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HARDY SPACES AND UNBOUNDED QUASIDISKS

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Abstract. We study the maximal number $0 \le h \le +\infty$ for a given plane domain Ω such that $f \in H^p$ whenever $p < h$ and f is analytic in the unit disk with values in Ω. One of our main contributions is an estimate of h for unbounded K-quasidisks.

1. Introduction

In his 1970 paper [10] Hansen introduced a number, denoted by $h(\Omega)$, for a domain Ω in the complex plane. The number $h = h(\Omega)$ is defined as the maximal one in $[0, +\infty]$ so that every holomorphic function on any plane domain D with values in Ω belongs to the Hardy class $H^p(D)$ whenever $0 < p < h$. The number was called by him the Hardy number of Ω. If Ω is bounded, then clearly $h(\Omega) = +\infty$. Therefore, the consideration of $h(\Omega)$ is meaningful only when Ω is unbounded.

Hansen [10] studied the number by using Ahlfors' distortion theorem. Also, in the same paper, he described it in terms of geometric quantities for starlike domains. Indeed, let $\Omega \neq \mathbf{C}$ be an unbounded starlike domain with respect to the origin. Let $\alpha_\Omega(t)$ be the length of maximal subarc of $\{z \in \mathbf{T} \colon tz \in \Omega\}$ for $t > 0$, where $\mathbf{T}$ stands for the unit circle $\{z \in \mathbf{C} \colon |z| = 1\}$. Observe that $\alpha_\Omega(t)$ is non-increasing in t by starlikeness. Hansen [10, Theorem 4.1] showed the formula $h(\Omega) = \lim_{t\to+\infty} \pi/\alpha_\Omega(t)$. Later he obtained a similar formula for spirallike domains [11]. These formulae cover only a family of good enough (necessarily simply connected) domains. In subsequent papers [12] and [13], lower bounds for $h(\Omega)$ are given in terms of growth of the image area.

Essén [7] gave a description of $h(\Omega)$ for general Ω in terms of harmonic measures and obtained almost necessary and sufficient conditions for $h(\Omega) > 0$ in terms of capacity. Practically, however, it is hard to compute or estimate the harmonic measure or capacity in terms of geometric quantities of the domain Ω. Thus it is desirable to have more geometric estimates of $h(\Omega)$.

It seems that after the work of Essén, only very few papers have been devoted to the study of the quantity $h(\Omega)$. Bourdon and Shapiro [4] and Poggi-Corradini [15] studied the range domains Ω of univalent Koenigs functions and found that the number $h(\Omega)$ can be described in terms of the essential norm of the associated composition operators.

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We will discuss below the change of $h(\Omega)$ under conformal mappings of domains. This sort of observation gives another way of estimation of $h(\Omega)$.

We briefly explain the organization of the present note. Section 2 is devoted to the basic properties of $h(\Omega)$ as well as preliminaries and necessary definitions. In Section 3, we introduce Essén's main lemma, by which we prove a couple of results given in Section 2. Section 4 is devoted to a study of local behaviour of quasiconformal mappings. One of our main results is Theorem 4.4 which gives a sharp estimate of $h(\varphi(\mathbf{H}))$ for a conformal mapping φ of the upper half-plane $\mathbf{H}$ with K-quasiconformal extension to the complex plane. The contents in Section 4 may be of independent interest.

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2. Basic properties of $h(\Omega)$

We denote by $\mathrm{Hol}(D,\Omega)$ the set of holomorphic functions on a domain D with values in a domain Ω. The (classical) Hardy space H^p is the set of holomorphic functions f on the unit disk $\mathbf{D}=\{z\in\mathbf{C}\colon |z|<1\}$ with finite norm

$$\|f\|_p=\sup_{0<r<1}\left(\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p\,d\theta\right)^{1/p}<\infty$$

for $0<p<\infty$ and

$$\|f\|_\infty=\sup_{z\in\mathbf{D}}|f(z)|<\infty$$

for $p=\infty$. (Note that $\|f\|_p$ is not really a norm when $0<p<1$.) For each holomorphic function f on $\mathbf{D}$, set

$$h(f)=\sup\{p>0\colon f\in H^p\}.$$

Here and hereafter, the supremum of the empty set is defined to be 0 unless otherwise stated. Since $H^p\subset H^q$ for $0<q<p\le\infty$, we have $f\notin H^p$ for $p>h(f)$.

The Hardy space $H^p(D)$ on a general plane domain D for $0<p<\infty$ is usually defined to be the set of holomorphic functions f such that $|f|^p$ has a harmonic majorant on D, that is, there is a harmonic function u satisfying $|f|^p\le u$ on D. The space $H^\infty(D)$ is defined to be the set of bounded holomorphic functions on D. When $D=\mathbf{D}$, the space $H^p(\mathbf{D})$ agrees with the classical H^p. See [6, Chap. 10] for details.

Lemma 2.1. *Let Ω be a domain in $\mathbf{C}$ with at least two boundary points. Then the number $h(\Omega)\in[0,+\infty]$ can be characterized by each of the following conditions:*

(1) $h(\Omega)=\sup\{p>0\colon |z|^p$ *has a harmonic majorant on* $\Omega\}$.
(2) $h(\Omega)$ *is the maximal number such that* $\mathrm{Hol}(D,\Omega)\subset H^p(D)$ *for any domain* D *in* $\mathbf{C}$ *and for any* $0<p<h(\Omega)$.
(3) $h(\Omega)=\sup\{p>0\colon \mathrm{Hol}(\mathbf{D},\Omega)\subset H^p\}$.
(4) $h(\Omega)=\inf\{h(f)\colon f\in\mathrm{Hol}(\mathbf{D},\Omega)\}$.
(5) $h(\Omega)=h(f)$ *for a holomorphic universal covering projection* f *of* $\mathbf{D}$ *onto* Ω.

The condition (1) is the original definition of the number $h=h(\Omega)$ due to Hansen [10, Definition 2.1]. Though part of this lemma is already noted in [10] and the others are obvious to experts, we indicate an outline of the proof for convenience of the reader.

Proof. For clarity, we use the notation h_j to designate $h(\Omega)$ which appears in the condition (j). If $u(z)$ is a harmonic majorant of $|z|^p$ on Ω, then $|f|^p \le u \circ f$ on D for $f \in \mathrm{Hol}(D, \Omega)$. Since $u \circ f$ is harmonic too, one has $h_1 \le h_2$. It is obvious that $h_2 \le h_3 = h_4 \le h_5$.

It is thus enough to show $h_5 \le h_1$. Suppose that f is a holomorphic universal covering projection of $\mathbf{D}$ onto Ω and $p < h_5$. Note that the radial limit f^* of f belongs to $L^p(\partial\mathbf{D})$. Let v be the Poisson integral of $|f^*|^p$. Then $|f|^p \le v$ on $\mathbf{D}$ because $|f|^p$ is subharmonic. Since the function f^* is invariant under the action of the Fuchsian group $\Gamma = \{\gamma \in \mathrm{Aut}(\mathbf{D})\colon f \circ \gamma = f\}$, so is v. Hence v is factored to $u \circ f$ with harmonic function u on $\Omega = \mathbf{D}/\Gamma$. It is now clear that u is the least harmonic majorant of $|z|^p$ on Ω. See the proof of Theorem 10.11 in [6] for the details of the last part. $\square$

It is well known that the conformal mapping $\varphi(z) = i(1+z)/(1-z)$ of $\mathbf{D}$ onto the upper half-plane $\mathbf{H}$ belongs to H^p precisely when $0 < p < 1$. In particular, $h(\mathbf{H}) = h(\varphi) = 1$. Since $\varphi(z)^\alpha$ maps $\mathbf{D}$ conformally onto the sector $0 < \arg w < \pi\alpha$ for $0 < \alpha \le 2$, we have the following, which is due to Cargo (cf. [10]).

Example 2.2. (Sectors) Let S_α be a sector with opening angle $\pi\alpha$ with $0 < \alpha \le 2$. Then $h(S_\alpha) = 1/\alpha$.

We also observe that $h(P) = +\infty$ for a parallel strip P since $f(z) = \log((1+z)/(1-z))$ belongs to BMOA and thus to H^p for all $0 < p < \infty$.

We collect basic properties of the number $h(\Omega)$. All properties but the last in the next lemma are found in [10].

Lemma 2.3. *Let Ω and Ω' be plane domains.*

(1) $h(\Omega) = +\infty$ *if Ω is bounded.*
(2) $h(\Omega') \le h(\Omega)$ *if $\Omega \subset \Omega'$.*
(3) $h(\varphi(\Omega)) = h(\Omega)$ *for a complex affine map $\varphi(z) = az + b, a \ne 0$.*
(4) $h(\Omega) = 0$ *if $\mathbf{C} \setminus \Omega$ is bounded.*
(5) $h(\Omega) \ge 1/2$ *if Ω is simply connected and $\Omega \ne \mathbf{C}$.*
(6) $h(\Omega) \ge 1$ *if Ω is convex and $\Omega \ne \mathbf{C}$.*

Proof. Assertions (1), (2) and (3) are trivial. To show (4), we may assume that $\mathbf{C} \setminus \Omega \subset \mathbf{D}$. Then the function $f(z) = \exp(\frac{1+z}{1-z})$ belongs to $\mathrm{Hol}(\mathbf{D}, \Omega)$ but does not belong to H^p for any $p > 0$. Thus $h(\Omega) = 0$. In view of Lemma 2.1 (5), assertion (5) follows from the fact that every univalent function on the unit disk belongs to H^p for $0 < p < 1/2$ (see [6, Theorem 3.16]). Since every convex proper subdomain Ω of $\mathbf{C}$ is contained in a half-plane, say, H, one can see that $h(\Omega) \ge h(H) = 1$. $\square$

In addition to the above lemma, we have the following deeper properties of the quantity $h(\Omega)$. We will give a proof for it in Section 3.

Theorem 2.4. *Let Ω and Ω' be plane domains.*

(i) $h(\Omega \setminus N) = h(\Omega)$ *for a locally closed polar set N in Ω.*
(ii) *Suppose that $0 \in \Omega$ and let Ω^* be the circular symmetrization of Ω with respect to the positive real axis. Then $h(\Omega) \ge h(\Omega^*)$.*

Here, Ω^* is defined to be $\{re^{i\theta}\colon 0 \le r < \infty, |\theta| < L(r)/2\}$, where $L(r)$ is the length of the set $\{\theta \in (-\pi, \pi]\colon re^{i\theta} \in \Omega\}$ if the circle $|z| = r$ is not entirely contained in Ω, and $L(r) = +\infty$ otherwise.

It is well known that a plane domain Ω does not admit Green's function if and only if $\partial\Omega$ is polar (cf. [2] or [5]). Therefore, as a consequence of (i) in the last theorem, we see that $h(\Omega) = 0$ when Ω does not admit Green's function. Frostman [8] even proved that there exists an analytic map $f\colon \mathbf{D} \to \Omega$ which does not belong to the Nevanlinna class if and only if Ω does not admit Green's function.

Remark. The authors proposed in [14] a quantity $W(\Omega)$, to which we named the *circular width* of Ω, for a plane domain Ω with $0 \notin \Omega$. Though the natures of the quantities $2h(\Omega)$ and $1/W(\Omega)$ are rather different, it is surprising that they share many properties. Compare with Theorem 3.2 and Example 5.1 in [14].

The following is useful to estimate the quantity $h(\Omega)$ by comparing with that of a standard domain.

Lemma 2.5. (Comparison lemma) *Let φ be a conformal homeomorphism of a domain Ω onto another domain Ω' and let α and β be positive numbers.*

(1) *If $|\varphi(z)| \le C(1+|z|^\alpha)$ for $z \in \Omega$ and for a positive constants C, then $h(\Omega) \le \alpha h(\Omega')$.*
(2) *If $c|z|^\beta \le |\varphi(z)| + 1$ for $z \in \Omega$ and for a positive constant c, then $\beta h(\Omega') \le h(\Omega)$.*

Proof. We first show (2). By assumption, there is a constant $A > 0$ such that $|z|^\beta \le A(|\varphi(z)|+1)$ holds for $z \in \Omega$. If $0 < p < h(\Omega')$, by definition, there exists a harmonic majorant $u(w)$ of $|w|^p$ on Ω', namely, $|w|^p \le u(w)$ on Ω'. Thus

$$|z|^{\beta p} \le (2A)^p(|\varphi(z)|^p + 1) \le (2A)^p(u(\varphi(z)) + 1), \quad z \in \Omega,$$

which means that $|z|^{\beta p}$ has the harmonic majorant $(2A)^p(u\circ\varphi+1)$. Hence, $h(\Omega) \ge \beta p$. Letting $p \to h(\Omega')$, we have assertion (2).

The proof of (1) is similar to (and even simpler than) the above. □

Example 2.6. (Spiral domains) For $\beta \in (-\pi/2, \pi/2)$, the image $\sigma_\beta = \gamma_\beta(\mathbf{R})$ of the curve $\gamma_\beta(t) = \exp(t(1+i\tan\beta))$ and its rotation $\sigma_{\beta,\theta} = e^{i\theta}\sigma_\beta$ are called a β-spiral. For $\alpha \in (0, 2]$, the domain

$$\mathrm{Sp}(\beta,\alpha) = \bigcup_{0<\theta<\pi\alpha} \sigma_{\beta,\theta}$$

will be called a β-spiral domain with width α. Note that $\mathrm{Sp}(0,\alpha) = S_\alpha$.

For a complex number $\lambda \ne 0$ with $|\lambda - 1| \le 1$, we consider the function $\varphi_\lambda(z) = z^\lambda = e^{\lambda \log z}$ on the upper half-plane $\mathbf{H}$, where we take the branch of $\log z$ so that $0 < \operatorname{Im}\log z < \pi$. Then one can easily see that φ_λ maps $\mathbf{H}$ conformally onto the domain $\mathrm{Sp}(\arg\lambda, |\lambda|^2/\operatorname{Re}\lambda)$. Then $|\varphi_\lambda(z)| = e^{-\operatorname{Im}\lambda \arg z}|z|^{\operatorname{Re}\lambda}$. Since the function $\operatorname{Im}\lambda\arg z$ is bounded on $\mathbf{H}$, Lemma 2.5 yields $h(\varphi_\lambda(\mathbf{H})) = h(\mathbf{H})/\operatorname{Re}\lambda = 1/\operatorname{Re}\lambda$.

For given $\beta \in (-\pi/2, \pi/2)$ and $\alpha \in (0, 2]$, we have $\mathrm{Sp}(\beta,\alpha) = \varphi_\lambda(\mathbf{H})$, where $\lambda = \alpha e^{i\beta}\cos\beta$. Since $\operatorname{Re}\lambda = \alpha\cos^2\beta$, we obtain

$$h(\mathrm{Sp}(\beta,\alpha)) = \frac{1}{\alpha\cos^2\beta}. \tag{2.1}$$

Note that the circular symmetrization of $\mathrm{Sp}(\beta,\alpha)$ is equal to $e^{-\pi i\alpha/2}S_\alpha$. Theorem 2.4 implies $h(\mathrm{Sp}(\beta,\alpha)) \ge h(S_\alpha) = 1/\alpha$. This agrees with the above computation. The formula (2.1) was already mentioned by Hansen [10, Example I, p. 245] for $\alpha = 2$ and can be deduced by the main result of [11].

3. Essén's main lemma

For a bounded domain D and a Borel measurable subset E of ∂D, we denote by $\omega(z, E, D)$ the harmonic measure of E viewed from z in D. In other words, $u(z) = \omega(z, E, D)$ is the bounded harmonic function on D determined by the boundary condition

$$u = \begin{cases} 1 & \text{on } E, \\ 0 & \text{on } \partial D \setminus E \end{cases}$$

in the sense of Perron–Wiener–Brelot (see [2] for details).

We now introduce Essén's main lemma in [7]. Let Ω be a domain in $\mathbf{C}$ with $0 \in \Omega$. For $R > 0$, let Ω_R be the connected component of $\Omega \cap \mathbf{D}_R$ containing 0 and set $\omega_R(z, \Omega) = \omega(z, \partial\Omega_R \cap \partial\mathbf{D}_R, \Omega_R)$, where $\mathbf{D}_R = \{z \in \mathbf{C} \colon |z| < R\}$. In view of its proof, the main lemma of Essén [7, §2] can be formulated as follows.

Lemma 3.1. *Let Ω be a domain with $0 \in \Omega$ and let $p_0 > 0$. If*

$$\omega_R(0, \Omega) = O(R^{-p_0}) \quad (R \to +\infty), \tag{3.1}$$

then $h(\Omega) \geq p_0$. Conversely, if $p_0 < h(\Omega)$, then (3.1) *holds.*

With the aid of Essén's main lemma, we are now able to show the following representation of $h(\Omega)$.

Lemma 3.2. *Let Ω be a plane domain containing the origin. Then*

$$h(\Omega) = -\limsup_{R \to +\infty} \frac{\log \omega_R(0, \Omega)}{\log R} = \liminf_{R \to +\infty} \frac{\log(1/\omega_R(0, \Omega))}{\log R}.$$

Proof. Let

$$q_0 = \limsup_{R \to +\infty} \frac{\log \omega_R(0, \Omega)}{\log R}.$$

For $q > q_0$, we have $\log \omega_R(0, \Omega) < q \log R$ for $R > R_0$, where R_0 is a large enough number. Then $\omega_R(0, \Omega) < R^q$ for $R > R_0$. By Lemma 3.1, we now have $h(\Omega) \geq -q$. Therefore, letting $q \to q_0$, we get $h(\Omega) \geq -q_0$.

We next take $p < h(\Omega)$. Then by Lemma 3.1 we have $\omega_R(0, \Omega) = O(R^{-p})$ as $R \to +\infty$. Hence, $\log \omega_R(0, \Omega) \leq -p \log R + O(1)$, which implies $q_0 \leq -p$. Letting $p \to h(\Omega)$, we get $q_0 \leq -h(\Omega)$, equivalently, $h(\Omega) \leq -q_0$.

Summarizing the above, we obtain $h(\Omega) = -q_0$ as required. $\square$

We are now ready to prove Theorem 2.4.

Proof of Theorem 2.4. We may assume that $0 \in \Omega \setminus N$ to show (i). Since N is polar and a polar set is removable for bounded harmonic functions (see [2, Cor. 5.2.3] for instance), we have $\omega_R(0, \Omega) = \omega_R(0, \Omega \setminus N)$ for $R > 0$. Hence, assertion (i) follows from Lemma 3.2.

Let now Ω^* be the circular symmetrization of a plane domain Ω with $0 \in \Omega$. We now fix $R > 0$. Since $\Omega_R \subset \Omega$, we have the relation $(\Omega_R)^* \subset \Omega^*$, and thus, $(\Omega_R)^* \subset (\Omega^*)_R$. A theorem of Baernstein II (see [3, Theorem 7]) asserts that

$$\omega(z, \partial\Omega_R \cap \partial\mathbf{D}_R, \Omega_R) \leq \omega(|z|, \partial(\Omega_R)^* \cap \partial\mathbf{D}_R, (\Omega_R)^*).$$

On the other hand, since $(\Omega_R)^* \subset (\Omega^*)_R$ and $\partial(\Omega_R)^* \cap \partial\mathbf{D}_R \subset \partial(\Omega^*)_R \cap \partial\mathbf{D}_R$, the maximum principle implies that

$$\omega(z, \partial(\Omega_R)^* \cap \partial\mathbf{D}_R, (\Omega_R)^*) \leq \omega(z, \partial(\Omega^*)_R \cap \partial\mathbf{D}_R, (\Omega^*)_R)$$

for $z \in (\Omega_R)^*$. Hence, we obtain

$$\begin{aligned}\omega_R(0,\Omega) = \omega(0, \partial\Omega_R \cap \partial\mathbf{D}_R, \Omega_R) &\le \omega(0, \partial(\Omega_R)^* \cap \partial\mathbf{D}_R, (\Omega_R)^*)\\ &\le \omega(0, \partial(\Omega^*)_R \cap \partial\mathbf{D}_R, (\Omega^*)_R) = \omega_R(0,\Omega^*).\end{aligned}$$

We now apply Lemma 3.2 to obtain the required assertion $h(\Omega) \ge h(\Omega^*)$. □

4. Local behaviour of quasiconformal mappings

Let $K \ge 1$ be a real number. A homeomorphism g of a subdomain Ω of the Riemann sphere $\widehat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ onto another one Ω' is called K-quasiconformal if g has locally square integrable partial derivatives (in the sense of distributions) on $\Omega \setminus \{\infty, g^{-1}(\infty)\}$ such that $|g_{\bar z}| \le k|g_z|$ a.e. on Ω, where $k = (K-1)/(K+1) \in [0,1)$. It is known (cf. [1]) that $g_z \ne 0$ a.e. on Ω and therefore the ratio $\mu_g = g_{\bar z}/g_z$ can be uniquely defined as a bounded measurable function on Ω with $\|\mu_g\|_\infty \le k$. The quantity μ_g is called the *complex dilatation* or *Beltrami coefficient* of g.

The local behaviour of quasiconformal mappings is well understood. If g is a K-quasiconformal mapping in a neighbourhood of the origin with $g(0) = 0$, then $c|z|^K \le |g(z)| \le C|z|^{1/K}$ for small enough z. (This can be seen, for example, in the following way. First we may assume that g is a bounded K-quasiconformal mapping of the unit disk $\mathbf{D}$. Let φ be the conformal homeomorphism of $g(\mathbf{D})$ onto $\mathbf{D}$ with $\varphi(0) = 0$. We can apply Mori's theorem [1] to the K-quasiconformal automorphism $G = \varphi \circ g$ of $\mathbf{D}$ to get $|z-w|^K/16^K \le |G(z) - G(w)| \le 16|z-w|^{1/K}$. Since φ is bi-Lipschitz continuous near the origin, we have the desired estimates.)

By the transformation $1/g(1/z)$, we obtain the following lemma.

Lemma 4.1. *Let g be a K-quasiconformal mapping on a neighbourhood of ∞ with $g(\infty) = \infty$. Then, there exist positive constants c and C such that*

$$c|z|^{1/K} \le |g(z)| \le C|z|^K$$

for large enough $|z|$.

We plug the last lemma with Lemma 2.5 to show the following.

Proposition 4.2. *Let φ be a conformal homeomorphism of an unbounded plane domain Ω onto another Ω' such that $z \to \infty$ in Ω precisely when $\varphi(z) \to \infty$ in Ω'. Suppose that there exists a K-quasiconformal mapping g around ∞ with $g(\infty) = \infty$ such that $\varphi(z) = g(z)$ for $z \in \Omega$ with large enough $|z|$. Then*

$$\frac{h(\Omega)}{K} \le h(\Omega') \le Kh(\Omega). \tag{4.1}$$

Note that the above assumption is always fulfilled when Ω is unbounded and φ has a K-quasiconformal extension to the complex plane $\mathbf{C}$.

Example 4.3. Fix a real number $K > 1$. Take $\alpha \in (0,1)$ and set $L = (1 - \alpha/2)K + \alpha/2$ and $\beta = \alpha/L$. We consider the conformal map $\varphi(z) = z^{\beta/\alpha}$ of the sector S_α onto S_β.

We now extend φ to the mapping $g\colon \mathbf{C} \to \mathbf{C}$ defined by $g(0) = 0$ and for $z \ne 0$ by

$$g(z) = \begin{cases} z^{\beta/\alpha}, & 0 \le \arg z \le \pi\alpha, \\ |z|^{\beta/\alpha} \exp\left(i\frac{2-\beta}{2-\alpha}\arg z\right), & -\pi(2-\alpha) \le \arg z < 0. \end{cases}$$

Then g is K-quasiconformal on $\mathbf{C}$. This can be seen by a straightforward computation or in the following way. The function g is nothing but $g_{\beta-1} \circ (g_{\alpha-1})^{-1}$, where g_κ is given in Example 4.5 below. Thus by (4.2) we have

$$\|\mu_g\|_\infty = \frac{\alpha - \beta}{1 - (\alpha - 1)(\beta - 1)} = \frac{L-1}{L+1-\alpha} = \frac{K-1}{K+1}.$$

Hence, we have confirmed that g is K-quasiconformal.

As we saw in Example 2.2, $h(S_\beta) = 1/\beta = L/\alpha = Lh(S_\alpha)$, namely, $L = h(S_\beta)/h(S_\alpha)$. This ratio $L = (1 - \alpha/2)K + \alpha/2$ tends to K as $\alpha \to 0$, which implies that the constant K cannot be replaced by any smaller number in Proposition 4.2.

As we have seen above, Proposition 4.2 is certainly sharp. However, for a specific domain Ω, we may improve the constant. For instance, if $\Omega = \mathbf{H}$, by Lemma 2.3 (5), $h(\Omega')$ is not less than $1/2$. On the other hand, $h(\mathbf{H})/K = 1/K$ may become much smaller. Indeed, we have a better estimate in this case.

Theorem 4.4. *Let $g\colon \mathbf{C} \to \mathbf{C}$ be a K-quasiconformal map which is conformal on the upper half-plane $\mathbf{H}$. Then the quasidisk $\Omega = g(\mathbf{H})$ satisfies*

$$\frac{K+1}{2K} \le h(\Omega) \le \frac{K+1}{2}.$$

The lower and upper bounds are both sharp.

Recall that a subdomain Ω of $\widehat{\mathbf{C}}$ is called a K-quasidisk if it is the image of the unit disk $\mathbf{D}$ under a K-quasiconformal homeomorphism of $\widehat{\mathbf{C}}$. Ω is called a quasidisk if it is a K-quasidisk for some $K \ge 1$.

The following examples show the sharpness.

Example 4.5. In Example 2.6, we set $\kappa = \lambda - 1$ and assume that $|\kappa| < 1$. Then the function $\varphi_{1+\kappa}$ extends to

$$g(z) = g_\kappa(z) = \begin{cases} z^{1+\kappa} & \text{for } \operatorname{Im} z \ge 0, \\ z\bar{z}^{\kappa} & \text{for } \operatorname{Im} z < 0. \end{cases}$$

Then $\mu_g = \kappa z/\bar{z}$ on $\operatorname{Im} z < 0$ and thus g is a $K(\kappa)$-quasiconformal automorphism of $\mathbf{C}$, where $K(\kappa) = (1 + |\kappa|)/(1 - |\kappa|)$. In particular, the image $\Omega = g(\mathbf{H}) = \varphi_{1+\kappa}(\mathbf{H})$ of $\mathbf{H}$ is an unbounded $K(\kappa)$-quasidisk. The following formula is sometimes useful:

$$(4.2) \qquad |\mu_{g_{\kappa'} \circ g_\kappa^{-1}}| = \left| \frac{\kappa' - \kappa}{1 - \bar{\kappa}\kappa'} \right| \quad \text{on } g_\kappa(\operatorname{Ext} \mathbf{H}).$$

For a given $K \ge 1$, we set $k = (K-1)/(K+1)$ as usual. If we let $\kappa = k$, then we have $K(k) = K$ and $\varphi_{1+k}(\mathbf{H}) = S_{1+k} = S_{2K/(K+1)}$. By Example 2.2, $h(\Omega) = (K+1)/2K$, which is the lower bound.

On the other hand, if we let $\kappa = -k$, then we have $K(-k) = K$ and $\varphi_{1-k}(\mathbf{H}) = S_{1-k} = S_{2/(K+1)}$. Similarly, we have $h(\Omega) = (K+1)/2$, which is the upper bound.

By the same argument used in the proof of Proposition 4.2, one can deduce Theorem 4.4 from the next proposition, which describes the local behaviour of quasiconformal mappings which are conformal on the upper half-plane.

Proposition 4.6. *Let g be a K-quasiconformal mapping of $\mathbf{D}$ onto a bounded domain with $g(0) = 0$. If g is conformal on $\mathbf{D}^+ = \{z \in \mathbf{D}\colon \operatorname{Im} z > 0\}$, then there* *exist positive constants c and C such that*

$$c|z|^{2K/(1+K)} \le |g(z)| \le C|z|^{2/(1+K)}, \quad z \in \mathbf{D}.$$

The exponents $2K/(1+K)$ and $2/(1+K)$ are both sharp.

For the proof, we need some preliminaries. A domain B in $\mathbf{C}$ is called a ring domain if the complement $\widehat{\mathbf{C}} \setminus B$ consists of two connected components. In the sequel, we always assume that both components are continua. Then B is known to be conformally equivalent to an annulus A of the form $\{r_2 < |z| < r_1\}$. The modulus of B is defined to be the number $\log(r_1/r_2)$ and denoted by $\operatorname{mod} B$.

The next lemma is essentially due to Teichmüller. The following form can be found in [9].

Lemma 4.7. *There exists an absolute constant $C_0 > 0$ with the following property. Let B be a ring domain in $\mathbf{C}$ separating 0 from ∞ with $\operatorname{mod} B > C_0$. Then B contains an annulus A of the form $\{r < |z| < R\}$ with $\operatorname{mod} B - \operatorname{mod} A \le C_0$.*

By using this lemma, one can show the following (see [9]).

Lemma 4.8. *There are positive absolute constants C_1 and C_2 with the following property. Let B be a ring domain in $\mathbf{C}$ with $\operatorname{mod} B > C_1$. Then*

$$\operatorname{diam} E_0 \le C_2 \operatorname{dist}(E_0, E_1) e^{-\operatorname{mod} B},$$

where E_0 and E_1 are bounded and unbounded components of $\widehat{\mathbf{C}} \setminus B$, respectively.

To state the next result, we introduce some quantities. Let g be a K-quasiconformal mapping of the unit disk $\mathbf{D}$ onto a bounded domain. Let μ be its complex dilatation, i.e., $\mu = g_{\bar z}/g_z$. This is a (Borel) measurable function on $\mathbf{D}$ with $|\mu| \le (K-1)/(K+1)$ a.e. in $\mathbf{D}$. We now define the measurable functions D_+ and D_- by

$$D_\pm(z) = \frac{|1 \pm \mu(z)\bar z/z|^2}{1 - |\mu(z)|^2}.$$

Note that $D_\pm \le K$ holds a.e. For $0 < r < R \le 1$, we set

$$I(r,R) = 2\pi \int_r^R \frac{1}{\int_0^{2\pi} D_-(te^{i\theta})d\theta} \frac{dt}{t},$$

and

$$J(r,R) = 2\pi \left(\int_0^{2\pi} \frac{d\theta}{\int_r^R D_+(te^{i\theta})\frac{dt}{t}} \right)^{-1}.$$

Then the following holds:

Lemma 4.9. (Reich and Walczak [16]) *Let g be a quasiconformal mapping of the unit disk, $I(r,R)$ and $J(r,R)$ be as above and $A = \{z : r < |z| < R\}$ for $0 < r < R \le 1$. Then*

$$I(r,R) \le \operatorname{mod} g(A) \le J(r,R).$$

We are now ready to prove Proposition 4.6.

Proof of Proposition 4.6. Let g satisfy the assumptions in the proposition. We may assume that $g(\mathbf{D}) \subset \mathbf{D}$ and set

$$r_1 = \operatorname{dist}(0, \partial g(\mathbf{D})).$$

Choose $\delta, \rho \in (0,1)$ so that $\frac{2}{K+1}\log(1/\delta) > C_0$ and $\frac{2}{K+1}\log(1/\rho) > C_1$, where C_0, C_1 are the constants in Lemmas 4.7 and 4.8, respectively. It is enough to show the inequality only for z with $|z| \le \rho$. Fix now an arbitrary point z_0 with $0 < r_0 = |z_0| \le \rho$ and put $w_0 = g(z_0)$. Set $A = \{z\colon r_0 < |z| < 1\}$ and $B = g(A)$. Let E_0 and E_1 be as in Lemma 4.8. Then $0, w_0 \in E_0$ while $1 \in E_1$. In particular, $|w_0| \le \operatorname{diam} E_0$ and $\operatorname{dist}(E_0, E_1) \le 1$.

Since $D_\pm = 1$ a.e. in $\mathbf{D}^+$ and $D_\pm \le K$ a.e. in $\mathbf{D} \setminus \mathbf{D}^+$, it is easily seen that

$$\frac{2}{K+1}\log\frac{R}{r} \le I(r,R) \quad \text{and} \quad J(r,R) \le \frac{2K}{K+1}\log\frac{R}{r}.$$

Therefore, Lemma 4.9 now implies

$$\frac{2}{K+1}\log\frac{R}{r} \le \operatorname{mod} g(\{r < |z| < R\}) \le \frac{2K}{K+1}\log\frac{R}{r}. \tag{4.3}$$

In particular, we have

$$\operatorname{mod} B \ge \frac{2}{K+1}\log\frac{1}{r_0} \ge \frac{2}{K+1}\log\frac{1}{\rho} > C_1.$$

Thus we can apply Lemma 4.8 to obtain the estimate

$$|w_0| \le \operatorname{diam} E_0 \le C_2 \operatorname{dist}(E_0, E_1) e^{-\operatorname{mod} B} \le C_2 e^{-(2/(K+1))\log(1/r_0)} = C_2 |z_0|^{2/(K+1)}.$$

Secondly, we make a lower estimate. We further set $\tilde{A} = \{z\colon \delta r_0 < |z| < 1\}$ and $A_0 = \{z\colon \delta r_0 < |z| < r_0\}$. Let

$$r_2 = \max\{|g(z)|\colon |z| = \delta r_0\}.$$

By (4.3) and the choice of δ, we now have

$$\operatorname{mod} g(A_0) \ge \frac{2}{K+1}\log\frac{1}{\delta} > C_0.$$

Thus, by Lemma 4.7, the annulus $\{r_2 < |w| < r_2 + \varepsilon\}$ is contained in $g(A_0)$ for a sufficiently small $\varepsilon > 0$. Since w_0 lies on the outer boundary of $g(A_0)$, one has $r_2 \le |w_0|$. Since the annulus $A' = \{w\colon r_2 < |w| < r_1\}$ is contained in $g(\tilde{A})$, we have the following estimate by monotonicity of the modulus and (4.3):

$$\operatorname{mod} A' \le \operatorname{mod} g(\tilde{A}) \le \frac{2K}{K+1}\log\frac{1}{\delta r_0}.$$

Taking into account the inequality $\log(r_1/|w_0|) \le \operatorname{mod} A'$, we have

$$r_1/|w_0| \le (\delta r_0)^{-2K/(K+1)}.$$

This is equivalent to

$$r_1(\delta r_0)^{2K/(K+1)} = c|z_0|^{2K/(K+1)} \le |w_0|,$$

where $c = r_1\delta^{2K/(K+1)}$. Thus we are done. □

Remark. With a slight modification, the above proof also yields a result for a sector S_α instead of the upper half-plane $\mathbf{H}$ in Theorem 4.4.

We conclude the present note by giving future problems. We discussed in this section the distortion of the number $h(\Omega)$ under conformal mappings which extend to K-quasiconformal automorphisms of $\mathbf{C}$. What can we say if we replace conformal mappings by quasiconformal mappings? For example, let g be a K-quasiconformal automorphism of $\mathbf{C}$ and let $\Omega' = g(\Omega)$ for an unbounded domain Ω. What is relationship between $h(\Omega)$ and $h(\Omega')$? Even the equivalence of the conditions $h(\Omega) > 0$ and $h(\Omega') > 0$ is not clear.

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