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ON REGULAR HOMEOMORPHISMS IN THE PLANE

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Abstract. A regular homeomorphism of the Sobolev class $W^{1,1}_{\mathrm{loc}}$ in the plane domain D is a ring Q-homeomorphism with $Q(z) = K^T_\mu(z, z_0)$ where $K^T_\mu(z, z_0)$ is the tangential dilatation of f at $z_0 \in D$.

1. Introduction

It has been established in [Sal] that a Q-homeomorphism in $\mathbf{R}^n$, $n \geq 2$, is in $W^{1,1}_{\mathrm{loc}}$ and differentiable with its Jacobian $J_f(z) \neq 0$ a.e. whenever $Q \in L^1_{\mathrm{loc}}$. These results were extended to ring Q-homeomorphisms in [SS$_1$] and [SS$_2$]. In the present paper it is conversely stated that every homeomorphism in the plane of the class $W^{1,1}_{\mathrm{loc}}$ with $J_f(z) > 0$ a.e. is a ring Q-homeomorphism with $Q(z) = K_f(z)$. Moreover, we give a pointwise characterization of this property.

Let D be a domain in the complex plane $\mathbf{C}$, i.e., a connected and open subset of $\mathbf{C}$. In what follows, we call a homeomorphism $f\colon D \to \mathbf{C}$ of the class $W^{1,1}_{\mathrm{loc}}$ *regular* if $J_f(z) > 0$ a.e. Note that every regular homeomorphism satisfies a Beltrami equation.

The *Beltrami equation* is the equation of the form

$$f_{\overline{z}} = \mu(z) \cdot f_z \tag{1.1}$$

where $f_{\overline{z}} = \overline{\partial} f = (f_x + i f_y)/2$, $f_z = \partial f = (f_x - i f_y)/2$, $z = x + iy$, and f_x and f_y are partial derivatives of f in x and y, correspondingly, and $\mu\colon D \to \mathbf{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. The function μ is called the *complex coefficient* and

$$K_\mu(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|} \tag{1.2}$$

the *dilatation* of the equation (1.1). The Beltrami equation (1.1) is said to be *degenerate* if $\operatorname{ess\,sup} K_\mu(z) = \infty$.

Given a point z_0 in $\overline{D}$, the *tangential dilatation* of (1.1) with respect to z_0 is the function

$$K^T_\mu(z, z_0) = \frac{\left|1 - \frac{\overline{z - z_0}}{z - z_0}\mu(z)\right|^2}{1 - |\mu(z)|^2}, \tag{1.3}$$

see [RSY$_1$]–[RSY$_2$], cf. the corresponding terms and notations in [An$_1$]–[An$_3$], [Ch], [GMSV], [Le] and [RW]. If f is a regular homeomorphism, then for a.e. $z \in D$

$$K^T_\mu(z, z_0) = \frac{|f_\theta(z)|^2}{r^2 J_f(z)} \tag{1.4}$$

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where $z = z_0 + re^{i\theta}$, see, e.g., [RSY$_1$, (2.4)].

Recall that a function $f\colon D \to \mathbf{C}$ is *absolutely continuous on lines*, abbr. $f \in$ ACL, if, for every closed rectangle R in D whose sides are parallel to the coordinate axes, $f|R$ is absolutely continuous on almost all line segments in R which are parallel to the sides of R. In particular, f is ACL (possibly modified on a set of Lebesgue measure zero) if it belongs to the Sobolev class $W^{1,1}_{\rm loc}$ of locally integrable functions with locally integrable first generalized derivatives and, conversely, if $f \in$ ACL has locally integrable first partial derivatives, then $f \in W^{1,1}_{\rm loc}$, see, e.g., [MP, 1.2.4]. Note that, if $f \in$ ACL, then f has partial derivatives f_x and f_y a.e. and, for a sense-preserving ACL homeomorphism $f\colon D \to \mathbf{C}$, the Jacobian $J_f(z) = |f_z|^2 - |f_{\overline{z}}|^2$ is nonnegative a.e. In this case, the *complex dilatation* μ_f of f is the ratio $\mu(z) = f_{\overline{z}}/f_z$, if $f_z \neq 0$ and $\mu(z) = 0$ otherwise, and the *dilatation* K_f of f is $K_\mu(z)$, see (1.2).

2. On ring Q-homeomorphisms

Recall that, given a family of paths Γ in $\overline{\mathbf{C}}$, a Borel function $\rho\colon \overline{\mathbf{C}} \to [0,\infty]$ is called *admissible* for Γ, abbr. $\rho \in \operatorname{adm}\Gamma$, if

$$\int\limits_\gamma \rho(z)\,|dz| \geq 1 \tag{2.1}$$

for each $\gamma \in \Gamma$. The *modulus* of Γ is defined by

$$M(\Gamma) = \inf_{\rho\in\operatorname{adm}\Gamma} \int\limits_{\mathbf{C}} \rho^2(z)\,dx\,dy. \tag{2.2}$$

Given a domain D and two sets E and F in $\overline{\mathbf{C}}$, $\Delta(E,F,D)$ denotes the family of all paths $\gamma\colon [a,b] \to \overline{\mathbf{C}}$ which join E and F in D, i.e., $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in D$ for $a < t < b$. We set $\Delta(E,F) = \Delta(E,F,\overline{\mathbf{C}})$ if $D = \overline{\mathbf{C}}$. Recall that a *ring domain*, or shortly a *ring* in $\overline{\mathbf{C}}$, is a domain R whose complement $\overline{\mathbf{C}} \setminus R$ consists of two components.

Motivated by the ring definition of quasiconformality in [Ge], the following notion was introduced in [RSY$_1$]–[RSY$_2$]. Let D be a domain in $\mathbf{C}$, $z_0 \in D$, and $Q\colon D \to [0,\infty]$ a measurable function. A homeomorphism $f\colon D \to \mathbf{C}$ is called a *ring Q-homeomorphism* at the point z_0 if

$$M(\Delta(fC_1, fC_2, fD)) \leq \int\limits_A Q(z)\cdot\eta^2(|z-z_0|)\,dx\,dy \tag{2.3}$$

for every ring

$$A = A(z_0, r_1, r_2) = \{z \in \mathbf{C}\colon r_1 < |z-z_0| < r_2\},\ 0 < r_1 < r_2 < \operatorname{dist}(z_0, \partial D),$$

and every measurable function $\eta\colon (r_1,r_2) \to [0,\infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)\,dr = 1 \tag{2.4}$$

and where $C_1 = \{z \in \mathbf{C}\colon |z-z_0| = r_1\}$ and $C_2 = \{z \in \mathbf{C}\colon |z-z_0| = r_2\}$.

Recall a criterion of ring Q-homeomorphisms obtained in [RS, Theorem 2.1], see also [MRSY, Theorem 7.2]. Below we use the standard conventions $a/\infty = 0$ for $a \neq \infty$ and $a/0 = \infty$ if $a > 0$ and $0\cdot\infty = 0$, see, e.g., [Sa, p. 6].

Lemma 2.1. *Let D be a domain in $\mathbf{C}$ and $Q\colon D \to [0,\infty]$ a measurable function. A homeomorphism $f\colon D \to \mathbf{C}$ is a ring Q-homeomorphism at a point $z_0 \in D$ if and only if for every $0 < r_1 < r_2 < d_0 = \operatorname{dist}(z_0, \partial D)$,*

$$M(\Delta(fC_1, fC_2, fD)) \le \frac{2\pi}{I}, \tag{2.5}$$

where $C_1 = \{z \in \mathbf{C}\colon |z - z_0| = r_1\}$, $C_2 = \{z \in \mathbf{C}\colon |z - z_0| = r_2\}$ and

$$I = I(r_1, r_2) = \int_{r_1}^{r_2} \frac{dr}{r q_{z_0}(r)}, \tag{2.6}$$

where $q_{z_0}(r)$ is the mean value of $Q(z)$ over the circle $|z - z_0| = r$.

Note that the infimum from the right hand side in (2.3) holds for the function

$$\eta_0(r) = \frac{1}{I r q_{z_0}(r)}. \tag{2.7}$$

3. The main results

Theorem 3.1. *Let $f\colon D \to \mathbf{C}$ be a regular homeomorphism. Then f is a ring Q-homeomorphisms at a point $z_0 \in D$ with $Q(z) = K_\mu^T(z, z_0)$, $\mu = \mu_f$.*

Proof. Without loss of generality, we may assume that $z_0 = 0 \in D$. Consider the ring $R = \{z \in \mathbf{C}\colon r_1 < |z| < r_2\}$. Then there is a conformal map h mapping the ring fR onto a ring $R^* = \{w\colon 1 < |w| < L\}$.

Let Γ^* be the family of paths joining boundary components $|w| = 1$ and $|w| = L$ of the ring R^*. Then, in view of conformal invariantnce of modulus, $M(\Gamma^*) = M(\Gamma)$, where Γ is the family of all path joining the boundary components of the ring fR . Thus,

$$M(\Gamma) = \frac{4\pi^2}{\int\limits_{R^*} \frac{du\,dv}{|w|^2}}.$$

Denote by C_r, $r_1 < r < r_2$, circles $\{z\colon |z| = r\}$. For $g = h \circ f$, we have that $g \in W^{1,1}_{\mathrm{loc}}(R)$, and hence g is a.e. differentiable and absolutely continuous on C_r for a.e. $r \in (r_1, r_2)$. The latter follows from the invariance of the class $W^{1,1}_{\mathrm{loc}}$ under locally quasi-isometric transformations of coordinates, see, e.g., [Ma, 1.1.7]. Note that

$$\int_{r_1}^{r_2} \int_0^{2\pi} \frac{J_g(re^{i\theta})}{|g(re^{i\theta})|^2}\, r\, dr\, d\theta \le \int\limits_{R^*} \frac{du\,dv}{|w|^2} = \frac{(2\pi)^2}{M(\Gamma)}, \tag{3.1}$$

where $w = u + iv$, J_g is the Jacobian of g, see, e.g., [LV, Lemma III.3.3].

Now, we have

$$2\pi \le \int\limits_{C_r} |d \arg g| \le \int\limits_{C_r} \frac{|dg(z)|}{|g(z)|} = \int_0^{2\pi} \frac{|g_\theta(re^{i\theta})|}{|g(re^{i\theta})|}\, d\theta$$

for a.e. $r \in (r_1, r_2)$ and applying the Schwarz inequality, see, e.g., [BB, Theorem I.4], we obtain that

$$(2\pi)^2 \le \left(\int_0^{2\pi} \frac{|g_\theta(re^{i\theta})|}{|g(re^{i\theta})|}\, d\theta \right)^2 \le \int_0^{2\pi} \frac{|g_\theta(re^{i\theta})|^2}{J(re^{i\theta})}\, d\theta \int_0^{2\pi} \frac{J(re^{i\theta})}{|g(re^{i\theta})|^2}\, d\theta,$$

i.e.,

$$\frac{2\pi}{r\frac{1}{2\pi}\int_0^{2\pi}\frac{|g_\theta(re^{i\theta})|^2}{r^2 J(re^{i\theta})}d\theta} \le r \int_0^{2\pi} \frac{J(re^{i\theta})}{|g(re^{i\theta})|^2}\, d\theta. \tag{3.2}$$

Setting, see (1.4),

$$k(r) = \frac{1}{2\pi}\int_0^{2\pi} \frac{|g_\theta(re^{i\theta})|^2}{r^2 J(re^{i\theta})} d\theta = \frac{1}{2\pi r}\int_{C_r} K_\mu^T(z, z_0)\, |dz|$$

and, integrating the both sides of the inequality (3.2) over $r \in (r_1, r_2)$, we see that

$$2\pi \int_{r_1}^{r_2} \frac{dr}{r\, k(r)} \le \int_{r_1}^{r_2} r\, dr \int_0^{2\pi} \frac{J(re^{i\theta})}{|g(re^{i\theta})|^2}\, d\theta.$$

Combining the last inequality and (3.1), we have by the Fubini theorem that

$$\int_{r_1}^{r_2} \frac{dr}{r\, k(r)} \le \frac{2\pi}{M(\Gamma)}.$$

Thus,

$$M(\Gamma) \le \frac{2\pi}{\int_{r_1}^{r_2} \frac{dr}{r\, k(r)}}.$$

Finally, applying Lemma 2.1, we obtain the conclusion of the theorem. □

Corollary 3.1. *Every regular homeomorphism $f\colon D \to \mathbf{C}$ is a ring Q-homeomorphism with $Q(z) = K_\mu(z)$, $\mu = \mu_f$, at each point $z_0 \in D$.*

Thus, the theory of ring Q-homeomorphisms can be applied to regular homeomorphisms of the Sobolev class $W^{1,1}_{\rm loc}$ in the plane, see, e.g., [MRSY, Chapter 7].

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