermore,

$$\mathbf{P}_0(M_n = 0,\ S_n = h, N_n = \nu) = \begin{cases} b_{0;n,h} & \text{for } \nu = 0, \\ 0 & \text{for } \nu \geq 1. \end{cases} \tag{4.22}$$

For $\mu \geq 0$, $|s| \leq 1$, $|w| \leq 1$, $|z| \leq 1$, let

$$\gamma_\mu(w,s,z) = \sum_{n=0}^{\infty} \sum_{\nu=0}^{n} \sum_{h=0}^{\infty} \mathbf{P}_0(M_n = -\mu,\ S_n = -\mu + h, N_n = \nu) z^h s^\nu w^n. \tag{4.23}$$

By exchanging the order of summation over n and ν one obtains from (4.21)

$$\gamma_\mu(w,s,z) = \sum_{\nu=0}^{\infty} \hat{\tilde{b}}_{0;\nu,\mu}(sw)^\nu \sum_{n=\nu}^{\infty} \sum_{h=0}^{\infty} b_{0;n-\nu,h} w^{n-\nu} z^h = \sum_{\nu=0}^{\infty} \hat{\tilde{b}}_{0;\nu,\mu}(sw)^\nu g_0^+(z,w)$$

for $\mu \geq 1$. From (4.22) follows that $\gamma_0(w,s,z) = g_0^+(z,w)$. Hence for $|\zeta| \leq 1$

$$\begin{aligned} \sum_{\mu=0}^{\infty} \gamma_\mu(w,s,z)\zeta^\mu &= \left(1 + \sum_{\mu=1}^{\infty} \sum_{\nu=0}^{\infty} \hat{\tilde{b}}_{0;\nu,\mu}(sw)^\nu \zeta^\mu\right) g_0^+(z,w) \\ &= \left(1 - \sum_{\nu=1}^{\infty} \hat{\tilde{b}}_{0;\nu,0}(sw)^\nu - \hat{\tilde{b}}_{0;0,0} + \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} \hat{\tilde{b}}_{0;\nu,\mu}(sw)^\nu \zeta^\mu\right) g_0^+(z,w) \\ &= \left(r(sw) - 1 + \hat{\tilde{g}}_0^+(\zeta, sw)\right) g_0^+(z,w), \end{aligned}$$

where we have used (2.42) and (2.17). With (2.44) follows that

$$\sum_{\mu=0}^{\infty} \gamma_\mu(w,s,z)\zeta^\mu = \frac{1}{p(1/\zeta, sw)} g_0^+(z,w) = \frac{1}{p(1/\zeta, sw)} \frac{p(z,w)}{1 - w\varphi(z)},$$

the latter by (3.1). Finally, (4.20) follows if we replace μ by $-\mu$ and ζ by $1/\zeta$. □

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Received 9 December 2008