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DIFFERENTIAL POLYNOMIALS AND SHARED VALUES

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Abstract. Let f and g be non-constant meromorphic functions in $\mathbf{C}$, a and b non-zero complex numbers and let n and k be natural numbers satisfying $n \geq 5k+17$. We show that if the differential polynomials $f^n + af^{(k)}$ and $g^n + ag^{(k)}$ share the value b CM, then f and g are either equal or at least closely related.

1. Introduction and statement of results

Inspired by the seminal work of Hayman [6], in recent decades lots of Picard type results on exceptional values of differential polynomials have been proved. In several subsequent papers, essentially from the last 15 years, it has been shown that to some of these results there are also corresponding uniqueness results, involving the concept of shared values. Here, if f and g are two non-constant meromorphic functions in a domain $D \subseteq \mathbf{C}$ and if $a \in \overline{\mathbf{C}} := \mathbf{C} \cup \{\infty\}$, we say that f and g share the value a IM (ignoring multiplicities) if f and g assume the value a at the same points. If f and g assume the value a at the same points and with the same multiplicities, then we say that f and g share a CM (counting multiplicities).

Using this concept, the uniqueness problems mentioned above usually have the following form: Assume that f and g are non-constant meromorphic functions in $\mathbf{C}$ and P is a certain differential polynomial such that $P[f]$ and $P[g]$ share one or possibly several values. Then the question arises under which assumptions (on P, on the number of shared values and so on) we can conclude that $f \equiv g$ or that f and g are closely related in some other way. One of the first and, in our opinion, most important results in this direction is due to Yang and Hua [8].

Theorem A. *Let f and g be non-constant meromorphic functions in $\mathbf{C}$ and $n \geq 11$ be an integer. Assume that $f^n f'$ and $g^n g'$ share a non-zero value CM. Then $f = cg$ for some $c \in \mathbf{C}$ satisfying $c^{n+1} = 1$ or fg is constant and $f(z) = e^{az+b}$ for certain $a, b \in \mathbf{C}$. If f and g are entire, this also holds for $n \geq 7$.*

Similar uniqueness results for entire and meromorphic functions involving differential polynomials like $P[u] := (u^n)^{(k)}$, $P[u] := (u^n(u-1))^{(k)}$ and $P[u] := u^n(u-1)^2u'$ (where n is sufficiently large) have been proved by Fang [2], Lin and Yi [7], among others.

In this paper, we consider another special case of the above problem which, to our best knowledge, hasn't been studied so far: the question whether there hold uniqueness theorems for meromorphic (or entire) functions and the differential polynomial

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$P[f] := f^n + af^{(k)}$. This is motivated by the well-known result of Hayman [6] which says that each function f meromorphic in $\mathbf{C}$ and satisfying $f^n(z) + af'(z) \neq b$ for all $z \in \mathbf{C}$ (where $n \geq 5$ and $a, b \in \mathbf{C}$ with $a \neq 0$) is constant; if f is entire, this holds also for $n \geq 3$ and for $n = 2$, $b = 0$. As Döringer [1] has shown, this remains valid for $f^n + af^{(k)}$ instead of $f^n + af'$ provided that $n \geq k + 4$; if f is entire, it suffices to assume $n \geq 3$ independently of k.

Our main result for meromorphic functions is the following.

Theorem 1. *Let f and g be non-constant meromorphic functions in $\mathbf{C}$, $a, b \in \mathbf{C} \setminus \{0\}$ and let n and k be natural numbers satisfying $n \geq 5k + 17$. Assume that the functions*

$$\psi_f := f^n + af^{(k)} \quad \text{and} \quad \psi_g := g^n + ag^{(k)} \tag{1.1}$$

share the value b CM. Then

$$\frac{\psi_f - b}{\psi_g - b} = \frac{f^n}{g^n} = \frac{af^{(k)} - b}{ag^{(k)} - b} \tag{1.2}$$

or

$$\frac{\psi_f - b}{\psi_g - b} = \frac{f^n}{ag^{(k)} - b} = \frac{af^{(k)} - b}{g^n} \tag{1.3}$$

or $f = g$, $f^{(k)} = g^{(k)} \equiv \frac{b}{a}$.

In fact, we believe that the case (1.3) cannot occur at all, but we were not able to prove this.

If we restrict ourselves to entire functions, we can weaken the assumption on n a bit, and we can exclude the case (1.3).

Theorem 2. *Let f and g be non-constant entire functions, $a, b \in \mathbf{C} \setminus \{0\}$ and let n and k be natural numbers satisfying $n \geq 11$ and $n \geq k + 2$. Assume that the functions ψ_f and ψ_g defined as in (1.1) share the value b CM. Then*

$$\frac{\psi_f - b}{\psi_g - b} = \frac{f^n}{g^n} = \frac{af^{(k)} - b}{ag^{(k)} - b} \tag{1.4}$$

or $f = g$, $f^{(k)} = g^{(k)} \equiv \frac{b}{a}$.

Here, if we impose the further assumption $k = 1$ we can conclude that f and g are identical.

Theorem 3. *Let f and g be non-constant entire functions, $a, b \in \mathbf{C} \setminus \{0\}$ and let $n \geq 11$ be a natural number. Assume that the functions*

$$\psi_f := f^n + af' \quad \text{and} \quad \psi_g := g^n + ag'$$

share the value b CM. Then $f \equiv g$ or f and g are polynomials of degree 1 with the same zero.

We do not know whether in the situation of (1.4) we can conclude that $f \equiv g$ (provided that f and g are transcendental) also for $k \geq 2$. But if we assume that two values are shared CM or that $g = f'$, we can deduce $f \equiv g$ as the following Corollaries show.

Corollary 4. *Let f and g be non-constant meromorphic functions in $\mathbf{C}$, $a, b_1, b_2 \in \mathbf{C} \setminus \{0\}$, $b_1 \neq b_2$ and let n and k be natural numbers satisfying $n \geq 5k + 17$. If the* *functions ψ_f and ψ_g defined as in (1.1) share the values b_1 and b_2 CM, then $f \equiv g$, or f and g are polynomials of degree at most $k-1$ and $f = e^{2\pi ij/n} g$ for some $j \in \mathbf{N}_0$. If f and g are entire, the same holds even for $n \geq \max\{11; k+2\}$.*

Corollary 5. *Let f be a non-constant meromorphic function in $\mathbf{C}$, $a, b \in \mathbf{C}\setminus\{0\}$ and let k and n be natural numbers satisfying $n \geq 5k+17$. If the functions ψ_f and $\psi_{f'}$ defined as in (1.1) share the value b CM, then $f \equiv f'$. If f is entire, this holds also for $n \geq \max\{11; k+2\}$.*

Since $f' \equiv f$ implies that f is entire, we can reformulate the "meromorphic case" of Corollary 5 also in the following way: If f is a meromorphic function in $\mathbf{C}$ with poles and $n \geq 5k+17$, then ψ_f and $\psi_{f'}$ do not share any non-zero value CM.

We do not know whether the restrictions on n and k in our results are best possible.

2. Lemmas

Besides the standard notations and results of Nevanlinna theory [5], we use the following notations: By $N_{p)}(r, f)$ we denote the counting function of those poles of f which have multiplicity at most p, each pole counted with its multiplicity. In the same way, $N_{(p}(r, f)$ will denote the counting function of those poles of f which have multiplicity at least p, each pole counted with its multiplicity. The corresponding reduced counting functions where each pole is counted only once are denoted by $\overline{N}_{p)}(r, f)$ and $\overline{N}_{(p}(r, f)$. Furthermore, by $N(r, f \,|\, g \neq c)$ we denote the counting function for those poles of f which are not zeros of $g - c$. Similar notations like $N(r, f \,|\, g = c)$ or $\overline{N}(r, f \,|\, g \neq c)$ which should be self-explanatory now are also used. By $S(r, f)$ we denote an arbitrary term of the form $o(T(r, f))$ for $r \to \infty$, r outside some set of finite measure.

The following famous estimate [5, Theorem 3.2] plays an important role in the proof of our main results.

Lemma 6. (Milloux's inequality) *If f is a meromorphic function in the complex plane and $k \in \mathbf{N}$, $c \neq 0$, then*

$$T(r, f) \leq \overline{N}(r, f) + N\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f^{(k)} - c}\right) - N\left(r, \frac{1}{f^{(k+1)}} \,\middle|\, f^{(k)} \neq c\right) + S(r, f)$$

provided that $f^{(k)} \not\equiv c$.

Additionally, we need the following extension of the famous Tumura Clunie Theorem due to Yi [9].

Lemma 7. *Let $n \geq 2$ be a natural number and P be a differential polynomial*[1] *of degree $\deg(P) \leq n-1$ and weight $w(P)$ with constant coefficients. Let f be a*

[1]For the convenience of the reader, we recall the definition of a differential polynomial and its degree and weight: By $\mathscr{M}(\mathbf{C})$ we denote the space of all functions meromorphic in $\mathbf{C}$. A mapping $M\colon \mathscr{M}(\mathbf{C}) \longrightarrow \mathscr{M}(\mathbf{C})$ given by

$$M[u] = a \cdot \prod_{\nu=1}^{d} u^{(k_\nu)} \quad \text{for all } u \in \mathscr{M}(\mathbf{C})$$

with $d \in \mathbf{N}_0$, $k_1, \ldots, k_d \in \mathbf{N}_0$ and a function $a \in \mathscr{M}(\mathbf{C})$, $a \not\equiv 0$ is called a *differential monomial* of *degree* $\deg(M) := d$ and *weight* $w(M) := \sum_{\nu=1}^{d}(1 + k_\nu)$. If $a \equiv 1$, we say that M is *normalized.*

non-constant meromorphic function and

$$\psi := f^n + P[f].$$

If $P[f] \not\equiv 0$*, then*

$$(n - \deg(P)) \cdot T(r,f) \le (1 + w(P) - \deg(P)) \cdot \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f}\right)$$
$$+ \overline{N}\left(r, \frac{1}{\psi}\right) + S(r,f).$$

The following extension of the lemma on the logarithmic derivative goes back to Döringer [1, Lemma 1 (i)] (see also [3, Lemma 5]).

Lemma 8. *Let* Q *be a differential polynomial with meromorphic coefficients* c_j $(j = 1, \ldots, p)$*. Then*

$$m(r, Q[f]) \le \deg(Q) \cdot m(r,f) + \sum_{j=1}^{p} m(r, c_j) + S(r,f)$$

holds for all meromorphic functions f *and all* $r > 0$*.*

Finally, the following result from [4, Theorem 9] is useful in the proof of Corollary 5.

Lemma 9. *Let*

$$H = \sum_{j=1}^{t} a_j M_j$$

be a homogeneous differential polynomial with normalized differential monomials M_j *and constant coefficients* a_j*. Assume that*

$$w(M_1) = \cdots = w(M_s) > w(M_j) \quad \text{for all } j = s+1, \ldots, t$$

with an $s \in \{1, \ldots, t\}$ *and that* $c := \sum_{j=1}^{s} a_j \ne 0$*. If* f *is an entire function without zeros in* $\mathbf{C}$ *and with* $H[f] \equiv 0$*, then* f *has the form* $f(z) = e^{az+b}$ *with certain* $a, b \in \mathbf{C}$*.*

3. Proof of Theorem 1 and Theorem 2

We prove Theorems 1 and 2 simultaneously. So we assume that f and g are meromorphic functions such that ψ_f and ψ_g share the value $b \ne 0$ CM and that $n \ge \max\{11; k+2\}$. Furthermore, we assume that f and g are entire or that $n \ge 5k+17$.

Since the proof is rather long, we first give a brief sketch of the main ideas.

Sketch of the proof. W.l.o.g. we may assume $a = 1$. We consider the functions

$$\varphi_f := \frac{f^n}{\psi_f - b} \quad \text{and} \quad \varphi_g := \frac{g^n}{\psi_g - b}$$

A sum $P := M_1 + \cdots + M_p$ of differential monomials $M_1, \ldots, M_p$ which are linearly independent over $\mathscr{M}(\mathbf{C})$ is called a *differential polynomial* of *degree* $\deg(P) := \max\{\deg(M_1), \ldots, \deg(M_p)\}$ and *weight* $w(P) := \max\{w(M_1), \ldots, w(M_p)\}$. Obviously, we have $\deg(P) \le w(P)$.

If $\deg(M_1) = \cdots = \deg(M_p) =: d$, we call P *homogeneous (of degree d)*.

For every differential polynomial P there exists a differential polynomial P' such that $P'[u](z) = (P[u])'(z)$ for all $u \in \mathscr{M}(\mathbf{C})$ and all $z \in \mathbf{C}$. It is easy to see that $\deg(P') = \deg(P)$ and $w(P') \ge w(P)$.

where ψ_f and ψ_g are defined as in (1.1). It is easy to see that $T(r,\varphi_f)$ behaves more or less like $(n \pm (k+1)) \cdot T(r,f)$; in particular, we have

$$T(r,\varphi_f) \geq (n-k-1) \cdot T(r,f) + S(r,f).$$

We want to apply the Second Fundamental Theorem to φ_f to deduce some estimate of the kind

$$T(r,\varphi_f) \leq c \cdot T(r,f) + S(r,f) \tag{3.1}$$

where $c > 0$ is some constant independent of n. From (3.1) we would obtain

$$T(r,f) \leq \frac{c}{n-k-1} \cdot T(r,f) + S(r,f)$$

which is a contradiction if n is large enough. (As stated in the Theorem, $n \geq 5k+17$ in the meromorphic case resp. $n \geq \max\{11; k+2\}$ in the entire case suffices.)

To get an estimate as in (3.1), we study the reduced (!) counting functions for the zeros and poles of φ_f and for the zeros of $\varphi_f - 1$. The zeros of φ_f are the zeros of f while the zeros of $\varphi_f - 1$ are the zeros of $f^{(k)} - b$ and the poles of f. So by the First Fundamental Theorem the reduced counting functions $\overline{N}\left(r, \frac{1}{\varphi_f}\right)$ and $\overline{N}\left(r, \frac{1}{\varphi_f - 1}\right)$ can be estimated by $T(r,f) + S(r,f)$ and by $(k+2) \cdot T(r,f) + S(r,f)$, resp.

The poles of φ_f are the zeros of $\psi_f - b$. It is the main difficulty in the proof to get some estimate for the corresponding counting function $\overline{N}\left(r, \frac{1}{\psi_f - b}\right)$.

Here, multiple zeros of $\psi_f - b$ are easy to control; their counting function turns out to be at most $(3+k) \cdot T(r,f) + S(r,f)$ (resp. at most $(2T(r,f) + S(r,f)$ in the entire case). So we can restrict our considerations to simple zeros of $\psi_f - b$.

Now it's helpful to introduce the auxiliary function

$$D := \frac{\psi_f'}{\psi_f - b} - \frac{\psi_g'}{\psi_g - b}$$

which has several nice properties: By the lemma on the logarithmic derivative, $m(r,D)$ is small, and since $\psi_f - b$ and $\psi_g - b$ share 0 CM, D has no other poles than possibly the poles of f and g, and all poles of D are simple (since D consists of logarithmic derivatives). If z_0 is a simple zero of $\psi_f - b$ and hence of $\psi_g - b$, then one can calculate

$$D(z_0) = \frac{1}{2} \cdot \left(\frac{\psi_f''}{\psi_f'} - \frac{\psi_g''}{\psi_g'} \right)(z_0),$$

so z_0 is a zero of

$$D - \frac{1}{2} \cdot \left(\frac{\psi_f''}{\psi_f'} - \frac{\psi_g''}{\psi_g'} \right) =: \widetilde{H}.$$

This means that we could estimate our counting function $\overline{N}_{1)}\left(r, \frac{1}{\psi_f - b}\right)$ by $T\left(r, \widetilde{H}\right)$. But here, one major problem occurs: $m(r,\widetilde{H})$ is small once more, but it seems that $N(r,\widetilde{H})$ cannot be controlled in the required way.

The solution to this problem is the following: If z_0 is a simple zero of $\psi_f - b$, then we use the equation $f^n(z_0) = b - f^{(k)}(z_0)$ to replace those terms in ψ_f' which are "large" in the sense of Nevanlinna theory (i.e. with characteristic $n \cdot T(r,f)$) by smaller ones (with characteristic $\sim c \cdot T(r,f)$ where c is independent of n); we obtain

$$\begin{aligned}\frac{\psi_f''}{\psi_f'}(z_0) &= \frac{n(n-1)f^n f'^2 + nf^{n+1}f'' + f^2 f^{(k+2)}}{nf^{n+1}f' + f^2 f^{(k+1)}}(z_0) \\ &= \frac{n(n-1)f'^2\left(b - f^{(k)}\right) + nff''\left(b - f^{(k)}\right) + f^2 f^{(k+2)}}{nff'\left(b - f^{(k)}\right) + f^2 f^{(k+1)}}(z_0).\end{aligned}$$

Therefore, instead of $\widetilde{H}$ we introduce the more complicated auxiliary function

$$H := D - Q[f] + Q[g]$$

where

$$Q[f] := \frac{1}{2} \cdot \frac{n(n-1)f'^2\left(b - f^{(k)}\right) + nff''\left(b - f^{(k)}\right) + f^2 f^{(k+2)}}{f^2 f^{(k+1)} + nff'\left(b - f^{(k)}\right)}.$$

Then every simple zero of $\psi_f - b$ is a zero of H. The main advantage of H is that it does not contain any terms involving f^n any more.

We assume that $H \not\equiv 0$. Then we obtain

$$N_{1)}\left(r, \frac{1}{\psi_f - b}\right) \le \overline{N}\left(r, \frac{1}{H}\right) \le T(r,H) + O(1).$$

Here, as already mentioned, D has no other poles than possibly the poles of f and g, and it consists of logarithmic derivatives, so $m(r,D)$ is small and $N(r,D) \le \overline{N}(r,f) + \overline{N}(r,g)$ is "not too large".

Therefore, it remains to consider $Q[f]$ and $Q[g]$. Using the First Fundamental Theorem, the counting function for the poles of $Q[f]$ (which are the zeros of the denominator of $Q[f]$ and the poles of f) can be estimated by $(k+5) \cdot T(r,f) + S(r,f)$. But what can we say about $m(r, Q[f])$ (and $m(r, Q[g])$)? Now something marvellous happens: It turns out that

$$Q[f] = n \cdot \frac{f'}{f} + \frac{f^{(k+1)}}{f^{(k)} - b} + \frac{(V[f])'}{V[f]} \quad \text{where} \quad V[f] := n \cdot \frac{f'}{f} - \frac{f^{(k+1)}}{f^{(k)} - b},$$

i.e. $Q[f]$ and $Q[g]$ are combinations of logarithmic derivatives once more, so their proximity functions are small.

This gives us the desired estimate (3.1), hence a contradiction. (In fact, the whole truth is a bit more complicated. To be precise, instead of $T(r,f)$ we have to deal with $\max\{T(r,f); T(r,g)\}$ in some of the fore-going considerations, and sometimes we use more intricate and refined estimates than in this outline.)

So we have shown $H \equiv 0$, i.e. we have deduced one first identity linking f and g. In the remaining parts of the proof, we gradually obtain stronger and stronger identities: First, we show that $V[f] \equiv c \cdot V[g]$ for some constant $c \in \mathbf{C} \setminus \{0\}$. Then we deduce that c is rational, i.e. $c = \frac{p}{q}$ for some $p, q \in \mathbf{Z}$. Integrating the identity $V[f] \equiv c \cdot V[g]$ and combining this with $H \equiv 0$ gives

$$\left(\frac{\psi_f - b}{\psi_g - b}\right)^{2q} = \alpha \cdot \frac{\left(f^{(k)} - b\right)^{2q}}{g^{n(q-p)}\left(g^{(k)} - b\right)^{q+p}}$$

for some $\alpha \in \mathbf{C}$ and two further identities of this kind. Here, by our assumption, the function on the left-hand side has no other zeros and poles than possibly the poles of f and g. This gives us certain connections between the zeros of f, g, $f^{(k)} - b$ and

$g^{(k)} - b$. A careful analysis of these connections yields a contradiction for $c \notin \{-1; 1\}$; here, the case $c > 0$, $c \neq 1$ proves to be the most recalcitrant. The main tool in this part of the proof is a repeated application of Milloux's inequality. So we end up with $c = \pm 1$, i.e.

$$\frac{f^n}{f^{(k)} - b} = d \cdot \frac{g^n}{g^{(k)} - b} \quad \text{or} \quad \frac{f^n}{f^{(k)} - b} = d \cdot \frac{g^{(k)} - b}{g^n}$$

for some $d \in \mathbf{C}$, and now it's easy to show that $d = 1$ which gives the assertion. Additionally, for entire functions the case $c = -1$ can be excluded.

After this outline we turn to the details of the proof.

Proof of Theorems 1 and 2. W.l.o.g. we may assume $a = 1$. (Otherwise we replace f and g by cf and cg and b by bc^n where c is an appropriate constant satisfying $c^{1-n} = a$.)

Of course, ψ_f is non-constant since otherwise from the lemma on the logarithmic derivative we would obtain

$$\begin{aligned} n \cdot T(r, f) &= T\left(r, f^n\right) = T\left(r, f^{(k)}\right) + O(1) \\ &\leq T(r, f) + m\left(r, \frac{f^{(k)}}{f}\right) + k \cdot \overline{N}(r, f) + O(1) \\ &\leq (k+1) \cdot T(r, f) + S(r, f), \end{aligned} \tag{3.2}$$

which in view of $n \geq k + 2$ would give $T(r, f) = S(r, f)$, a contradiction. For the same reason, ψ_g is non-constant, too.

The value sharing assumption implies that $\frac{\psi_f - b}{\psi_g - b}$ has no zeros and poles, with the possible exception of the poles of f and of g. In view of $n \geq k + 2$, a pole of f of order p is a pole of ψ_f of order np, and the same holds for ψ_g. These facts will be used repeatedly in the following considerations.

(1) First, we consider the case that $f^{(k)} \equiv b$. Then f is a polynomial of degree k, and the functions

$$\widetilde{\psi}_f := \psi_f - b = f^n \quad \text{and} \quad \widetilde{\psi}_g := \psi_g - b = g^n + g^{(k)} - b$$

share the value 0 CM. Since in this case f is a polynomial, $\widetilde{\psi}_g$ has only finitely many zeros, and all zeros of $\widetilde{\psi}_g$ have multiplicity at least n (since every such zero is a zero of f^n of the same multiplicity).

We assume by negation $g^{(k)} \not\equiv b$. If we apply Yi's extension of the Tumura–Clunie Theorem (Lemma 7) with $P[u] = u^{(k)} - b$ and Döringer's Lemma (Lemma 8), we obtain

$$\begin{aligned} (n-1) \cdot T(r, g) &\leq \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{\psi_g - b}\right) + (k+1) \cdot \overline{N}(r, g) + S(r, g) \\ &\leq T\left(r, g\right) + \frac{1}{n} \cdot N\left(r, \frac{1}{\psi_g - b}\right) + (k+1) \cdot \overline{N}(r, g) + S(r, g) \\ &\leq T\left(r, g\right) + \frac{1}{n} \cdot T\left(r, \psi_g - b\right) + (k+1) \cdot \overline{N}(r, g) + S(r, g) \\ &\leq T\left(r, g\right) + \frac{1}{n} \cdot n \cdot T\left(r, g\right) + (k+1) \cdot \overline{N}(r, g) + S(r, g) \\ &= 2 \cdot T(r, g) + (k+1) \cdot \overline{N}(r, g) + S(r, g), \end{aligned}$$

hence

$$(n-3)\cdot T(r,g) \le (k+1)\cdot \overline{N}(r,g) + S(r,g).$$

This gives a contradiction both in the meromorphic case (where $n-3 > k+1$) and in the entire case (where $\overline{N}(r,g)=0$). So $g^{(k)} \equiv b$.

Since $\widetilde{\psi}_f = f^n$ and $\widetilde{\psi}_g = g^n$ share the value 0 CM and since f and g have turned out to be polynomials, there exists some $\alpha \in \mathbf{C}$ such that $f = \alpha g$. From this and $f^{(k)} \equiv b \equiv g^{(k)} \not\equiv 0$ we see that even $f \equiv g$. So the assertion of the theorem holds in this case.

The case that $g^{(k)} \equiv b$ can be treated in the same way.

(2) From now on, we assume $f^{(k)} \not\equiv b$ and $g^{(k)} \not\equiv b$. We define

$$\varphi_f := \frac{f^n}{\psi_f - b} \quad \text{and} \quad \varphi_g := \frac{g^n}{\psi_g - b}.$$

If φ_f would be constant, $\varphi_f \equiv c$, then we would have

$$f^n(1-c) \equiv c\cdot\left(f^{(k)} - b\right); \tag{3.3}$$

here $c \ne 1$ in view of our assumption that $f^{(k)} \not\equiv b$, so as in (3.2) we would obtain

$$n\cdot T(r,f) = T\left(r, f^{(k)}\right) + O(1) \le (k+1)\cdot T(r,f) + S(r,f),$$

i.e. $T(r,f) = S(r,f)$, a contradiction. Therefore φ_f is not constant, and neither is φ_g. Since φ_f is analytic at the poles of f, we have

$$\overline{N}\left(r,\varphi_f\right) \le \overline{N}\left(r, \frac{1}{\psi_f - b}\right), \tag{3.4}$$

and from

$$\frac{1}{\varphi_f} = 1 + \frac{f^{(k)} - b}{f^n} \quad \text{and} \quad \frac{1}{\varphi_f - 1} = \frac{\psi_f - b}{b - f^{(k)}} = -1 - \frac{f^n}{f^{(k)} - b}$$

we obtain

$$\overline{N}\left(r, \frac{1}{\varphi_f}\right) = \overline{N}\left(r, \frac{f^{(k)} - b}{f^n}\right) \quad \text{and} \quad \overline{N}\left(r, \frac{1}{\varphi_f - 1}\right) = \overline{N}\left(r, \frac{f^n}{f^{(k)} - b}\right). \tag{3.5}$$

Furthermore, we note that each zero of f which is not a zero of $f^{(k)} - b$ is a zero of φ_f of order at least n, i.e.

$$\overline{N}\left(r, \frac{1}{f} \,\Big|\, f^{(k)} \ne b\right) \le \frac{1}{n}\cdot N\left(r, \frac{1}{\varphi_f}\right). \tag{3.6}$$

We apply the Second Fundamental Theorem and use (3.4), (3.5) and (3.6) to obtain

$$\begin{aligned}
T(r,\varphi_f) &\le \overline{N}\left(r,\varphi_f\right) + \overline{N}\left(r, \frac{1}{\varphi_f}\right) + \overline{N}\left(r, \frac{1}{\varphi_f - 1}\right) + S(r,\varphi_f) \\
&\le \overline{N}\left(r, \frac{1}{\psi_f - b}\right) + \overline{N}\left(r, \frac{f^{(k)} - b}{f^n}\right) + \overline{N}\left(r, \frac{f^n}{f^{(k)} - b}\right) + S(r,f) \\
&\le \overline{N}\left(r, \frac{1}{\psi_f - b}\right) + \overline{N}\left(r, \frac{1}{f} \,\Big|\, f^{(k)} \ne b\right) + \overline{N}\left(r, \frac{1}{f^{(k)} - b}\right) \\
&\quad + \overline{N}(r,f) + S(r,f) \\
&\le \overline{N}\left(r, \frac{1}{\psi_f - b}\right) + \frac{1}{n}\cdot N\left(r, \frac{1}{\varphi_f}\right) + \overline{N}\left(r, \frac{1}{f^{(k)} - b}\right) + \overline{N}(r,f) + S(r,f);
\end{aligned} \tag{3.7}$$

here in the third estimate we have used the fact that a common zero of f and $f^{(k)}-b$ can be a pole of at most one of the functions $\frac{f^{(k)}-b}{f^n}$ and $\frac{f^n}{f^{(k)}-b}$ (since these two functions have no common poles at all).

As mentioned in the sketch of proof on p. 51, the main difficulty in the following will be to obtain an estimate for $\overline{N}\left(r, \frac{1}{\psi_f - b}\right)$.

For meromorphic w we define

$$P[w] := 2w^2 w^{(k+1)} + 2nww'\left(b - w^{(k)}\right).$$

We want to show that $P[f] \not\equiv 0$. Assume that $P[f] \equiv 0$. Then in view of $f^{(k)} - b \not\equiv 0$ we have

$$\frac{f^{(k+1)}}{f^{(k)} - b} \equiv n \cdot \frac{f'}{f},$$

and integration gives

$$f^n = c \cdot \left(f^{(k)} - b\right)$$

for an appropriate constant $c \neq 0$. In the same way as with (3.3) this leads to a contradiction.

So $P[f] \not\equiv 0$, and by the same reasoning we obtain $P[g] \not\equiv 0$.

For meromorphic w we set

$$\begin{aligned} Q[w] &:= \frac{1}{P[w]} \cdot \left(n(n-1)w'^2\left(b - w^{(k)}\right) + nww''\left(b - w^{(k)}\right) + w^2 w^{(k+2)}\right) \\ &= \frac{n(n-1)w'^2\left(b - w^{(k)}\right) + nww''\left(b - w^{(k)}\right) + w^2 w^{(k+2)}}{2w^2 w^{(k+1)} + 2nww'\left(b - w^{(k)}\right)}. \end{aligned}$$

Furthermore, we define

$$D := \frac{\psi_f'}{\psi_f - b} - \frac{\psi_g'}{\psi_g - b} \quad \text{and} \quad H := D - Q[f] + Q[g].$$

(3) We consider the case $H \not\equiv 0$.

(3.1) At first, we deduce an estimate for the counting function of the simple zeros of $\psi_f - b$.

(a) Let z_0 be a simple zero of $\psi_f - b$ and hence of $\psi_g - b$. Then $\frac{\psi_f'}{\psi_f - b}$ has the Laurent expansion

$$\frac{\psi_f'}{\psi_f - b}(z) = \frac{\psi_f'(z_0) + \psi_f''(z_0)(z - z_0) + \ldots}{\psi_f'(z_0)(z - z_0) + \frac{1}{2}\psi_f''(z_0)(z - z_0)^2 + \ldots} = \frac{1}{z - z_0} + \frac{1}{2} \cdot \frac{\psi_f''}{\psi_f'}(z_0) + \ldots.$$

Since an analogous expansion holds for $\frac{\psi_g'}{\psi_g - b}$, we obtain

$$D(z_0) = \frac{1}{2} \cdot \frac{\psi_f''}{\psi_f'}(z_0) - \frac{1}{2} \cdot \frac{\psi_g''}{\psi_g'}(z_0). \tag{3.8}$$

We insert

$$b = \psi_f(z_0) = f^n(z_0) + f^{(k)}(z_0)$$

into

$$\frac{\psi_f''}{\psi_f'} = \frac{n(n-1)f^{n-2}f'^2 + nf^{n-1}f'' + f^{(k+2)}}{nf^{n-1}f' + f^{(k+1)}} = \frac{n(n-1)f^n f'^2 + nf^{n+1}f'' + f^2 f^{(k+2)}}{nf^{n+1}f' + f^2 f^{(k+1)}}$$

and deduce

$$\frac{\psi_f''}{\psi_f'}(z_0) = \frac{n(n-1)f'^2\left(b - f^{(k)}\right) + nff''\left(b - f^{(k)}\right) + f^2 f^{(k+2)}}{nff'\left(b - f^{(k)}\right) + f^2 f^{(k+1)}}(z_0) = 2Q[f](z_0).$$

In the same way we get $\frac{\psi_g''}{\psi_g'}(z_0) = 2Q[g](z_0)$. Inserting this into (3.8), we see

$$D(z_0) = Q[f](z_0) - Q[g](z_0),$$

hence $H(z_0) = 0$. This consideration shows

$$N_{1)}\left(r, \frac{1}{\psi_f - b}\right) = N_{1)}\left(r, \frac{1}{\psi_g - b}\right) \le \overline{N}\left(r, \frac{1}{H}\right) \le T(r, H) + O(1). \tag{3.9}$$

We discuss $m(r, H)$ and $N(r, H)$ separately, keeping in mind that $H = D - Q[f] + Q[g]$.

(b) By the lemma on the logarithmic derivative we have

$$m(r, D) \le S(r, \psi_f) + S(r, \psi_g) \le S(r, f) + S(r, g). \tag{3.10}$$

The estimate for $m(r, Q[f])$ and $m(r, Q[g])$ is more complicated. We set

$$L[w] := \frac{w^{(k+1)}}{w^{(k)} - b}, \quad \widetilde{L}[w] := \frac{w^{(k+2)}}{w^{(k)} - b} \quad \text{and} \quad V[w] := n \cdot \frac{w'}{w} - L[w].$$

Then we have

$$(L[w])' = \widetilde{L}[w] - (L[w])^2$$

for all non-constant meromorphic functions w with $w^{(k)} \not\equiv b$, hence

$$\begin{aligned} 2Q[w] &= \frac{n(n-1)w'^2 + nww'' - w^2\widetilde{L}[w]}{nww' - w^2 L[w]} \\ &= n \cdot \frac{w'}{w} + \frac{nww'' - w^2(L[w])' - w^2(L[w])^2 - nw'^2 + nww'L[w]}{nww' - w^2 L[w]} \\ &= n \cdot \frac{w'}{w} + L[w] + \frac{n\frac{w''}{w} - n\frac{w'^2}{w^2} - (L[w])'}{n\frac{w'}{w} - L[w]} \\ &= n \cdot \frac{w'}{w} + L[w] + \frac{(V[w])'}{V[w]}, \end{aligned} \tag{3.11}$$

provided that $P[w] \not\equiv 0$ and therefore $V[w] \not\equiv 0$. In particular, this holds for $w = f$ and $w = g$. By the lemma on the logarithmic derivative and the First Fundamental Theorem we easily obtain

$$T(r, V[f]) \le C \cdot T(r, f) + S(r, f),$$

for some $C > 0$. (In fact, one can choose $C = k + 3$, but this exact value of C is not needed in the following.) Therefore we deduce $S(r, V[f]) \le S(r, f)$. Now from (3.11) we conclude that

$$m(r, Q[f]) = S(r, f) + S\left(r, f^{(k)}\right) + S(r, V[f]) = S(r, f).$$

Similarly, we have $m(r, Q[g]) = S(r, g)$. Combining this with (3.10) gives

$$m(r, H) = S(r, f) + S(r, g). \tag{3.12}$$

(c) Since ψ_f and ψ_g share b CM, the only possible poles of D are the poles of f and g, and of course all poles of D are simple. But, as we can see from (3.11), the poles of f resp. g are poles of $Q[f]$ resp. $Q[g]$. Therefore we have

$$N(r,H) \le N(r,Q[f]) + N(r,Q[g]). \tag{3.13}$$

To estimate the counting function for the poles of $Q[f]$, let's take a look at a simple zero z_0 of $f^{(k)} - b$ which is not a zero of f. Then z_0 is a simple pole of $L[f]$ and hence of $V[f]$ and $\frac{(V[f])'}{V[f]}$, and so the residues are

$$\operatorname{Res}(L[f];z_0) = 1, \quad \operatorname{Res}\left(\frac{(V[f])'}{V[f]};z_0\right) = -1.$$

Thus, in view of (3.11), we conclude that $Q[f]$ is analytic at z_0.

Therefore, each pole of $Q[f]$ is a zero or pole of f, a multiple zero of $f^{(k)} - b$ or a zero or pole of $V[f]$. But the poles of $V[f]$ are zeros or poles of f or zeros of $f^{(k)} - b$ again. Furthermore, all poles of $Q[f]$ are simple. So we deduce

$$N(r,Q[f]) = \overline{N}(r,Q[f]) \le \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}_{(2}\left(r,\frac{1}{f^{(k)}-b}\right) + \overline{N}\left(r,\frac{1}{V[f]}\right).$$

Here, by the First Fundamental Theorem,

$$\begin{aligned} &\overline{N}_{(2}\left(r,\frac{1}{f^{(k)}-b}\right) + \overline{N}\left(r,\frac{1}{V[f]}\right) \\ &\le \overline{N}_{(2}\left(r,\frac{1}{f^{(k)}-b}\right) + T\left(r, n\cdot\frac{f'}{f} - \frac{f^{(k+1)}}{f^{(k)}-b}\right) + O(1) \\ &\le \overline{N}_{(2}\left(r,\frac{1}{f^{(k)}-b}\right) + \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f^{(k)}-b}\right) \\ &\quad + S(r,f) + S\left(r,f^{(k)}\right) \\ &\le \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f^{(k)}-b}\right) + S(r,f), \end{aligned} \tag{3.14}$$

and we arrive at

$$N(r,Q[f]) \le 2\overline{N}(r,f) + 2\overline{N}\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f^{(k)}-b}\right) + S(r,f).$$

An analogous estimate holds for $Q[g]$, too:

$$N(r,Q[g]) \le 2\overline{N}(r,g) + 2\overline{N}\left(r,\frac{1}{g}\right) + N\left(r,\frac{1}{g^{(k)}-b}\right) + S(r,g).$$

(d) Now from (3.9), (3.12), (3.13) and the latter two estimates we obtain

$$\begin{aligned} N_{1)}\left(r,\frac{1}{\psi_f-b}\right) &\le T(r,H) + O(1) \\ &\le N(r,Q[f]) + N(r,Q[g]) + S(r,f) + S(r,g) \\ &\le 2\overline{N}(r,f) + 2\overline{N}(r,g) + 2\overline{N}\left(r,\frac{1}{f}\right) + 2\overline{N}\left(r,\frac{1}{g}\right) \\ &\quad + N\left(r,\frac{1}{f^{(k)}-b}\right) + N\left(r,\frac{1}{g^{(k)}-b}\right) + S(r,f) + S(r,g). \end{aligned} \tag{3.15}$$

(3.2) Now we turn to the multiple zeros of $\psi_f - b$. Assume that z_0 is such a zero. Then $\psi_f'(z_0) = 0$, hence

$$f^n(z_0) + f^{(k)}(z_0) = b \quad \text{and} \quad nf^{n-1}f'(z_0) + f^{(k+1)}(z_0) = 0,$$

and we conclude that either

$$f(z_0) = 0 = f^{(k)}(z_0) - b = f^{(k+1)}(z_0)$$

(i.e. z_0 is a multiple zero of $f^{(k)} - b$) or $f(z_0) \neq 0$, $f^{(k)}(z_0) \neq b$ and

$$0 = n \cdot \frac{f'}{f}(z_0) - \frac{f^{(k+1)}}{f^{(k)} - b}(z_0) = V[f](z_0).$$

Together with (3.14) this shows

$$\begin{aligned} \overline{N}_{(2}\left(r, \frac{1}{\psi_f - b}\right) &\leq \overline{N}_{(2}\left(r, \frac{1}{f^{(k)} - b}\right) + \overline{N}\left(r, \frac{1}{V[f]}\right) \\ &\leq \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{(k)} - b}\right) + S(r,f). \end{aligned} \tag{3.16}$$

(3.3) If we combine (3.16) with (3.15) we obtain the desired estimate for the counting function of the zeros of $\psi_f - b$:

$$\begin{aligned} \overline{N}\left(r, \frac{1}{\psi_f - b}\right) &= N_{1)}\left(r, \frac{1}{\psi_f - b}\right) + \overline{N}_{(2}\left(r, \frac{1}{\psi_f - b}\right) \\ &\leq 3\overline{N}(r,f) + 2\overline{N}(r,g) + 3\overline{N}\left(r, \frac{1}{f}\right) + 2\overline{N}\left(r, \frac{1}{g}\right) \\ &\quad + 2N\left(r, \frac{1}{f^{(k)} - b}\right) + N\left(r, \frac{1}{g^{(k)} - b}\right) + S(r,f) + S(r,g). \end{aligned} \tag{3.17}$$

Inserting this into (3.7) gives

$$\begin{aligned} T(r, \varphi_f) &\leq \frac{1}{n} \cdot N\left(r, \frac{1}{\varphi_f}\right) + 4\overline{N}(r,f) + 2\overline{N}(r,g) + 3\overline{N}\left(r, \frac{1}{f}\right) + 2\overline{N}\left(r, \frac{1}{g}\right) \\ &\quad + 3N\left(r, \frac{1}{f^{(k)} - b}\right) + N\left(r, \frac{1}{g^{(k)} - b}\right) + S(r,f) + S(r,g) \\ &\leq \frac{1}{n} \cdot T(r, \varphi_f) + 6T(r,f) + 3T(r,g) \\ &\quad + (3k+4) \cdot \overline{N}(r,f) + (k+2) \cdot \overline{N}(r,g) + S(r,f) + S(r,g). \end{aligned}$$

Setting

$$T(r) := \max\{T(r,f); T(r,g)\}$$

and denoting by $S(r)$ any term of the form $S(r) = o(T(r))$ for $r \to \infty$ outside a set of finite Lebesgue measure, we conclude that

$$\left(1 - \frac{1}{n}\right) \cdot T(r, \varphi_f) \leq 9T(r) + (3k+4) \cdot \overline{N}(r,f) + (k+2) \cdot \overline{N}(r,g) + S(r). \tag{3.18}$$

From

$$\varphi_f = \frac{1}{1 + \frac{f^{(k)} - b}{f^n}}$$

we get

$$\begin{aligned} n \cdot T(r,f) &\le T\left(r, \frac{f^{(k)}-b}{f^n}\right) + T\left(r, \frac{1}{f^{(k)}-b}\right) + O(1) \\ &= T(r,\varphi_f) + T\left(r, f^{(k)}\right) + O(1) \\ &\le T(r,\varphi_f) + T(r,f) + k \cdot \overline{N}(r,f) + S(r,f), \end{aligned}$$

hence

$$T(r,f) \le \frac{1}{n-k-1} \cdot T(r,\varphi_f) + S(r,f). \tag{3.19}$$

We insert this estimate this into (3.18). This yields

$$\begin{aligned} &(n-k-1)(n-1) \cdot T(r,f) \\ &\le (n-1) \cdot T(r,\varphi_f) + S(r,f) \\ &\le 9n \cdot T(r) + n(3k+4) \cdot \overline{N}(r,f) + n(k+2) \cdot \overline{N}(r,g) + S(r). \end{aligned}$$

If we combine this with the analogous estimate for $T(r,g)$, we obtain

$$(n-k-1)(n-1) \cdot T(r) \le (15+4k)n \cdot T(r) + S(r). \tag{3.20}$$

Here, by our assumption we have

$$(n-k-1)(n-1) - (15+4k)n = n^2 - (5k+17)n + k + 1 \ge k+1 > 0,$$

and we arrive at $T(r) = S(r)$, a contradiction.

If f and g are entire, i.e. if all terms $\overline{N}(r,f)$ and $\overline{N}(r,g)$ vanish, then instead of (3.19) we even have

$$T(r,f) \le \frac{1}{n-1} \cdot T(r,\varphi_f) + S(r,f)$$

(and a similar estimate for $T(r,g)$), and instead of (3.20) we obtain

$$(n-1)^2 \cdot T(r) \le 9n \cdot T(r) + S(r),$$

so the weaker assumption $n \ge 11$ of Theorem 2 (which implies $(n-1)^2 > 9n$) suffices to generate a contradiction.

So both in the meromorphic and in the entire case $H \not\equiv 0$ is impossible.

(4) Now, we turn to the case $H \equiv 0$. From (3.11) and

$$0 = 2H = 2 \cdot \frac{\psi_f'}{\psi_f - b} - 2 \cdot \frac{\psi_g'}{\psi_g - b} - 2Q[f] + 2Q[g],$$

by integration we deduce the existence of a constant $c \in \mathbf{C} \setminus \{0\}$ such that

$$\left(\frac{\psi_f - b}{\psi_g - b}\right)^2 = c \cdot \frac{f^n}{g^n} \cdot \frac{f^{(k)}-b}{g^{(k)}-b} \cdot \frac{V[f]}{V[g]}. \tag{3.21}$$

We discuss the cases $V[f] \equiv c \cdot V[g]$ and $V[f] \not\equiv c \cdot V[g]$ separately.

(4.1) Assume first that $V[f] \not\equiv c \cdot V[g]$. Let z_0 be a simple zero of $\psi_f - b$ (and hence of $\psi_g - b$), but not a zero of f or g. Then in view of $f^n(z_0) = b - f^{(k)}(z_0)$ and

$g^n(z_0) = b - g^{(k)}(z_0)$, z_0 isn't a zero of $f^{(k)} - b$ or of $g^{(k)} - b$, and clearly it isn't a pole of f or g. So we compute

$$\begin{aligned}\frac{\psi_f - b}{\psi_g - b}(z_0) &= \frac{\psi_f'}{\psi_g'}(z_0) = \frac{nf^{n-1}f' + f^{(k+1)}}{ng^{n-1}g' + g^{(k+1)}}(z_0)\\ &= \frac{f^n}{g^n}(z_0) \cdot \frac{n \cdot \frac{f'}{f} - \frac{f^{(k+1)}}{f^{(k)}-b}}{n \cdot \frac{g'}{g} - \frac{g^{(k+1)}}{g^{(k)}-b}}(z_0) = \frac{f^n}{g^n}(z_0) \cdot \frac{V[f]}{V[g]}(z_0).\end{aligned}$$

Inserting this into (3.21) gives

$$\frac{f^{2n}}{g^{2n}}(z_0) \cdot \left(\frac{V[f]}{V[g]}\right)^2(z_0) = c \cdot \frac{f^n}{g^n}(z_0) \cdot \frac{f^{(k)} - b}{g^{(k)} - b}(z_0) \cdot \frac{V[f]}{V[g]}(z_0) = c \cdot \frac{f^{2n}}{g^{2n}}(z_0) \cdot \frac{V[f]}{V[g]}(z_0).$$

Using the fact that z_0 is neither a zero or pole of $\frac{f^n}{g^n}$ nor a zero or pole of $\frac{\psi_f - b}{\psi_g - b}$, it is easy to see from (3.21) that $\frac{V[f]}{V[g]}$ has no zero or pole at z_0. Furthermore, from the definition of V we see that neither $V[f]$ nor $V[g]$ has a pole at z_0. Therefore we can conclude that

$$\frac{V[f]}{V[g]}(z_0) = c \quad \text{and} \quad V[f](z_0) - c \cdot V[g](z_0) = 0.$$

If z_0 is a multiple zero of $\psi_f - b$ (and hence of $\psi_g - b$), then the consideration leading to (3.16) shows that z_0 is a zero of f or g or a common zero of $V[f]$ and $V[g]$, and hence a zero of $V[f] - c \cdot V[g]$ as well. Therefore, we arrive at

$$\begin{aligned}\overline{N}\left(r, \frac{1}{\psi_f - b}\right) &\le \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{V[f] - cV[g]}\right)\\ &\le \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{g}\right) + T(r, V[f]) + T(r, V[g]) + O(1)\\ &\le \overline{N}(r, f) + \overline{N}(r, g) + 2\overline{N}\left(r, \frac{1}{f}\right) + 2\overline{N}\left(r, \frac{1}{g}\right)\\ &\quad + \overline{N}\left(r, \frac{1}{f^{(k)} - b}\right) + \overline{N}\left(r, \frac{1}{g^{(k)} - b}\right) + S(r, f) + S(r, g),\end{aligned}$$

hence at an even better estimate than in (3.17). Now we can use the same argument as in part (3.3) of the proof to deduce a contradiction.

(4.2) Finally, we assume $V[f] \equiv c \cdot V[g]$, i.e.

$$n \cdot \frac{f'}{f} - \frac{f^{(k+1)}}{f^{(k)} - b} \equiv c \cdot \left(n \cdot \frac{g'}{g} - \frac{g^{(k+1)}}{g^{(k)} - b}\right). \tag{3.22}$$

Since $V[g] \not\equiv 0$, we can simplify (3.21) to obtain

$$\left(\frac{\psi_f - b}{\psi_g - b}\right)^2 = c^2 \cdot \frac{f^n}{g^n} \cdot \frac{f^{(k)} - b}{g^{(k)} - b}. \tag{3.23}$$

Now we distinguish several cases according to the nature of c.

(a) If c is not rational, comparing the residues of the left hand side and of the right hand side of (3.22) and keeping in mind that the residues of logarithmic derivatives are integers, we deduce that $n \cdot \frac{f'}{f} - \frac{f^{(k+1)}}{f^{(k)}-b}$ has no poles at all, so $\frac{f^n}{f^{(k)}-b}$ is a non-vanishing entire function. This implies that f itself is entire (since a pole of f of multiplicity p would be a pole of $\frac{f^n}{f^{(k)}-b}$ of multiplicity $np-(p+k)>0$). Furthermore, all zeros of $f^{(k)}-b$ have multiplicity at least n, and we obtain

$$N\left(r,\frac{1}{f}\right)=\frac{1}{n}\cdot N\left(r,\frac{1}{f^{(k)}-b}\right),\quad \overline{N}\left(r,\frac{1}{f^{(k)}-b}\right)\leq\frac{1}{n}\cdot N\left(r,\frac{1}{f^{(k)}-b}\right).$$

Inserting these estimates into Milloux's inequality (Lemma 6) yields

$$\begin{aligned} T(r,f) &\leq \overline{N}\left(r,f\right)+N\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}-b}\right)+S(r,f)\\ &\leq 0+\frac{2}{n}\cdot N\left(r,\frac{1}{f^{(k)}-b}\right)+S(r,f)\\ &\leq \frac{2}{n}\cdot m\left(r,f^{(k)}\right)+S(r,f)\leq\frac{2}{n}\cdot T\left(r,f\right)+S(r,f). \end{aligned}$$

In view of $n\geq 3$ this means $T(r,f)=S(r,f)$, a contradiction. Hence, we conclude that c is rational (and $c\neq 0$).

(b) If $c<0$ (and c is rational), then there exist $p,q\in\mathbf{N}$ such that $c=-\frac{p}{q}$. Integrating (3.22) gives

$$\left(\frac{f^n}{f^{(k)}-b}\right)^q=d\cdot\left(\frac{g^{(k)}-b}{g^n}\right)^p \tag{3.24}$$

for a certain constant $d\in\mathbf{C}\setminus\{0\}$. Combining this with (3.23), we get

$$\left(\frac{\psi_f-b}{\psi_g-b}\right)^{2p}=\frac{c^{2p}}{d}\cdot\frac{f^{n(p+q)}}{\left(f^{(k)}-b\right)^{q-p}\left(g^{(k)}-b\right)^{2p}} \tag{3.25}$$

and

$$\left(\frac{\psi_f-b}{\psi_g-b}\right)^{2q}=c^{2q}d\cdot\frac{\left(f^{(k)}-b\right)^{2q}}{g^{n(p+q)}\left(g^{(k)}-b\right)^{q-p}}. \tag{3.26}$$

(i) First we consider the case where both f and g are entire functions. W.l.o.g. we may assume $p\leq q$. Since now $\frac{\psi_f-b}{\psi_g-b}$ has no zeros and poles at all, from (3.25) and (3.26) we deduce

$$\overline{N}\left(r,\frac{1}{g^{(k)}-b}\right)\leq\overline{N}\left(r,\frac{1}{f}\right),\quad \overline{N}\left(r,\frac{1}{f^{(k)}-b}\right)\leq\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{g^{(k)}-b}\right),$$
$$N\left(r,\frac{1}{f}\right)=\frac{q-p}{n(q+p)}\cdot N\left(r,\frac{1}{f^{(k)}-b}\right)+\frac{2p}{n(p+q)}\cdot N\left(r,\frac{1}{g^{(k)}-b}\right)$$

and

$$N\left(r,\frac{1}{g}\right)=\frac{2q}{n(q+p)}\cdot N\left(r,\frac{1}{f^{(k)}-b}\right)-\frac{q-p}{n(p+q)}\cdot N\left(r,\frac{1}{g^{(k)}-b}\right).$$

Inserting this into Milloux's inequality, we arrive at

$$\begin{aligned}
&T(r,f)+T(r,g)\\
&\le N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}-b}\right)+\overline{N}\left(r,\frac{1}{g^{(k)}-b}\right)+S(r,f)+S(r,g)\\
&\le 3N\left(r,\frac{1}{f}\right)+2N\left(r,\frac{1}{g}\right)+S(r,f)+S(r,g)\\
&\le \frac{7q-3p}{n(p+q)}\cdot N\left(r,\frac{1}{f^{(k)}-b}\right)+\frac{8p-2q}{n(p+q)}\cdot N\left(r,\frac{1}{g^{(k)}-b}\right)+S(r,f)+S(r,g)\\
&\le \frac{7}{n}\cdot T\left(r,f^{(k)}\right)+\frac{6p}{n(p+q)}\cdot T\left(r,g^{(k)}\right)+S(r,f)+S(r,g)\\
&\le \frac{7}{n}\cdot\left(T\left(r,f\right)+T\left(r,g\right)\right)+S(r,f)+S(r,g),
\end{aligned}$$

which gives $T(r,f)+T(r,g)=S(r,f)+S(r,g)$ since $n\ge 8$. This is a contradiction.

(ii) Now we consider the case where either f or g has a pole. W.l.o.g. we assume that $z_0\in\mathbf{C}$ is a pole of g of multiplicity β.[2] Then z_0 is a zero of the right hand side and hence of the left hand side of (3.24), and we deduce that it is a zero of f, say of multiplicity α. On the other hand, z_0 can't be a zero of $f^{(k)}-b$, for otherwise it would be a zero of ψ_f-b and hence of ψ_g-b, contradicting the fact that the poles of g are also poles of ψ_g. So comparing the multiplicities on both sides of (3.24) we deduce

$$nq\alpha = np\beta - p(\beta+k),$$

and by (3.25) we have

$$2np\beta = n(p+q)\alpha + 2p(\beta+k).$$

Combining these two identities we obtain $p=q$. Hence from (3.24) we have

$$\frac{f^n}{f^{(k)}-b}=\widetilde{d}\cdot\frac{g^{(k)}-b}{g^n} \tag{3.27}$$

for some $\widetilde{d}\neq 0$.

We claim that $\widetilde{d}=1$. Indeed, if every zero of ψ_f-b (and hence of ψ_g-b) would be a zero of $f^{(k)}-b$ or a zero of $g^{(k)}-b$, we would obtain

$$\overline{N}\left(r,\frac{1}{\psi_f-b}\right)\le N\left(r,\frac{1}{f^{(k)}-b}\right)+N\left(r,\frac{1}{g^{(k)}-b}\right), \tag{3.28}$$

i.e. an estimate even stronger than (3.17), and a similar estimate would hold for $\overline{N}\left(r,\frac{1}{\psi_g-b}\right)$. So exactly the same argument as in part (3.3) of the proof would lead to a contradiction. This shows that there exists a $\widetilde{z}\in\mathbf{C}$ such that $\psi_f(\widetilde{z})=b=\psi_g(\widetilde{z})$, $f^{(k)}(\widetilde{z})\neq b$ and $g^{(k)}(\widetilde{z})\neq b$, hence

$$\frac{f^n}{f^{(k)}-b}(\widetilde{z})=-1=\frac{g^{(k)}-b}{g^n}(\widetilde{z}).$$

[2] This is possible since in Case (ii) we do not assume $q\ge p$ any longer.

Inserting this into (3.27) gives $\widetilde{d}=1$, thus

$$\frac{f^n}{f^{(k)}-b}=\frac{g^{(k)}-b}{g^n}.$$

So we arrive at

$$\psi_f-b=f^n\left(1+\frac{f^{(k)}-b}{f^n}\right)=f^n\left(1+\frac{g^n}{g^{(k)}-b}\right)=\frac{f^n}{g^{(k)}-b}\cdot(\psi_g-b),$$

hence

$$\frac{\psi_f-b}{\psi_g-b}=\frac{f^n}{g^{(k)}-b}=\frac{f^{(k)}-b}{g^n},$$

and (1.3) holds.

(c) Therefore, it remains to study the case that $c>0$ (and c is rational). Assume that $c\neq 1$. Then $c=\frac{p}{q}$ for certain $p,q\in\mathbf{N}$, $p\neq q$. From (3.22) we obtain by integration

$$\left(\frac{f^n}{f^{(k)}-b}\right)^q=d\cdot\left(\frac{g^n}{g^{(k)}-b}\right)^p \tag{3.29}$$

with appropriate $d\in\mathbf{C}\setminus\{0\}$. Combining this with (3.23) yields

$$\left(\frac{\psi_f-b}{\psi_g-b}\right)^{2q}=c^{2q}d\cdot\frac{\left(f^{(k)}-b\right)^{2q}}{g^{n(q-p)}\left(g^{(k)}-b\right)^{q+p}} \tag{3.30}$$

$$=\frac{c^{2q}}{d}\cdot\frac{f^{2nq}}{g^{n(q+p)}\left(g^{(k)}-b\right)^{q-p}} \tag{3.31}$$

and

$$\left(\frac{\psi_f-b}{\psi_g-b}\right)^{2p}=\frac{c^{2p}}{d}\cdot\frac{f^{n(p+q)}}{g^{2np}\left(f^{(k)}-b\right)^{q-p}}. \tag{3.32}$$

(i) We start with the case that either f or g has a pole in $\mathbf{C}$. W.l.o.g. we assume that $z_0\in\mathbf{C}$ is a pole of f of multiplicity α. Considering (3.29) yields that z_0 is a pole of g or a zero of $g^{(k)}-b$.

First we assume that z_0 is a pole of g of multiplicity β. So from (3.29) and (3.30) we deduce

$$nq\alpha-q(\alpha+k)=np\beta-p(\beta+k)$$

and

$$2nq(\alpha-\beta)=2q(\alpha+k)-n(q-p)\beta-(p+q)(\beta+k).$$

Combining these two identities gives

$$n(q-p)\beta=(q-p)(\beta+k),$$

and in view of $p\neq q$ we obtain $k=(n-1)\beta\geq n-1\geq k+1$, a contradiction.

Now we turn to the case that z_0 is a zero of $g^{(k)}-b$ of multiplicity δ. Then from our value sharing condition we see $g(z_0)\neq 0$ (since otherwise z_0 would be a zero of ψ_g-b and hence of ψ_f-b, contradicting the fact that z_0 is a pole of f). Comparing the multiplicities of the poles on both sides of (3.29) gives

$$nq\alpha-q(\alpha+k)=p\delta,$$

and from (3.30) we get

$$2nq\alpha=2q(\alpha+k)+(p+q)\delta.$$

Combining these two identities gives $(q-p)\delta = 0$, a contradiction once more.

(ii) Now we consider the case that both f and g are entire functions. W.l.o.g. we may assume $q > p$. If z_0 is a zero of $f^{(k)} - b$ but not a zero of g, then from (3.32) we see that it is a zero of f, thus the multiplicity of z_0 as a zero of $f^{(k)} - b$ is at least $\frac{n(q+p)}{q-p} > n$, i.e. at least $n+1$. Therefore,

$$\overline{N}\left(r, \frac{1}{f^{(k)} - b}\right) \le \overline{N}\left(r, \frac{1}{g}\right) + \frac{1}{n+1} \cdot N\left(r, \frac{1}{f^{(k)} - b} \;\Big|\; g \ne 0\right). \tag{3.33}$$

Likewise, if z_0 is a zero of $g^{(k)} - b$ but not a zero of g, then from (3.31) we get that it is a zero of f and that its multiplicity as a zero of $g^{(k)} - b$ is at least $\frac{2nq}{q-p} > 2n$, i.e. at least $2n+1$. This shows

$$\overline{N}\left(r, \frac{1}{g^{(k)} - b}\right) \le \overline{N}\left(r, \frac{1}{g}\right) + \frac{1}{2n+1} \cdot N\left(r, \frac{1}{g^{(k)} - b} \;\Big|\; g \ne 0\right). \tag{3.34}$$

Furthermore, from (3.23) we deduce

$$N\left(r, \frac{1}{f}\right) \le N\left(r, \frac{1}{g}\right) + \frac{1}{n} \cdot N\left(r, \frac{1}{g^{(k)} - b}\right). \tag{3.35}$$

Next, assume that z_0 is a zero of g of multiplicity $\beta \ge 1$. Then from (3.31) and (3.30) we see that z_0 is a zero of f of multiplicity, say, $\alpha \ge 1$, and a zero of $f^{(k)} - b$ of multiplicity $\gamma \ge 1$. So it is a zero of $\psi_f - b$ and hence of $\psi_g - b$. From this and $g(z_0) = 0$ we finally obtain that z_0 is a zero of $g^{(k)} - b$, say of multiplicity $\delta \ge 1$. Since f^n and g^n have a zero of order at least n at z_0, we have

$$\psi_f(z_0) = f^{(k)}(z_0) - b, \qquad \psi_g(z_0) = g^{(k)}(z_0) - b$$

and

$$\psi_f^{(j)}(z_0) = f^{(k+j)}(z_0), \quad \psi_g^{(j)}(z_0) = g^{(k+j)}(z_0) \;\text{ for } j = 1, \ldots, n-1. \tag{3.36}$$

Assume that $\gamma < n$ or $\delta < n$. Then from (3.36) and the fact that $\psi_f - b$ and $\psi_g - b$ share the value 0 CM we would deduce $\gamma = \delta$. Together with (3.23) this would imply $\alpha = \beta$, and so (3.29) would give

$$q(n\alpha - \gamma) = p(n\beta - \delta) = p(n\alpha - \gamma),$$

hence $n\alpha - \gamma = 0$ since $q \ne p$. So we arrive at $\gamma = n\alpha \ge n$, a contradiction to $\gamma = \delta < n$. This shows that $\gamma \ge n$ and $\delta \ge n$. Furthermore, from $f^{(k)}(z_0) = g^{(k)}(z_0) = b \ne 0$ we see that $\alpha \le k$ and $\beta \le k$.

These considerations show

$$\overline{N}\left(r, \frac{1}{g}\right) \le \frac{1}{n} \cdot \min\left\{N\left(r, \frac{1}{f^{(k)} - b} \;\Big|\; g = 0\right) ; N\left(r, \frac{1}{g^{(k)} - b} \;\Big|\; g = 0\right)\right\} \tag{3.37}$$

and

$$\begin{aligned} N\left(r, \frac{1}{g}\right) &\le k \cdot \overline{N}\left(r, \frac{1}{g}\right) \\ &\le \frac{k}{n} \cdot \min\left\{N\left(r, \frac{1}{f^{(k)} - b} \;\Big|\; g = 0\right) ; N\left(r, \frac{1}{g^{(k)} - b} \;\Big|\; g = 0\right)\right\}. \end{aligned} \tag{3.38}$$

Inserting (3.33), (3.34), (3.35), (3.37) and (3.38) into Milloux's inequality, we deduce

$$\begin{aligned}
&T(r,f)+T(r,g)\\
&\le N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}-b}\right)+\overline{N}\left(r,\frac{1}{g^{(k)}-b}\right)+S(r,f)+S(r,g)\\
&\le 2N\left(r,\frac{1}{g}\right)+2\overline{N}\left(r,\frac{1}{g}\right)+\frac{1}{n}\cdot N\left(r,\frac{1}{g^{(k)}-b}\right)+\frac{1}{n+1}\cdot N\left(r,\frac{1}{f^{(k)}-b}\;\Big|\; g\neq 0\right)\\
&\quad+\frac{1}{2n+1}\cdot N\left(r,\frac{1}{g^{(k)}-b}\;\Big|\; g\neq 0\right)+S(r,f)+S(r,g)\\
&\le \frac{2}{n}\cdot(k+1)\cdot\min\left\{N\left(r,\frac{1}{f^{(k)}-b}\;\Big|\; g=0\right);\,N\left(r,\frac{1}{g^{(k)}-b}\;\Big|\; g=0\right)\right\}\\
&\quad+\frac{1}{n}\cdot N\left(r,\frac{1}{g^{(k)}-b}\right)+\frac{1}{n+1}\cdot N\left(r,\frac{1}{f^{(k)}-b}\;\Big|\; g\neq 0\right)\\
&\quad+\frac{1}{2n+1}\cdot N\left(r,\frac{1}{g^{(k)}-b}\;\Big|\; g\neq 0\right)+S(r,f)+S(r,g)\\
&\le \frac{2k+3}{2n}\cdot N\left(r,\frac{1}{f^{(k)}-b}\right)+\frac{2k+3}{2n}\cdot N\left(r,\frac{1}{g^{(k)}-b}\right)+S(r,f)+S(r,g)\\
&\le \frac{2k+3}{2n}\cdot(T(r,f)+T(r,g))+S(r,f)+S(r,g).
\end{aligned}$$

Since $2n>2k+3$, this gives $T(r,f)+T(r,g)\le S(r,f)+S(r,g)$, a contradiction.

Therefore, we finally can conclude that $c=1$.

By integration, we have

$$\frac{f^n}{f^{(k)}-b}=d\cdot\frac{g^n}{g^{(k)}-b} \tag{3.39}$$

for a constant $d\neq 0$.

We show that $d=1$. If every zero of ψ_f-b would be a zero of $f^{(k)}-b$ or a zero of $g^{(k)}-b$, the same reasoning as in (3.28) would lead to a contradiction. So there exists a $z_0\in\mathbf{C}$ such that $\psi_f(z_0)=b=\psi_g(z_0)$, $f^{(k)}(z_0)\neq b$ and $g^{(k)}(z_0)\neq b$, hence

$$\frac{f^n}{f^{(k)}-b}(z_0)=-1=\frac{g^n}{g^{(k)}-b}(z_0).$$

Inserting this into (3.39) gives $d=1$, thus

$$\frac{f^n}{f^{(k)}-b}=\frac{g^n}{g^{(k)}-b}$$

and

$$\psi_f-b=f^n\cdot\left(1+\frac{f^{(k)}-b}{f^n}\right)=f^n\cdot\left(1+\frac{g^{(k)}-b}{g^n}\right)=\frac{f^n}{g^n}\cdot(\psi_g-b)$$

and therefore

$$\frac{\psi_f-b}{\psi_g-b}=\frac{f^n}{g^n}=\frac{f^{(k)}-b}{g^{(k)}-b}.$$

This completes the proof of Theorem 1 and Theorem 2. $\square$

4. Proof of Theorem 3 and of the Corollaries

Proof of Theorem 3. We assume that $\frac{f}{g}$ is not constant. Then from Theorem 2 we obtain that

$$\frac{f^n}{g^n} = \frac{af'-b}{ag'-b},$$

and that f and g share the value 0 CM. We define

$$H := \frac{af'-b}{af^n}.$$

Then

$$f' = f^n \cdot H + \frac{b}{a}, \quad g' = g^n \cdot H + \frac{b}{a},$$

i.e. f and g satisfy the same differential equation $u' = u^n \cdot H + \frac{b}{a}$. The main idea of the proof is to apply the uniqueness theorem for differential equations. This requires some point z_0 where f and g coincide and a Lipschitz condition for the right hand side of the differential equation which is valid in some neighborhood of z_0. Unfortunately, the only points where f and g coincide might be the common zeros of f and g where no Lipschitz condition for H holds. To avoid this problem we make some transformation and consider f^j and g^j instead of f and g for some appropriate j.

Since $\frac{f}{g}$ omits the values 0 and ∞, by Picard's theorem $\frac{f}{g}$ assumes every value $w \in \partial\mathbf{D}$. Since the set $\partial\mathbf{D}$ is uncountable while f and g have only countably many zeros, there exists some $\widetilde{z} \in \mathbf{C}$ such that $w_0 := \frac{f}{g}(\widetilde{z}) \in \partial\mathbf{D}$ while $f(\widetilde{z}) \neq 0$, $g(\widetilde{z}) \neq 0$. There is some open neighborhood U of $\widetilde{z}$ such that f and g are non-vanishing in U. Since $\frac{f}{g}(U)$ is an open neighborhood of $w_0 \in \partial\mathbf{D}$, there is some $z_0 \in U$ such that $\eta := \frac{f}{g}(z_0)$ is some root of unity, i.e. $\eta^j = 1$ for some $j \in \mathbf{N}$. We choose an open disk U_0 with center z_0 such that f and g are non-vanishing in U_0. Now we set

$$F := f^{jn} \quad \text{and} \quad G := g^{jn}.$$

Then in the simply connected domain U_0 we have

$$F' = jn \cdot f^{jn-1} f' = jn \cdot f^{jn-1}\left(f^n \cdot H + \frac{b}{a}\right) = jn \cdot F^{1+\frac{1}{j}-\frac{1}{jn}} \cdot H + \frac{b}{a} \cdot jn \cdot F^{1-\frac{1}{jn}}$$

and in the same way

$$G' = jn \cdot G^{1+\frac{1}{j}-\frac{1}{jn}} \cdot H + \frac{b}{a} \cdot jn \cdot G^{1-\frac{1}{jn}}$$

provided that we choose appropriate branches of $F^{1+\frac{1}{j}-\frac{1}{jn}}$ and the other roots. Hence in U_0 the functions F and G satisfy the differential equation

$$u' = jn \cdot u^{1+\frac{1}{j}-\frac{1}{jn}} \cdot H + \frac{b}{a} \cdot jn \cdot u^{1-\frac{1}{jn}}$$

and the initial value condition

$$F(z_0) = f^{jn}(z_0) = \eta^{jn} g^{jn}(z_0) = G(z_0),$$

and H is analytic in U_0. So by the uniqueness theorem for differential equations we obtain $F \equiv G$ in U_0 and hence in $\mathbf{C}$. This of course means that $\frac{f}{g}$ is constant, a contradiction.

So we have shown that there exists some $c \in \mathbf{C} \setminus \{0\}$ such that $f \equiv cg$. Now we obtain

$$cg' - 1 = f' - 1 = (g' - 1) \cdot \frac{f^n}{g^n} = (g' - 1) \cdot c^n,$$

hence

$$(c - c^n) \cdot g' = 1 - c^n.$$

If $c \neq c^n$, then g' turns out to be constant, so g and f are polynomials of degree 1 with the same zero. If $c = c^n$, then also $c^n = 1$, hence $c = 1$ and $f \equiv g$. This proves the theorem. □

Proof of Corollary 4. If $f \not\equiv g$, Theorem 1 gives (for $n \geq 5k + 17$)

$$\text{(4.1)} \qquad \text{(a)} \quad \frac{f^n}{g^n} = \frac{af^{(k)} - b_1}{ag^{(k)} - b_1} \quad \text{or} \quad \text{(b)} \quad \frac{f^n}{af^{(k)} - b_1} = \frac{ag^{(k)} - b_1}{g^n}$$

and

$$\text{(4.2)} \qquad \text{(a)} \quad \frac{f^n}{g^n} = \frac{af^{(k)} - b_2}{ag^{(k)} - b_2} \quad \text{or} \quad \text{(b)} \quad \frac{f^n}{af^{(k)} - b_2} = \frac{ag^{(k)} - b_2}{g^n}$$

Under the additional assumption that f and g are entire, from Theorem 2 we deduce for $n \geq \max\{11; k + 2\}$ that (4.1a) and (4.2a) hold.

Now there are four cases to consider.

Case 1: If (4.1b) and (4.2b) hold, we obtain

$$\left(af^{(k)} - b_1\right) \cdot \left(ag^{(k)} - b_1\right) = f^n \cdot g^n = \left(af^{(k)} - b_2\right) \cdot \left(ag^{(k)} - b_2\right),$$

hence

$$a(b_2 - b_1) \cdot \left(f^{(k)} + g^{(k)}\right) = b_2^2 - b_1^2.$$

In view of $b_1 \neq b_2$, this shows that $f + g$ is a polynomial. In particular, f and g have the same poles with the same multiplicities. In view of (4.1b) this means that f and g don't have any poles at all, i.e. they are entire. But now, as mentioned above, from Theorem 2 we deduce that instead of (4.1b) and (4.2b) we must have (4.1a) and (4.2a). So this case can be ruled out.

Case 2: Assume that (4.1a) and (4.2b) hold. Once more, we want to show that f and g are entire functions. So w.l.o.g. we assume that z_0 is a pole f of multiplicity α. Then from (4.2b) we obtain that z_0 is a zero of g, say of multiplicity β. On the other hand, $g^{(k)}(z_0) \neq \frac{b_1}{a}$ since otherwise $\psi_g(z_0) = b_1$, hence $\psi_f(z_0) = b_1$, contradicting the fact that z_0 is a pole of f. Therefore, (4.1a) yields

$$n\alpha + n\beta = \alpha + k,$$

hence $n - 1 \leq (n - 1)\alpha \leq k$, a contradiction.

So f and g are entire, and as in Case 2 we deduce that instead of (4.2b) we must have (4.2a). So this case can be ruled out as well.

Case 3: The case where (4.1b) and (4.2a) hold is of course essentially the same as Case 2 and can be ruled out.

Case 4: If (4.1a) and (4.2a) hold, then we obtain

$$\frac{af^{(k)} - b_1}{ag^{(k)} - b_1} = \frac{f^n}{g^n} = \frac{af^{(k)} - b_2}{ag^{(k)} - b_2},$$

hence

$$ab_1\left(f^{(k)}-g^{(k)}\right)=ab_2\left(f^{(k)}-g^{(k)}\right), \quad \text{i.e.} \quad f^{(k)}=g^{(k)} \quad \text{and} \quad f^n=g^n.$$

So we have $f=e^{2\pi ij/n}g$ for some integer j. If $g^{(k)}\not\equiv 0$, from $g^{(k)}=f^{(k)}=e^{2\pi ij/n}g^{(k)}$ we could even deduce $e^{2\pi ij/n}=1$, i.e. $f\equiv g$, which contradicts our assumption that $f\not\equiv g$. So $f^{(k)}\equiv g^{(k)}\equiv 0$ which means that f and g are polynomials of degree at most $k-1$.

This proves Corollary 4. □

Proof of Corollary 5. If f would be a polynomial, then ψ_f and $\psi_{f'}$ would be polynomials of unequal degree, so they could not share the value b CM.

Therefore, from Theorem 1, resp. Theorem 2, we deduce that either

$$\frac{\psi_{f'}-b}{\psi_f-b}=\left(\frac{f'}{f}\right)^n=\frac{af^{(k+1)}-b}{af^{(k)}-b} \tag{4.3}$$

or

$$f^n(f')^n=\left(af^{(k)}-b\right)\cdot\left(af^{(k+1)}-b\right). \tag{4.4}$$

In both cases it is easy to see from our assumptions on n and k that f cannot have any poles. So f is an entire transcendental function. In view of Theorem 2 this means that (4.3) holds. In particular, f and f' share the value 0 CM. Of course, this is only possible if f and f' have no zeros at all.

Therefore, $q:=\frac{f'}{f}$ is a non-vanishing entire function.

We assume that q is not constant.

Claim. *For all $j\ge 0$ there exists a differential polynomial P_j of degree at most $j+1$ with constant coefficients and without any terms of degree 0 or 1 such that*

$$q^{(j)}=\frac{f^{(j+1)}}{f}+P_j[q]$$

and such that each differential monomial $M(\not\equiv 0)$ appearing in P_j satisfies

$$w(M)=j+1.$$

Proof. For $j=0$ this obviously holds with $P_0\equiv 0$. We assume that our claim holds for one $j\ge 0$. Then by differentiating we obtain

$$\begin{aligned} q^{(j+1)} &= \frac{f^{(j+2)}}{f}-\frac{f^{(j+1)}}{f}\cdot\frac{f'}{f}+P_j'[q] \\ &= \frac{f^{(j+2)}}{f}-q\cdot\left(q^{(j)}-P_j[q]\right)+P_j'[q]=\frac{f^{(j+2)}}{f}+P_{j+1}[q] \end{aligned}$$

with $P_{j+1}[u]:=P_j'[u]+u\cdot P_j[u]-uu^{(j)}$. It is easy to see that the required properties carry over from P_j to P_{j+1}. By induction, our claim follows for all $j\ge 0$. □

For all $j\ge 0$ we can write

$$P_j=\sum_{\mu=2}^{j+1}H_{j,\mu}$$

with certain homogeneous differential polynomials $H_{j,\mu}$ of degree μ (or $H_{j,\mu}\equiv 0$).

Now we define

$$L := \frac{q^{(k)}}{q} - \frac{q^{(k-1)}}{q} - \sum_{\mu=2}^{k+1} \frac{H_{k,\mu}[q]}{q^\mu} + \sum_{\mu=2}^{k} \frac{H_{k-1,\mu}[q]}{q^\mu}.$$

Then L is an entire function, and by the lemma on the logarithmic derivative and the properties of $H_{j,\mu}$ we have $m(r, L) = S(r, q)$.

We consider two cases:

Case 1: $L \not\equiv 0$. Assume that $q(z_0) = 1$. Then $f(z_0) = f'(z_0) \neq 0$, and from (4.3) we see $f^{(k)}(z_0) = f^{(k+1)}(z_0)$. Therefore we conclude that

$$\begin{aligned} L(z_0) &= q^{(k)}(z_0) - q^{(k-1)}(z_0) - \sum_{\mu=2}^{k+1} H_{k,\mu}[q](z_0) + \sum_{\mu=2}^{k} H_{k-1,\mu}[q](z_0) \\ &= \frac{f^{(k+1)}}{f}(z_0) - \frac{f^{(k)}}{f}(z_0) = 0. \end{aligned}$$

This consideration shows

$$\overline{N}\left(r, \frac{1}{q-1}\right) \le \overline{N}\left(r, \frac{1}{L}\right) \le T(r, L) + O(1) = m(r, L) + O(1) = S(r, q).$$

Applying the Second Fundamental Theorem we obtain

$$T(r, q) \le \overline{N}(r, q) + \overline{N}\left(r, \frac{1}{q}\right) + \overline{N}\left(r, \frac{1}{q-1}\right) + S(r, q) = S(r, q),$$

a contradiction.

Case 2: $L \equiv 0$. We set

$$H[u] := u^k \cdot u^{(k)} - u^k \cdot u^{(k-1)} - \sum_{\mu=2}^{k+1} u^{k+1-\mu} \cdot H_{k,\mu}[u] + \sum_{\mu=2}^{k} u^{k+1-\mu} \cdot H_{k-1,\mu}[u].$$

Then H is a *homogeneous* differential polynomial of degree $k + 1$ and

$$H[q] = q^{k+1} \cdot L \equiv 0.$$

In view of the properties of P_j, we have $w(H_{j,\mu}) = j + 1$ (or possibly[3] $H_{j,\mu} \equiv 0$) for all $j \ge 1$ and all $\mu = 2, \dots, j + 1$. Hence all terms in $H[u]$ with the exception of $u^k \cdot u^{(k)}$ have weight at most $2k$ while $u^k \cdot u^{(k)}$ has weight $2k + 1$. Therefore Lemma 9 (applied with $s = 1$) shows that $q(z) = e^{\alpha z + \beta}$ with certain $\alpha, \beta \in \mathbf{C}$.

From $f' = qf$ and the assumption that q is not constant we deduce that f has infinite order.

On the other hand, from

$$f^{(j+1)} = f \cdot \left(q^{(j)} - P_j[q]\right)$$

and (4.3) we obtain

$$q^n = \left(\frac{f'}{f}\right)^n = \frac{af \cdot \left(q^{(k)} - P_k[q]\right) - b}{af \cdot (q^{(k-1)} - P_{k-1}[q]) - b},$$

hence

$$af\left(q^{(k)} - q^n \cdot q^{(k-1)} - P_k[q] + q^n \cdot P_{k-1}[q]\right) = b \cdot (1 - q^n).$$

[3]In fact, one can show that the case $H_{j,\mu} \equiv 0$ never occurs, but this is not required for our purposes.

Since $q^{(k)} - q^n \cdot q^{(k-1)} - P_k[q] + q^n \cdot P_{k-1}[q]$ and $1 - q^n$ have finite order while f has infinite order, we conclude that

$$q^{(k)} - q^n \cdot q^{(k-1)} - P_k[q] + q^n \cdot P_{k-1}[q] \equiv 0 \quad \text{and} \quad 1 - q^n \equiv 0.$$

This contradicts our assumption that q is non-constant.

So we have shown that q is constant. From $f' = qf$ and (4.3) we get

$$aqf^{(k)} = af^{(k+1)} = b + q^n \left(af^{(k)} - b\right), \quad \text{hence} \ \ a\,(q - q^n) \cdot f^{(k)} = b \cdot (1 - q^n)\,.$$

Since f is transcendental this implies $q - q^n = 0$ and $1 - q^n = 0$, hence $q = 1$. This shows $f' = f$ and therefore completes the proof of the Corollary. □

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