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REAL INTERPOLATION FOR GRAND BESOV AND TRIEBEL–LIZORKIN SPACES ON RD-SPACES

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Abstract. Let $\mathscr{X}$ be an RD-space, namely, a metric space enjoying both doubling and reverse doubling properties. In this paper, for all $s \in [-1,1]$ and $p,q \in (0,\infty]$, the authors introduce the grand Besov spaces $\mathscr{A}\dot{B}^s_{p,q}(\mathscr{X})$ and grand Triebel–Lizorkin spaces $\mathscr{A}\dot{F}^s_{p,q}(\mathscr{X})$, and prove that when $\epsilon \in (0,1)$, $|s| < \epsilon$ and $p \in (\max\{n/(n+\epsilon), n/(n+\epsilon+s)\},\infty]$, $\mathscr{A}\dot{B}^s_{p,q}(\mathscr{X}) \cap (\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))' = \dot{B}^s_{p,q}(\mathscr{X})$ with $q \in (0,\infty]$ and $\mathscr{A}\dot{F}^s_{p,q}(\mathscr{X}) \cap (\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))' = \dot{F}^s_{p,q}(\mathscr{X})$ with $q \in (\max\{n/(n+\epsilon), n/(n+\epsilon+s)\},\infty]$ for all admissible β and γ, where $\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma)$ is the space of test functions. As applications, the authors obtain some real interpolation results on these grand Besov and Triebel–Lizorkin spaces. The corresponding results for inhomogeneous spaces are also presented.

1. Introduction

The theory of Besov and Triebel–Lizorkin spaces on metric spaces has developed rapidly in recent years. In particular, a theory of Besov spaces $B^s_{p,q}(\mathscr{X})$ and Triebel–Lizorkin spaces $F^s_{p,q}(\mathscr{X})$ on the so-called RD-spaces $\mathscr{X}$, namely, metric spaces enjoying both doubling and reverse doubling properties, was established in [14, 15] and further developed in [20, 24]; see these papers and their references for part of the history of these spaces on metric measure spaces. Very recently, in [18, 19], a class of grand Besov spaces $\mathscr{A}\dot{B}^s_{p,q}$ and grand Triebel–Lizorkin spaces $\mathscr{A}\dot{F}^s_{p,q}$, and their inhomogeneous counterparts, $\mathscr{A}B^s_{p,q}$ and $\mathscr{A}F^s_{p,q}$, on both $\mathbf{R}^n$ for full range of parameters and RD-spaces for $s \in (0,1)$ and $p,q \in (0,\infty]$ were introduced and proved therein that these spaces coincide with, respectively, Besov spaces and Triebel–Lizorkin spaces for some parameters s, p and q. Furthermore, the grand Triebel–Lizorkin spaces $\mathscr{A}\dot{F}^s_{p,q}$ and $\mathscr{A}F^s_{p,q}$ were also proved to cover, respectively, the Hajłasz–Sobolev spaces $\dot{M}^{s,p}$ and $M^{s,p}$. Recall that the Hajłasz–Sobolev space when $s = 1$ was introduced by Hajłasz in [11, 12] and when $s \in (0,1)$ in [22]. In recent years, a lot of attention has

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been paid to the study of Hajłasz–Sobolev spaces on metric spaces; see, for example, [9, 13, 17].

In this paper, motivated by [17, 18, 19], for all parameters $s \in [-1, 1]$ and $p, q \in (0, \infty]$, we introduce the grand Besov spaces $\mathscr{A}\dot{B}^s_{p,q}(\mathscr{X})$ and grand Triebel–Lizorkin spaces $\mathscr{A}\dot{F}^s_{p,q}(\mathscr{X})$ on an RD-space $\mathscr{X}$ via the grand Littlewood–Paley g-function, and prove that when $\epsilon \in (0, 1)$, $|s| < \epsilon$ and $p \in (\max\{n/(n + \epsilon), n/(n + \epsilon + s)\}, \infty]$, $\mathscr{A}\dot{B}^s_{p,q}(\mathscr{X}) \cap (\mathring{\mathscr{G}}^\epsilon_0(\beta, \gamma))' = \dot{B}^s_{p,q}(\mathscr{X})$ with $q \in (0, \infty]$, and $\mathscr{A}\dot{F}^s_{p,q}(\mathscr{X}) \cap (\mathring{\mathscr{G}}^\epsilon_0(\beta, \gamma))' = \dot{F}^s_{p,q}(\mathscr{X})$ with $q \in (\max\{n/(n + \epsilon), n/(n + \epsilon + s)\}, \infty]$ in the sense of equivalent quasi-norms for all admissible β and γ, where $\mathring{\mathscr{G}}^\epsilon_0(\beta, \gamma)$ is the space of test functions introduced in [15]; see Theorem 1.1 below. This generalizes [18, Theorem 1.4] and [19, Theorem 4.1] by taking $s \in (0, 1)$, $p \in (\max\{n/(n + \epsilon), n/(n + \epsilon + s)\}, \infty]$ and $q \in (0, \infty]$. As an application of these coincidences and via the Calderón reproducing formulae in [15], we establish some real interpolation conclusions of the spaces $\mathscr{A}\dot{B}^s_{p,q}(\mathscr{X})$ and $\mathscr{A}\dot{F}^s_{p,q}(\mathscr{X})$, which generalize the real interpolation theorems of Besov and Triebel–Lizorkin spaces on Ahlfors n-regular metric spaces in [23] and RD-spaces in [15]; see Theorem 1.2 below. The corresponding results on inhomogeneous grand Besov spaces $\mathscr{A}B^s_{p,q}(\mathscr{X})$ and grand Triebel–Lizorkin spaces $\mathscr{A}F^s_{p,q}(\mathscr{X})$ are also obtained; see Theorems 3.1 and 3.2.

We begin with the notion of RD-spaces in [15] (see also [24]).

Definition 1.1. Let $(\mathscr{X}, d, \mu)$ be a metric space with a regular Borel measure μ such that all balls defined by the metric d have finite and positive measures. For any $x \in \mathscr{X}$ and $r \in (0, \infty)$, let $B(x, r) \equiv \{y \in \mathscr{X} : d(x, y) < r\}$. The triple $(\mathscr{X}, d, \mu)$ is called an RD-*space* if there exist constants $0 < \kappa \le n$ and $0 < C_1 \le 1 \le C_2 < \infty$ such that for all $x \in \mathscr{X}$, $0 < r < 2\mathrm{diam}(\mathscr{X})$ and $1 \le \lambda < 2\mathrm{diam}(\mathscr{X})/r$,

$$C_1\lambda^\kappa \mu(B(x, r)) \le \mu(B(x, \lambda r)) \le C_2\lambda^n \mu(B(x, r)), \tag{1.1}$$

where $\mathrm{diam}(\mathscr{X}) \equiv \sup_{x,y\in\mathscr{X}} d(x, y)$.

We remark that a connected space of homogeneous type in the sense of Coifman and Weiss [7, 8] (with the quasi-metric replaced by metric) is an RD-space; see [24].

In what follows, we *always assume that* $(\mathscr{X}, d, \mu)$ *is an* RD-*space*. Let $V(x, y) \equiv \mu(B(x, d(x, y)))$ and $V_r(x) \equiv \mu(B(x, r))$ for any x, $y \in \mathscr{X}$ and $r \in (0, \infty)$. It is easy to see that $V(x, y) \sim V(y, x)$.

Definition 1.2. Let $x_1 \in \mathscr{X}$, $r \in (0, \infty)$, $\beta \in (0, 1]$ and $\gamma \in (0, \infty)$. A function φ on $\mathscr{X}$ is said to be in the *space* $\mathscr{G}(x_1, r, \beta, \gamma)$ if there exists a nonnegative constant C such that

(i) $|\varphi(x)| \le C\frac{1}{V_r(x_1)+V(x_1,x)}[\frac{r}{r+d(x_1,x)}]^\gamma$ for all $x \in \mathscr{X}$;

(ii) $|\varphi(x)-\varphi(y)| \le C[\frac{d(x,y)}{r+d(x_1,x)}]^\beta \frac{1}{V_r(x_1)+V(x_1,x)}[\frac{r}{r+d(x_1,x)}]^\gamma$ for all x, $y \in \mathscr{X}$ satisfying $d(x, y) \le (r + d(x_1, x))/2$.

Moreover, for any $\varphi \in \mathscr{G}(x_1, r, \beta, \gamma)$, its *norm* in $\mathscr{G}(x_1, r, \beta, \gamma)$ is defined by

$$\|\varphi\|_{\mathscr{G}(x_1,r,\beta,\gamma)} \equiv \inf\{C : \text{(i) and (ii) hold}\}.$$

Throughout the whole paper, we fix $x_1 \in \mathscr{X}$ and let $\mathscr{G}(\beta, \gamma) \equiv \mathscr{G}(x_1, 1, \beta, \gamma)$. The space $\mathscr{G}(\beta, \gamma)$ is a Banach space with respect to the norm $\|\cdot\|_{\mathscr{G}(\beta,\gamma)}$; see [15, Section 2.1].

For any given $\epsilon \in (0,1]$, let $\mathscr{G}_0^\epsilon(\beta,\gamma)$ be the *completion of the space* $\mathscr{G}(\epsilon,\epsilon)$ *in* $\mathscr{G}(\beta,\gamma)$ when $\beta,\gamma \in (0,\epsilon]$. Obviously, $\mathscr{G}_0^\epsilon(\epsilon,\epsilon)=\mathscr{G}(\epsilon,\epsilon)$. We also let $\mathring{\mathscr{G}}(x_1,r,\beta,\gamma) \equiv \{f \in \mathscr{G}(x_1,r,\beta,\gamma)\colon \int_{\mathscr{X}} f(x)\,dx = 0\}$, and the *space* $\mathring{\mathscr{G}}_0^\epsilon(\beta,\gamma)$ is defined to be the completion of the space $\mathring{\mathscr{G}}(\epsilon,\epsilon)$ in $\mathring{\mathscr{G}}(\beta,\gamma)$ when $\beta,\gamma \in (0,\epsilon]$. Moreover, if $f \in \mathring{\mathscr{G}}_0^\epsilon(\beta,\gamma)$, we then define $\|f\|_{\mathring{\mathscr{G}}_0^\epsilon(\beta,\gamma)} \equiv \|f\|_{\mathscr{G}(\beta,\gamma)}$. Let $(\mathscr{G}_0^\epsilon(\beta,\gamma))'$ and $(\mathring{\mathscr{G}}_0^\epsilon(\beta,\gamma))'$ be respectively the *dual spaces* of $\mathscr{G}_0^\epsilon(\beta,\gamma)$ and $\mathring{\mathscr{G}}_0^\epsilon(\beta,\gamma)$, endowed with the weak $*$-topology. It is easy to see that $(\mathring{\mathscr{G}}_0^\epsilon(\beta,\gamma))' = (\mathscr{G}_0^\epsilon(\beta,\gamma))'/\mathbf{C}$; see [15].

We now recall the notion of approximations of the identity on RD-spaces in [15, Definition 2.3].

Definition 1.3. Let $\epsilon_1 \in (0,1]$. A sequence $\{S_k\}_{k\in\mathbf{Z}}$ of bounded linear integral operators on $L^2(\mathscr{X})$ is called an *approximation of the identity of order* ϵ_1 (for short, ϵ_1-AOTI) with bounded support, if there exist positive constants C_3 and C_4 such that for all $k\in\mathbf{Z}$ and all x, x', y and $y' \in \mathscr{X}$, $S_k(x,y)$, the *integral kernel* of S_k, is a measurable function from $\mathscr{X}\times\mathscr{X}$ into $\mathbf{C}$ satisfying

(i) $S_k(x,y)=0$ if $d(x,y) > C_4 2^{-k}$ and $|S_k(x,y)| \le C_3 \frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(x)}$;

(ii) $|S_k(x,y)-S_k(x',y)| \le C_3 2^{k\epsilon_1} \frac{[d(x,x')]^{\epsilon_1}}{V_{2^{-k}}(x)+V_{2^{-k}}(x)}$ for $d(x,x') \le \max\{C_4,1\}2^{1-k}$;

(iii) Property (ii) holds with x and y interchanged;

(iv) $|[S_k(x,y)-S_k(x,y')]-[S_k(x',y)-S_k(x',y')]| \le C_3 2^{2k\epsilon_1} \frac{[d(x,x')]^{\epsilon_1}[d(y,y')]^{\epsilon_1}}{V_{2^{-k}}(x)+V_{2^{-k}}(x)}$ for $d(x,x') \le \max\{C_4,1\}2^{1-k}$ and $d(x,x') \le \max\{C_4,1\}2^{1-k}$;

(v) $\int_{\mathscr{X}} S_k(x,z)\,d\mu(z) = 1 = \int_{\mathscr{X}} S_k(z,y)\,d\mu(z)$.

It was proved in [15] that, for any $\epsilon_1 \in (0,1]$, there always exists an ϵ_1-AOTI with bounded support on an RD-space $\mathscr{X}$. In what follows, for all $k\in\mathbf{Z}$, we set $D_k \equiv S_k - S_{k-1}$, and for any $\epsilon\in(0,1)$ and $|s|<\epsilon$, we let $p(s,\epsilon) \equiv \max\{n/(n+\epsilon), n/(n+\epsilon+s)\}$.

Let $\mathscr{X}$ be an RD-spaces with $\mu(\mathscr{X})=\infty$. We recall the homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathscr{X})$ and Triebel–Lizorkin spaces $\dot{F}^s_{p,q}(\mathscr{X})$ on RD-spaces; see [15, Definition 5.8].

Definition 1.4. Let $\epsilon\in(0,1)$, $|s|<\epsilon$ and $p\in(p(s,\epsilon),\infty]$. Let $\{S_k\}_{k\in\mathbf{Z}}$ be an ϵ-AOTI with bounded support as in Definition 1.3.

(i) Let $q\in(0,\infty]$. The *homogeneous Besov space* $\dot{B}^s_{p,q}(\mathscr{X})$ is defined to be the set of all $f\in(\mathring{\mathscr{G}}_0^\epsilon(\beta,\gamma))'$, for some β, γ satisfying that

$$\begin{aligned}&\max\{s,0,-s+n(1/p-1)_+\} < \beta < \epsilon,\\ &\max\{s-\kappa/p, n(1/p-1)_+, -s+n(1/p-1)_+-\kappa(1-1/p)_+)\} < \gamma < \epsilon,\end{aligned} \tag{1.2}$$

such that $\|f\|_{\dot{B}^s_{p,q}(\mathscr{X})} \equiv \{\sum_{k\in\mathbf{Z}} 2^{ksq}\|D_k(f)\|^q_{L^p(\mathscr{X})}\}^{1/q} < \infty$ with the usual modifications made when $p=\infty$ or $q=\infty$.

(ii) Let $q\in(p(s,\epsilon),\infty]$. The *homogeneous Triebel–Lizorkin space* $\dot{F}^s_{p,q}(\mathscr{X})$ is defined to be the set of all $f\in(\mathring{\mathscr{G}}_0^\epsilon(\beta,\gamma))'$ for some β, γ satisfying (1.2) such that $\|f\|_{\dot{F}^s_{p,q}(\mathscr{X})}<\infty$, where when $p<\infty$,

$$\|f\|_{\dot{F}^s_{p,q}(\mathscr{X})} \equiv \left\| \left\{ \sum_{k\in\mathbf{Z}} 2^{ksq}|D_k(f)|^q \right\}^{1/q} \right\|_{L^p(\mathscr{X})}$$

with the usual modification made when $q=\infty$, and when $p=\infty$,

$$\|f\|_{\dot F^s_{\infty,q}(\mathscr{X})}\equiv\sup_{l\in\mathbf{Z}}\sup_{x\in\mathscr{X}}\left\{\frac{1}{\mu(B(x,2^{-l}))}\int_{B(x,2^{-l})}\sum_{k=l}^{\infty}2^{ksq}|D_k(f)(x)|^q\,d\mu(x)\right\}^{1/q}$$

with the usual modification made when $q=\infty$.

It was proved in [15] that the spaces $\dot B^s_{p,q}(\mathscr{X})$ and $\dot F^s_{p,q}(\mathscr{X})$ are independent of the choices of the approximations of the identity and the distribution spaces $(\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))'$ with β, γ as in (1.2). Many properties of the spaces $\dot B^s_{p,q}(\mathscr{X})$ and $\dot F^s_{p,q}(\mathscr{X})$, such as the frame characterization, the real interpolation and the dual theory, were also established in [15]. Recently, Müller and Yang [20] characterized the spaces $\dot B^s_{p,q}(\mathscr{X})$ and $\dot F^s_{p,q}(\mathscr{X})$ in terms of differences. To be precise, it was proved in [20] that when $s\in(0,1)$, $p\in[1,\infty]$, and $q\in(0,\infty]$, the space $\dot B^s_{p,q}(\mathscr{X})$ coincides with both the *space of all locally p-integrable functions f on $\mathscr{X}$ satisfying that*

$$\left\{\sum_{v\in\mathbf{Z}}2^{vsq}\left[\int_{\mathscr{X}}\frac{1}{\mu(B(x,C_12^{-v}))}\int_{B(x,C_12^{-v})}|f(x)-f(y)|^p\,d\mu(y)\,d\mu(x)\right]^{q/p}\right\}^{1/q}<\infty$$

and the *space of all locally integrable functions f on $\mathscr{X}$ satisfying that*

$$\left\{\sum_{v\in\mathbf{Z}}2^{vsq}\left[\int_{\mathscr{X}}\left[\frac{1}{\mu(B(x,C_12^{-v}))}\int_{B(x,C_12^{-v})}|f(x)-f(y)|d\mu(y)\right]^p d\mu(x)\right]^{q/p}\right\}^{1/q}<\infty,$$

and when $s\in(0,1)$, $p\in(1,\infty)$ and $q\in(1,\infty]$, the space $\dot F^s_{p,q}(\mathscr{X})$ coincides with the *space of all locally integrable functions f on $\mathscr{X}$ satisfying that*

$$\left\|\left\{\sum_{v\in\mathbf{Z}}2^{vsq}\left(\frac{1}{\mu(B(\cdot,C_22^{-v}))}\int_{B(\cdot,C_22^{-v})}|f(\cdot)-f(y)|\,d\mu(y)\right)^q\right\}^{1/q}\right\|_{L^p(\mathscr{X})}<\infty,$$

where C_1 and C_2 are positive constants independent of f.

Remark 1.1. Let $\alpha\in[0,\infty)$, $q\in(0,\infty]$, $p\in[1,\infty)$ and (X,d,μ) be a doubling metric measure space. Recently, Gogatishvili, Koskela and Shanmugalingam [10] introduced the *Besov space* $B^\alpha_{p,q}(\mathscr{X})$, which is defined to be space of all locally p-integrable functions f on $\mathscr{X}$ such that

$$\left\{\int_0^\infty\left[\int_{\mathscr{X}}\frac{1}{\mu(B(x,t))}\int_{B(x,t)}|f(x)-f(y)|^p\,d\mu(y)\,d\mu(x)\right]^{q/p}\frac{dt}{t^{\alpha q+1}}\right\}^{1/q}<\infty.$$

This definition can be regarded as a "continuous" variant of the Besov spaces introduced in [20] and, as was pointed by Gogatishvili, Koskela and Shanmugalingam [10, p. 216], when $\alpha\in(0,1)$ and $\mathscr{X}$ is an RD-space, this space also coincides with the Besov space defined via test functions in Definition 1.4.

Following the ideas in [18] and [19], we define the homogeneous grand Besov and Triebel–Lizorkin space as follows:

Definition 1.5. Let $s\in[-1,1]$, $q\in(0,\infty]$ and $\mathscr{A}\equiv\{\mathscr{A}_k(x)\}_{k\in\mathbf{Z},x\in\mathscr{X}}$ with $\mathscr{A}_k(x)\equiv\{\phi\in\mathring{\mathscr{G}}(1,2)\colon\|\phi\|_{\mathscr{G}(x,2^{-k},1,2)}\le1\}$ for all $x\in\mathscr{X}$ and all $k\in\mathbf{Z}$.

(i) The *homogeneous grand Besov space* $\mathscr{A}\dot{B}^s_{p,q}(\mathscr{X})$ with $p\in(0,\infty]$ is defined to be the space of all $f\in(\mathscr{G}(1,2))'$ such that

$$\|f\|_{\mathscr{A}\dot{B}^s_{p,q}(\mathscr{X})}\equiv\left\{\sum_{k\in\mathbf{Z}}2^{kqs}\left\|\sup_{\phi\in\mathscr{A}_k(\cdot)}|\langle f,\phi\rangle|\right\|^q_{L^p(\mathscr{X})}\right\}^{1/q}<\infty$$

with the usual modifications made when $p=\infty$ or $q=\infty$.

(ii) The *homogeneous grand Triebel–Lizorkin space* $\mathscr{A}\dot{F}^s_{p,q}(\mathscr{X})$ is defined to be the space of all $f\in(\mathscr{G}(1,2))'$ such that $\|f\|_{\mathscr{A}\dot{F}^s_{p,q}(\mathscr{X})}<\infty$, where when $p\in(0,\infty)$,

$$\|f\|_{\mathscr{A}\dot{F}^s_{p,q}(\mathscr{X})}\equiv\left\|\left\{\sum_{k\in\mathbf{Z}}2^{kqs}\sup_{\phi\in\mathscr{A}_k(\cdot)}|\langle f,\phi\rangle|^q\right\}^{1/q}\right\|_{L^p(\mathscr{X})}$$

with the usual modification made when $q=\infty$, and when $p=\infty$,

$$\|f\|_{\mathscr{A}\dot{F}^s_{\infty,q}(\mathscr{X})}\equiv\sup_{l\in\mathbf{Z}}\sup_{x\in\mathscr{X}}\left\{\frac{1}{\mu(B(x,2^{-l}))}\int_{B(x,2^{-l})}\sum_{k=l}^{\infty}2^{ksq}\sup_{\phi\in\mathscr{A}_k(x)}|\langle f,\phi\rangle|^q\,d\mu(x)\right\}^{1/q}$$

with the usual modification made when $q=\infty$.

By the same reason as in [18, Remark 4.1] (see also [19, Remark 4.2], noticing that $(\mathring{\mathscr{G}}(1,2))'=(\mathscr{G}(1,2))'/\mathbf{C}$, if we replace $(\mathscr{G}(1,2))'$ by $(\mathring{\mathscr{G}}(1,2))'$ in Definition 1.5, the obtained new spaces, modulo constants, are respectively equivalent to the original spaces in Definition 1.5.

Remark 1.2. Let all the notation be as in Definition 1.5.

(i) Similarly to the proof of [15, Proposition 5.7], we obtain that if $f\in(\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))'$ with $\|f\|_{\mathscr{A}\dot{B}^s_{p,q}(\mathscr{X})}<\infty$ (resp. $\|f\|_{\mathscr{A}\dot{F}^s_{p,q}(\mathscr{X})}<\infty$), then $f\in(\mathring{\mathscr{G}}^\epsilon_0(\widetilde{\beta},\widetilde{\gamma}))'$ for every $\widetilde{\beta}$, $\widetilde{\gamma}$ satisfying (1.2), namely, for any $h\in\mathring{\mathscr{G}}^\epsilon_0(\widetilde{\beta},\widetilde{\gamma})$, $|\langle f,h\rangle|\le C\|f\|_{\mathscr{A}\dot{B}^s_{p,q}(\mathscr{X})}\|h\|_{\mathring{\mathscr{G}}^\epsilon_0(\widetilde{\beta},\widetilde{\gamma})}$ (resp. $|\langle f,h\rangle|\le C\|f\|_{\mathscr{A}\dot{F}^s_{p,q}(\mathscr{X})}\|h\|_{\mathring{\mathscr{G}}^\epsilon_0(\widetilde{\beta},\widetilde{\gamma})}$), where C is a positive constant independent of f and h.

(ii) Recall that when $s\in(0,1)$ and $p\in(p(s,\epsilon),\infty]$, it was proved, respectively, in [19, Theorem 4.1] and [18, Theorem 1.4] that $\mathscr{A}\dot{B}^s_{p,q}(\mathscr{X})=\dot{B}^s_{p,q}(\mathscr{X})$ for $q\in(0,\infty]$ and $\mathscr{A}\dot{F}^s_{p,q}(\mathscr{X})=\dot{F}^s_{p,q}(\mathscr{X})$ for $q\in(p(s,\epsilon),\infty]$ with equivalent quasi-norms via a Sobolev embedding theorem (see Lemmas 4.1 and 4.2 in [18]). On the other hand, from [15, Proposition 5.10], it follows that $\dot{B}^s_{p,q}(\mathscr{X})\subset(\mathring{\mathscr{G}}^\epsilon_0(\widetilde{\beta},\widetilde{\gamma}))'$ and $\dot{F}^s_{p,q}(\mathscr{X})\subset(\mathring{\mathscr{G}}^\epsilon_0(\widetilde{\beta},\widetilde{\gamma}))'$ for every $\widetilde{\beta}$, $\widetilde{\gamma}$ satisfying (1.2). Thus, the statement (i) of this remark when $s\in(0,1)$ still holds if we only assume that $f\in\mathscr{A}\dot{B}^s_{p,q}(\mathscr{X})$ (resp. $f\in\mathscr{A}\dot{F}^s_{p,q}(\mathscr{X})$).

We have the following coincidences.

Theorem 1.1. *Let all notation be as in Definition 1.4. Then*

$$\mathscr{A}\dot{B}^s_{p,q}(\mathscr{X})\cap(\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))'=\dot{B}^s_{p,q}(\mathscr{X})\text{ and }\mathscr{A}\dot{F}^s_{p,q}(\mathscr{X})\cap(\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))'=\dot{F}^s_{p,q}(\mathscr{X})$$

with equivalent quasi-norms.

When $s\in(0,1)$, by Remark 1.2(ii), we know that $\mathscr{A}\dot{B}^s_{p,q}(\mathscr{X})\cap(\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))'=\mathscr{A}\dot{B}^s_{p,q}(\mathscr{X})$ and $\mathscr{A}\dot{F}^s_{p,q}(\mathscr{X})\cap(\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))'=\mathscr{A}\dot{F}^s_{p,q}(\mathscr{X})$ with β and γ as in (1.2). Thus, Theorem 1.1 generalizes [19, Theorem 4.1] and [18, Theorem 1.4] by taking $s\in(0,1)$,

$p \in (p(s,\epsilon),\infty]$ and $q \in (0,\infty]$. However, when $s \in (-1,0]$, it is still unclear so far if we can replace $\mathscr{A}\dot{B}^s_{p,q}(\mathscr{X}) \cap (\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))'$ (resp. $\mathscr{A}\dot{F}^s_{p,q}(\mathscr{X}) \cap (\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))'$) by $\mathscr{A}\dot{B}^s_{p,q}(\mathscr{X})$ (resp. $\mathscr{A}\dot{F}^s_{p,q}(\mathscr{X})$) in Theorem 1.1.

Let us now recall some general background on the real interpolation for quasi-Banach spaces in [21]; see also [4] for the case of Banach spaces. Let $\mathscr{H}$ be a linear complex Hausdorff space, and let A_0, A_1 be two complex quasi-Banach spaces such that $A_0 \subset \mathscr{H}$ and $A_1 \subset \mathscr{H}$. Let $A_0 + A_1$ be the set of all elements $a \in \mathscr{H}$ which can be represented as $a = a_0 + a_1$ with $a_0 \in A_0$ and $a_1 \in A_1$. Then Peetre's K-functional of $a = a_0 + a_1$ at $t \in (0,\infty)$ is given by

$$K(t,a) \equiv K(t,a;A_0,A_1) \equiv \inf\left\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} \colon a = a_0 + a_1,\ a_0 \in A_0 \text{ and } a_1 \in A_1\right\}.$$

Definition 1.6. Let $\sigma \in (0,1)$ and $q \in (0,\infty)$. The interpolation space $(A_0,A_1)_{\sigma,q}$ is defined by

$$(A_0,A_1)_{\sigma,q} \equiv \left\{a\colon a \in A_0 + A_1,\ \|a\|_{(A_0,A_1)_{\sigma,q}} \equiv \left(\int_0^\infty [t^{-\sigma}K(t,a)]^q \,\frac{dt}{t}\right)^{1/q} < \infty\right\}.$$

If $\sigma \in (0,1)$ and $q = \infty$, then define

$$(A_0,A_1)_{\sigma,\infty} \equiv \left\{a\colon a \in A_0 + A_1,\ \|a\|_{(A_0,A_1)_{\sigma,\infty}} \equiv \sup_{0<t<\infty} t^{-\sigma}K(t,a) < \infty\right\}.$$

Using the Calderón reproducing formulae obtained in [15], we establish the following interpolation theorem. By Remark 1.2(i), in the below proof of Theorem 1.2, we choose $\mathscr{H} = (\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))'$ with β and γ as in (1.2).

Theorem 1.2. *Let ϵ, β and γ be as in Definition 1.4, $\sigma \in (0,1)$ and $q \in (0,\infty]$.*

(i) *Let $s \equiv 1 - 2\sigma$. Then for $p \in (p(s,\epsilon),\infty]$,*

$$\left(\mathscr{A}\dot{B}^1_{p,\infty}(\mathscr{X}) \cap (\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))', \mathscr{A}\dot{B}^{-1}_{p,\infty}(\mathscr{X}) \cap (\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))'\right)_{\sigma,q} = \dot{B}^s_{p,q}(\mathscr{X})$$

and

$$\left(\mathscr{A}\dot{F}^1_{p,\infty}(\mathscr{X}) \cap (\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))', \mathscr{A}\dot{F}^{-1}_{p,\infty}(\mathscr{X}) \cap (\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))'\right)_{\sigma,q} = \dot{B}^s_{p,q}(\mathscr{X}).$$

(ii) *Let $s_1 \in (-\epsilon,\epsilon)$, $s \equiv (1-\sigma) + \sigma s_1$ and $p \in (p(s_1,\epsilon),\infty]$. If $q_1 \in (0,\infty]$, then*

$$\left(\mathscr{A}\dot{B}^1_{p,\infty}(\mathscr{X}) \cap (\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))', \mathscr{A}\dot{B}^{s_1}_{p,q_1}(\mathscr{X}) \cap (\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))'\right)_{\sigma,q} = \dot{B}^s_{p,q}(\mathscr{X});$$

if $q_1 \in (p(s_1,\epsilon),\infty]$, then

$$\left(\mathscr{A}\dot{F}^1_{p,\infty}(\mathscr{X}) \cap (\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))', \mathscr{A}\dot{F}^{s_1}_{p,q_1}(\mathscr{X}) \cap (\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))'\right)_{\sigma,q} = \dot{B}^s_{p,q}(\mathscr{X}).$$

(iii) *Let $s_0 \in (-\epsilon,\epsilon)$, $s \equiv (1-\sigma)s_0 - \sigma$ and $p \in (p(s_0,\epsilon),\infty]$. If $q_0 \in (0,\infty]$, then*

$$\left(\mathscr{A}\dot{B}^{s_0}_{p,q_0}(\mathscr{X}) \cap (\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))', \mathscr{A}\dot{B}^{-1}_{p,\infty}(\mathscr{X}) \cap (\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))'\right)_{\sigma,q} = \dot{B}^s_{p,q}(\mathscr{X});$$

if $q_0 \in (p(s_0,\epsilon),\infty]$, then

$$\left(\mathscr{A}\dot{F}^{s_0}_{p,q_0}(\mathscr{X}) \cap (\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))', \mathscr{A}\dot{F}^{-1}_{p,\infty}(\mathscr{X}) \cap (\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))'\right)_{\sigma,q} = \dot{B}^s_{p,q}(\mathscr{X}).$$

(iv) *Let* $s_0, s_1 \in (-\epsilon, \epsilon)$, $s \equiv (1-\sigma)s_0 + \sigma s_1$ *and* $p \in (\max\{p(s_0,\epsilon), p(s_1,\epsilon)\}, \infty]$. *If* $q_0, q_1 \in (0,\infty]$, *then*

$$\Big(\mathscr{A}\dot{B}^{s_0}_{p,q_0}(\mathscr{X}) \cap (\mathring{\mathscr{G}}^{\epsilon}_0(\beta,\gamma))', \mathscr{A}\dot{B}^{s_1}_{p,q_1}(\mathscr{X}) \cap (\mathring{\mathscr{G}}^{\epsilon}_0(\beta,\gamma))'\Big)_{\sigma,q} = \dot{B}^{s}_{p,q}(\mathscr{X});$$

if $q_0, q_1 \in (p(s_i,\epsilon), \infty]$, *then*

$$\Big(\mathscr{A}\dot{F}^{s_0}_{p,q_0}(\mathscr{X}) \cap (\mathring{\mathscr{G}}^{\epsilon}_0(\beta,\gamma))', \mathscr{A}\dot{F}^{s_1}_{p,q_1}(\mathscr{X}) \cap (\mathring{\mathscr{G}}^{\epsilon}_0(\beta,\gamma))'\Big)_{\sigma,q} = \dot{B}^{s}_{p,q}(\mathscr{X}).$$

We point out that by Remark 1.2(ii), when $s_0 \in (0,1)$ (resp. $s_1 \in (0,1)$), the space $\mathscr{A}\dot{B}^{s_0}_{p,q_0}(\mathscr{X}) \cap (\mathring{\mathscr{G}}^{\epsilon}_0(\beta,\gamma))'$ (resp. $\mathscr{A}\dot{B}^{s_1}_{p,q_1}(\mathscr{X}) \cap (\mathring{\mathscr{G}}^{\epsilon}_0(\beta,\gamma))'$) in Theorem 1.2 can be replaced by $\mathscr{A}\dot{B}^{s_0}_{p,q_0}(\mathscr{X})$ (resp. $\mathscr{A}\dot{B}^{s_1}_{p,q_1}(\mathscr{X})$).

Similarly, we also obtain the inhomogeneous counterparts of Theorems 1.1 and 1.2; see Theorems 3.1 and 3.2 below.

From Theorems 1.1 and 3.1, we deduce that Theorems 1.2 and 3.2 generalize [23, Theorem 2.3] and [15, Theorems 8.3–8.6] by taking $s_0, s_1 \in (0,1)$ and $q_0, q_1, q \in (0,\infty]$ and $p \in (\max\{p(s_0,\epsilon), p(s_1,\epsilon)\}, \infty]$.

Recently, Gogatishvili, Koskela and Shanmugalingam [10] proved that if a doubling metric space $\mathscr{X}$ supports a $(1,p)$-Poincaré inequality, then for all $p \in [1,\infty)$, $s_0, s_1, \theta \in (0,1)$ and $q_0, q_1, q \in [1,\infty]$, or $s_0, \theta \in (0,1)$, $s_1 = 0$, $q_0, q \in [1,\infty]$ and $q_1 = \infty$,

$$(B^{s_0}_{p,q_0}(\mathscr{X}), B^{s_1}_{p,q_1}(\mathscr{X}))_{\theta,q} = B^{s}_{p,q}(\mathscr{X}),\ s = (1-\theta)s_0 + \theta s_1,\ s_0 \neq s_1$$

and

$$(KS^{1,p}(\mathscr{X}), B^{s_1}_{p,q_1}(\mathscr{X}))_{\theta,q} = B^{s}_{p,q}(\mathscr{X}),\ s = (1-\theta) + \theta s_1,$$

where $KS^{1,p}(\mathscr{X})$ is the Sobolev space in the sense of Korevaar and Schoen [16]. It was also pointed out in [10] that $KS^{1,p}(\mathscr{X})$ coincides with the Hajłasz–Sobolev space $M^{1,p}(\mathscr{X})$ when the doubling metric space $\mathscr{X}$ supports a $(1,p)$-Poincaré inequality. From this and the coincidence $\mathscr{A}F^{1}_{p,\infty}(\mathscr{X}) = M^{1,p}(\mathscr{X})$ for RD-space $\mathscr{X}$ obtained in [18], we deduce that if an RD-space $\mathscr{X}$ supports a $(1,p)$-Poincaré inequality, then $\mathscr{A}F^{1}_{p,\infty}(\mathscr{X}) = KS^{1,p}(\mathscr{X})$, and hence Theorem 3.2 generalizes [10, Theorem 4.4] in this case.

Some other recent developments on the real interpolation theory of Sobolev spaces on metric spaces were made by Badr [1, 2] and Badr–Bernicot [3]. Badr in [1, 2] obtained the interpolation properties between two Sobolev spaces both with order 1 on some classes of manifolds, Lie groups and metric spaces satisfying certain doubling properties, while Badr and Bernicot [3] studied the real interpolation between Hardy–Sobolev spaces and Sobolev spaces both with order 1 on doubling Riemannian manifolds via an atomic decomposition. Notice that the Triebel–Lizorkin spaces coincide with Sobolev spaces for parameters $s = 1$ and certain p, q. In comparison with Theorems 1.2 and 3.2, Badr and Bernicot's interpolation results can be seen as the endpoint case of Theorems 1.2 and 3.2 with $s_0 = s_1 = 1$ and $q_0 = q_1 = q = \infty$, which are not included in Theorems 1.2 and 3.2.

The organization of this paper is as follows. Section 2 is devoted to the proofs of Theorems 1.1 and 1.2. The key tools used in the whole paper are the dyadic cubes of Christ [6] and the Calderón reproducing formulae established in [15]. In Section 3, we establish the counterparts of Theorems 1.1 and 1.2 for the inhomogeneous spaces; see Theorems 3.1 and 3.2 below.

Finally, we make some conventions. Throughout this paper, we always use C to denote a *positive constant* that is independent of the main parameters involved and whose value may differ form line to line. *Constants with subscripts*, such as C_1, do not change in different occurrence. If $f \le Cg$, we then write $f \lesssim g$ or $g \gtrsim f$; and if $f \lesssim g \lesssim f$, we then write $f \sim g$. Denote the set of *integers* by $\mathbf{Z}$, the set of *positive integers* by $\mathbf{N}$ and $\mathbf{N} \cup \{0\}$ by $\mathbf{Z}_+$. For $a, b \in \mathbf{R}$, we denote $\min\{a, b\}$, $\max\{a, b\}$ and $\max\{a, 0\}$ by $a \wedge b$, $a \vee b$ and a_+, respectively. If E is a subset of $\mathscr{X}$, we denote by χ_E the *characteristic function* of E.

2. Proofs of Theorems 1.1 and 1.2

We first recall the following construction given by Christ in [6], which provides an analogue of the grid of Euclidean dyadic cubes on spaces of homogeneous type in the sense of Coifman and Weiss [7, 8].

Lemma 2.1. *Let $\mathscr{X}$ be a space of homogeneous type. Then there exists a collection $\{Q_\alpha^k \subset \mathscr{X} : k \in \mathbf{Z}, \alpha \in I_k\}$ of open subsets, where I_k is some index set, and constants $\delta \in (0,1)$ and C_5, $C_6 > 0$ such that*

(i) *$\mu(\mathscr{X} \setminus \cup_\alpha Q_\alpha^k) = 0$ for each fixed k and $Q_\alpha^k \cap Q_\beta^k = \emptyset$ if $\alpha \ne \beta$;*

(ii) *for any α, β, k, l with $l \ge k$, either $Q_\beta^l \subset Q_\alpha^k$ or $Q_\beta^l \cap Q_\alpha^k = \emptyset$;*

(iii) *for each (k, α) and each $l < k$, there exists a unique β such that $Q_\alpha^k \subset Q_\beta^l$;*

(iv) *$\mathrm{diam}(Q_\alpha^k) \le C_5 \delta^k$;*

(v) *each Q_α^k contains some ball $B(z_\alpha^k, C_6 \delta^k)$, where $z_\alpha^k \in \mathscr{X}$.*

In fact, we can think of Q_α^k as being a dyadic cube with diameter rough δ^k and centered at z_α^k. In what follows, to simplify our presentation, we always suppose $\delta = 1/2$; otherwise, we need to replace 2^{-k} in the definition of approximations of the identity by δ^k and some other changes are also necessary; see [15] for details.

In the following, for $k \in \mathbf{Z}$ and $\tau \in I_k$, we denote by $\{Q_\tau^{k,\nu} : \nu = 1, \cdots, N(k, \tau)\}$ the set of all cubes $Q_{\tau'}^{k+j} \subset Q_\tau^k$, where Q_τ^k is a dyadic cube as in Lemma 2.1 and j is a fixed positive large integer such that $2^{-j} C_5 < 1/3$. Denote by $z_\tau^{k,\nu}$ the "center" of $Q_\tau^{k,\nu}$ as in Lemma 2.1 and by $y_\tau^{k,\nu}$ a point in $Q_\tau^{k,\nu}$. From (1.1), it follows that

$$\mu(Q_\tau^{k,\nu}) \sim V_{2^{-(k+j)}}(y_\tau^{k,\nu}) \sim V_{2^{-k}}(y_\tau^{k,\nu}) \tag{2.1}$$

with equivalent constants depending on j.

The following discrete Calderón reproducing formula on RD-spaces and its variants were established in [15].

Lemma 2.2. *Let $\epsilon \in (0,1)$ and $\{S_k\}_{k \in \mathbf{Z}}$ be a 1-AOTI with bounded support. Then, for any fixed $j \in \mathbf{N}$ large enough, there exists a family $\{\widetilde{D}_k\}_{k \in \mathbf{Z}}$ of linear operators such that for any fixed $y_\tau^{k,\nu} \in Q_\tau^{k,\nu}$ with $k \in \mathbf{Z}$, $\tau \in I_k$ and $\nu \in \{1, \cdots, N(k,\tau)\}$, $x \in \mathscr{X}$, and for all $f \in (\mathring{\mathscr{G}}_0^\epsilon(\beta, \gamma))'$ with β, $\gamma \in (0, \epsilon)$,*

$$f(x) = \sum_{k \in \mathbf{Z}} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) \widetilde{D}_k(x, y_\tau^{k,\nu}) D_k(f)(y_\tau^{k,\nu}),$$

where the series converges in both the norm of $(\mathring{\mathscr{G}}_0^\epsilon(\beta, \gamma))'$ and the norm of $L^p(\mathscr{X})$ for $p \in (1, \infty)$. Moreover, for any $\epsilon' \in (\epsilon, 1)$, there exists a positive constant C, depending on ϵ' and j, such that the kernels, denoted by $\widetilde{D}_k(x, y)$, of the operators $\widetilde{D}_k$ satisfy

(i) *for all* $x,\, y \in \mathscr{X}$, $|\widetilde{D}_k(x,y)| \le C\frac{1}{V_{2^{-k}}(x)+V(x,y)}[\frac{2^{-k}}{2^{-k}+d(x,y)}]^{\epsilon'}$;

(ii) *for all* $x,\, x',\, y \in \mathscr{X}$ *with* $d(x,x') \le (2^{-k}+d(x,y))/2$,

$$|\widetilde{D}(x,y)-\widetilde{D}(x',y)| \le C\left[\frac{d(x,x')}{2^{-k}+d(x,y)}\right]^{\epsilon'}\frac{1}{V_{2^{-k}}(x)+V(x,y)}\left[\frac{2^{-k}}{2^{-k}+d(x,y)}\right]^{\epsilon'};$$

(iii) *for all* $k \in \mathbf{Z}$, $\int_{\mathscr{X}} \widetilde{D}_k(x,z)\,d\mu(z) = 0 = \int_{\mathscr{X}} \widetilde{D}_k(z,y)\,d\mu(z)$.

Lemma 2.3. *Let* $\epsilon \in (0,1)$ *and* $\{S_k\}_{k\in\mathbf{Z}}$ *be a 1-AOTI with bounded support as in Definition 1.3. Then, for any fixed* $j \in \mathbf{N}$ *large enough, there exists a family* $\{\overline{D}_k\}_{k\in\mathbf{Z}}$ *of linear operators such that for any fixed* $y_\tau^{k,\nu} \in Q_\tau^{k,\nu}$ *with* $k \in \mathbf{Z}$, $\tau \in I_k$ *and* $\nu \in \{1,\cdots,N(k,\tau)\}$, $x \in \mathscr{X}$, *and for all* $f \in (\mathring{\mathscr{G}}_0^\epsilon(\beta,\gamma))'$ *with* $\beta,\, \gamma \in (0,\epsilon)$,

$$f(x) = \sum_{k\in\mathbf{Z}}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu})D_k(x,y_\tau^{k,\nu})\overline{D}_k(f)(y_\tau^{k,\nu}),$$

where the series converges in both the norm of $(\mathring{\mathscr{G}}_0^\epsilon(\beta,\gamma))'$ *and the norm of* $L^p(\mathscr{X})$ *for* $p \in (1,\infty)$. *Moreover, for any* $\epsilon' \in (\epsilon,1)$, *there exists a positive constant* C, *depending on* ϵ' *and* j, *such that the kernels, denoted by* $\overline{D}_k(x,y)$, *of the operators* $\overline{D}_k$ *satisfy*

(i) *for all* $x,\, y \in \mathscr{X}$, $|\overline{D}_k(x,y)| \le C\frac{1}{V_{2^{-k}}(x)+V(x,y)}[\frac{2^{-k}}{2^{-k}+d(x,y)}]^{\epsilon'}$;

(ii) *for all* $x,\, y,\, y' \in \mathscr{X}$ *with* $d(y,y') \le (2^{-k}+d(x,y))/2$,

$$|\overline{D}(x,y)-\overline{D}(x,y')| \le C\left[\frac{d(y,y')}{2^{-k}+d(x,y)}\right]^{\epsilon'}\frac{1}{V_{2^{-k}}(x)+V(x,y)}\left[\frac{2^{-k}}{2^{-k}+d(x,y)}\right]^{\epsilon'};$$

(iii) *for all* $k \in \mathbf{Z}$, $\int_{\mathscr{X}} \overline{D}_k(x,z)\,d\mu(z) = 0 = \int_{\mathscr{X}} \overline{D}_k(z,y)\,d\mu(z)$.

We now present some basic estimates which are used throughout the whole paper; see [15, Lemmas 2.1, 5.2 and 5.3].

Lemma 2.4. (i) *If* $a > \eta \ge 0$ *and* $\delta \in (0,\infty)$, *then there exists a positive constant* C, *independent of* δ, η *and* a, *such that for all* $x \in \mathscr{X}$,

$$\int_{\mathscr{X}} \frac{1}{V_\delta(x)+V(x,y)}\left(\frac{\delta}{\delta+d(x,y)}\right)^a [d(x,y)]^\eta\,d\mu(y) \le C\delta^\eta.$$

(ii) *If* $a \in (0,\infty)$ *and* $\delta \in (0,\infty)$, *then there exists a positive constant* C, *independent of* a *and* δ, *such that for all* $f \in L^1_{\rm loc}(\mathscr{X})$ *and all* $x \in \mathscr{X}$,

$$\int_{d(x,y)>\delta} \frac{1}{V(x,y)}\left(\frac{\delta}{\delta+d(x,y)}\right)^a |f(y)|\,d\mu(y) \le CM(f)(x),$$

where M *is the Hardy–Littlewood maximal function on* $\mathscr{X}$.

(iii) *Let* $\epsilon \in (0,\infty)$, $k,\, k' \in \mathbf{Z}$, *and* $y_\tau^{k,\nu}$ *be any point in* $Q_\tau^{k,\nu}$ *for* $\tau \in I_k$ *and* $\nu \in \{1,\cdots,N(k,\tau)\}$. *If* $p \in (n/(n+\epsilon),1]$, *then there exists a positive constant, independent of* k, k', τ *and* ν, *such that for all* $x \in \mathscr{X}$,

$$\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} \frac{\mu(Q_\tau^{k,\nu})}{[V_{2^{-k\wedge k'}}(x)+V(x,y_\tau^{k,\nu})]^p}\left[\frac{2^{-k\wedge k'}}{2^{-k\wedge k'}+d(x,y_\tau^{k,\nu})}\right]^{\epsilon p} \le C\,[V_{2^{-k\wedge k'}}(x)]^{1-p}.$$

(iv) *Let* $\epsilon \in (0,\infty)$, $k,\, k' \in \mathbf{Z}$, *and* $y_\tau^{k,\nu}$ *be any point in* $Q_\tau^{k,\nu}$ *for* $\tau \in I_k$ *and* $\nu \in \{1,\cdots,N(k,\tau)\}$. *If* $r \in (n/(n+\epsilon),1]$, *then there exists a positive constant* C, *depending on r but independent of k, k', τ and ν, such that for all $a_\tau^{k,\nu}\in\mathbf{C}$ and all $x\in\mathscr{X}$,*

$$\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\frac{\mu(Q_\tau^{k,\nu})}{V_{2^{-k\wedge k'}}(x)+V(x,y_\tau^{k,\nu})}\left[\frac{2^{-k\wedge k'}}{2^{-k\wedge k'}+d(x,y_\tau^{k,\nu})}\right]^\epsilon|a_\tau^{k,\nu}|$$
$$\le C2^{[k\wedge k'-k]n(1-1/r)}\left\{M\left(\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}|a_\tau^{k,\nu}|^r\chi_{Q_\tau^{k,\nu}}\right)(x)\right\}^{1/r}.$$

With these tools, we are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $\{S_k\}_{k\in\mathbf{Z}}$ be a 1-AOTI with bounded support. It is easy to check that $D_k(x,\cdot)\in\mathscr{A}_k(x)$ for all $k\in\mathbf{Z}$ and all $x\in\mathscr{X}$, which further implies that the quasi-norms $\|\cdot\|_{\dot B^s_{p,q}(\mathscr{X})}$ and $\|\cdot\|_{\dot F^s_{p,q}(\mathscr{X})}$ are, respectively, dominated by $\|\cdot\|_{\mathscr{A}\dot B^s_{p,q}(\mathscr{X})}$ and $\|\cdot\|_{\mathscr{A}\dot F^s_{p,q}(\mathscr{X})}$. Thus, $\mathscr{A}\dot B^s_{p,q}(\mathscr{X})\cap(\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))'$ and $\mathscr{A}\dot F^s_{p,q}(\mathscr{X})\cap(\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))'$ are continuously included in $\dot B^s_{p,q}(\mathscr{X})$ and $\dot F^s_{p,q}(\mathscr{X})$, respectively.

To complete the proof of Theorem 1.1, it suffices to show that $\dot B^s_{p,q}(\mathscr{X})\subset\mathscr{A}\dot B^s_{p,q}(\mathscr{X})$ and $\dot F^s_{p,q}(\mathscr{X})\subset\mathscr{A}\dot F^s_{p,q}(\mathscr{X})$. For $f\in\dot B^s_{p,q}(\mathscr{X})$ or $f\in\dot F^s_{p,q}(\mathscr{X})$, since $f\in(\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))'$, by Lemma 2.2, for all $x\in\mathscr{X}$, $l\in\mathbf{Z}$ and $\phi\in\mathscr{A}_l(x)$, we have

$$\langle f,\phi\rangle=\sum_{k\in\mathbf{Z}}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\mu(Q_\tau^{k,\nu})D_k(f)(y_\tau^{k,\nu})\int_{\mathscr{X}}\widetilde D_k(z,y_\tau^{k,\nu})\phi(z)\,d\mu(z),$$

where we fix $y_\tau^{k,\nu}\in Q_\tau^{k,\nu}$ such that $|D_k(f)(y_\tau^{k,\nu})|\le 2\inf_{z\in Q_\tau^{k,\nu}}|D_k(f)(z)|$. By the definition of $\mathscr{A}_l(x)$, we know that $\phi\in\mathring{\mathscr{G}}(1,2)\subset\mathring{\mathscr{G}}(\epsilon,\epsilon)\subset\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma)$. From this and the uniform estimates of $\widetilde D_k$ in Lemma 2.2, it follows that for any fixed $\beta'\in(|s|,\beta)$ and $\gamma'\in(|s|,\gamma)$ satisfying (1.2) in Definition 1.4,

$$\left|\int_{\mathscr{X}}\widetilde D_k(z,y_\tau^{k,\nu})\phi(z)\,d\mu(z)\right|\lesssim\frac{2^{-|k-l|\beta'}}{V_{2^{-(k\wedge l)}}(x)+V(x,y_\tau^{k,\nu})}\left(\frac{2^{-(k\wedge l)}}{2^{-(k\wedge l)}+d(x,y_\tau^{k,\nu})}\right)^{\gamma'};\tag{2.2}$$

see [15, Proposition 5.7] and also [18] for a detailed proof. Thus,

$$|\langle f,\phi\rangle|\lesssim\sum_{k\in\mathbf{Z}}2^{-|k-l|\beta'}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\frac{\mu(Q_\tau^{k,\nu})|D_k(f)(y_\tau^{k,\nu})|}{V_{2^{-(k\wedge l)}}(x)+V(x,y_\tau^{k,\nu})}\left(\frac{2^{-(k\wedge l)}}{2^{-(k\wedge l)}+d(x,y_\tau^{k,\nu})}\right)^{\gamma'}.\tag{2.3}$$

We first prove that $\dot B^s_{p,q}(\mathscr{X})\subset\mathscr{A}\dot B^s_{p,q}(\mathscr{X})$ in the case when $p\in(1,\infty]$. Notice that for any $z\in Q_\tau^{k,\nu}$, $V_{2^{-(k\wedge l)}}(x)+V(x,y_\tau^{k,\nu})\sim V_{2^{-(k\wedge l)}}(x)+V(x,z)$ and $2^{-(k\wedge l)}+d(x,y_\tau^{k,\nu})\sim 2^{-(k\wedge l)}+d(x,z)$. These estimates, together with the choice of $y_\tau^{k,\nu}$, (2.3) and Lemma 2.4(ii), yield that

$$|\langle f,\phi\rangle|\lesssim\sum_{k\in\mathbf{Z}}2^{-|k-l|\beta'}\int_{\mathscr{X}}\frac{|D_k(f)(z)|}{V_{2^{-(k\wedge l)}}(x)+V(x,z)}\left(\frac{2^{-(k\wedge l)}}{2^{-(k\wedge l)}+d(x,z)}\right)^{\gamma'}d\mu(z)$$
$$\lesssim\sum_{k\in\mathbf{Z}}2^{-|k-l|\beta'}M(|D_k(f)|)(x),$$

where M is the Hardy–Littlewood maximal function. Then applying the Minkowski inequality and the boundedness of the Hardy–Littlewood maximal function on $L^p(\mathscr{X})$

with $p \in (1, \infty]$, we have

$$\begin{aligned}\|f\|^q_{\mathscr{A}\dot{B}^s_{p,q}(\mathscr{X})} &= \sum_{l\in\mathbf{Z}} 2^{lqs} \left\| \sup_{\phi\in\mathscr{A}_l(\cdot)} |\langle f, \phi\rangle| \right\|^q_{L^p(\mathscr{X})} \\ &\lesssim \sum_{l\in\mathbf{Z}} 2^{lqs} \left\| \sum_{k\in\mathbf{Z}} 2^{-|k-l|\beta'} M(|D_k(f)|) \right\|^q_{L^p(\mathscr{X})} \\ &\lesssim \sum_{l\in\mathbf{Z}} 2^{lqs} \left(\sum_{k\in\mathbf{Z}} 2^{-|k-l|\beta'} \|D_k(f)\|_{L^p(\mathscr{X})} \right)^q .\end{aligned}$$

If $q \in (0, 1]$, by the inequality that for all $\{a_k\}_{k\in\mathbf{Z}} \subset \mathbf{C}$ and $r \in (0, 1]$,

$$\left(\sum_{k\in\mathbf{Z}} |a_k| \right)^r \le \sum_{k\in\mathbf{Z}} |a_k|^r \tag{2.4}$$

and the fact that $\beta' > |s|$, we obtain

$$\begin{aligned}\|f\|^q_{\mathscr{A}\dot{B}^s_{p,q}(\mathscr{X})} &\lesssim \sum_{k\in\mathbf{Z}} 2^{kqs} \left(\sum_{l\le k} 2^{(l-k)q(s+\beta')} + \sum_{l>k} 2^{(k-l)q(\beta'-s)} \right) \|D_k(f)\|^q_{L^p(\mathscr{X})} \\ &\lesssim \sum_{k\in\mathbf{Z}} 2^{kqs} \|D_k(f)\|^q_{L^p(\mathscr{X})} \sim \|f\|^q_{\dot{B}^s_{p,q}(\mathscr{X})}.\end{aligned}$$

If $q \in (1, \infty]$, choosing $\delta > 0$ such that $|s| + \delta < \beta'$ and then using the Hölder inequality, we see that

$$\begin{aligned}\|f\|_{\mathscr{A}\dot{B}^s_{p,q}(\mathscr{X})} &\lesssim \left\{ \sum_{l\in\mathbf{Z}} 2^{lqs} \left(\sum_{k\le l} 2^{(k-l)q\beta'-kq\delta} \|D_k(f)\|^q_{L^p(\mathscr{X})} 2^{lq\delta} \right.\right. \\ &\qquad \left.\left. + \sum_{k>l} 2^{(l-k)q\beta'+kq\delta} \|D_k(f)\|^q_{L^p(\mathscr{X})} 2^{-lq\delta} \right) \right\}^{1/q} \\ &\lesssim \left\{ \sum_{k\in\mathbf{Z}} 2^{kqs} \left(\sum_{l\ge k} 2^{(l-k)q(s+\delta-\beta')} + \sum_{l<k} 2^{(k-l)q(\delta-\beta'-s)} \right) \|D_k(f)\|^q_{L^p(\mathscr{X})} \right\}^{1/q} \\ &\lesssim \left\{ \sum_{k\in\mathbf{Z}} 2^{kqs} \|D_k(f)\|^q_{L^p(\mathscr{X})} \right\}^{1/q} \lesssim \|f\|_{\dot{B}^s_{p,q}(\mathscr{X})},\end{aligned}$$

which completes the proof in the case when $p \in (1, \infty]$.

For the case when $p \in (p(s, \epsilon), 1]$, by (2.3), the fact that

$$V_{2^{-(k\wedge l)}}(x) + V(x, y^{k,\nu}_\tau) \sim V_{2^{-(k\wedge l)}}(y^{l,\nu'}_{\tau'}) + V(y^{l,\nu'}_{\tau'}, y^{k,\nu}_\tau) + V_{2^{-(k\wedge l)}}(y^{k,\nu}_\tau)$$

and $2^{-(k\wedge l)}+d(x,y_\tau^{k,\nu})\sim 2^{-(k\wedge l)}+d(y_{\tau'}^{l,\nu'},y_\tau^{k,\nu})$ when $x\in Q_{\tau'}^{l,\nu'}$, the choice of $y_\tau^{k,\nu}$, Lemma 2.4(iii) and (2.4), we have

$$\begin{aligned}
&\left\|\sup_{\phi\in\mathscr{A}_l(\cdot)}|\langle f,\phi\rangle|\right\|_{L^p(\mathscr{X})}^p\\
&\lesssim\sum_{k\in\mathbf{Z}}2^{-|k-l|p\beta'}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}[\mu(Q_\tau^{k,\nu})]^p|D_k(f)(y_\tau^{k,\nu})|^p\\
&\quad\times\int_{\mathscr{X}}\frac{1}{[V_{2^{-(k\wedge l)}}(x)+V(x,y_\tau^{k,\nu})]^p}\left(\frac{2^{-(k\wedge l)}}{2^{-(k\wedge l)}+d(x,y_\tau^{k,\nu})}\right)^{\gamma'p}d\mu(x)\\
&\sim\sum_{k\in\mathbf{Z}}2^{-|k-l|p\beta'}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}[\mu(Q_\tau^{k,\nu})]^p|D_k(f)(y_\tau^{k,\nu})|^p\\
&\quad\times\sum_{\tau'\in I_l}\sum_{\nu'=1}^{N(l,\tau')}\frac{\mu(Q_{\tau'}^{l,\nu'})}{[V_{2^{-(k\wedge l)}}(y_{\tau'}^{l,\nu'})+V(y_{\tau'}^{l,\nu'},y_\tau^{k,\nu})+V_{2^{-(k\wedge l)}}(y_\tau^{k,\nu})]^p}\\
&\quad\times\left(\frac{2^{-(k\wedge l)}}{2^{-(k\wedge l)}+d(y_{\tau'}^{l,\nu'},y_\tau^{k,\nu})}\right)^{\gamma'p}\\
&\lesssim\sum_{k\in\mathbf{Z}}2^{-|k-l|p\beta'}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}[\mu(Q_\tau^{k,\nu})]^p|D_k(f)(y_\tau^{k,\nu})|^p\left[V_{2^{-(k\wedge l)}}(y_\tau^{k,\nu})\right]^{1-p}\\
&\lesssim\sum_{k\in\mathbf{Z}}2^{-|k-l|p\beta'+[k-(k\wedge l)]n(1-p)}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\mu(Q_\tau^{k,\nu})|D_k(f)(y_\tau^{k,\nu})|^p\\
&\lesssim\sum_{k\in\mathbf{Z}}2^{-|k-l|p\beta'+[k-(k\wedge l)]n(1-p)}\|D_k(f)\|_{L^p(\mathscr{X})}^p.
\end{aligned}$$

If $q/p\in(0,1]$, by (2.4) and (1.2), we see that

$$\begin{aligned}
\|f\|_{\mathscr{A}\dot{B}^s_{p,q}(\mathscr{X})}^q&=\sum_{l\in\mathbf{Z}}2^{lqs}\left\|\sup_{\phi\in\mathscr{A}_l(\cdot)}|\langle f,\phi\rangle|\right\|_{L^p(\mathscr{X})}^q\\
&\lesssim\sum_{l\in\mathbf{Z}}2^{lqs}\sum_{k\in\mathbf{Z}}2^{-|k-l|q\beta'+[k-(k\wedge l)]qn(1/p-1)}\|D_k(f)\|_{L^p(\mathscr{X})}^q\\
&\lesssim\sum_{k\in\mathbf{Z}}2^{kqs}\left(\sum_{l\in\mathbf{Z}}2^{(l-k)qs-|k-l|q\beta'+[k-(k\wedge l)]qn(1/p-1)}\right)\|D_k(f)\|_{L^p(\mathscr{X})}^q\\
&\lesssim\sum_{k\in\mathbf{Z}}2^{kqs}\left(\sum_{l\le k}2^{(l-k)q[s+\beta'+n(1-1/p)]}+\sum_{l>k}2^{(l-k)q(s-\beta')}\right)\|D_k(f)\|_{L^p(\mathscr{X})}^q\\
&\lesssim\sum_{k\in\mathbf{Z}}2^{kqs}\|D_k(f)\|_{L^p(\mathscr{X})}^q\sim\|f\|_{\dot{B}^s_{p,q}(\mathscr{X})}^q.
\end{aligned}$$

If $q/p \in (1,\infty]$, choosing $\delta > 0$ satisfying $|s|+\delta < \beta'$ and $\beta'+s+n(1-1/p)-\delta > 0$, by the Hölder inequality, we have

$$\begin{aligned}\|f\|_{\mathscr{A}\dot{B}^s_{p,q}(\mathscr{X})} &= \left\{\sum_{l\in\mathbf{Z}} 2^{lqs}\left\|\sup_{\phi\in\mathscr{A}_l(\cdot)}|\langle f,\phi\rangle|\right\|^q_{L^p(\mathscr{X})}\right\}^{1/q}\\ &\lesssim \left\{\sum_{k\in\mathbf{Z}} 2^{kqs}\sum_{l\in\mathbf{Z}} 2^{(l-k)qs-|k-l|q(\beta'-\delta)+[(k\wedge l)-k]qn(1-1/p)}\|D_k(f)\|^q_{L^p(\mathscr{X})}\right\}^{1/q}\\ &\lesssim \left\{\sum_{k\in\mathbf{Z}} 2^{kqs}\|D_k(f)\|^q_{L^p(\mathscr{X})}\right\}^{1/q} \sim \|f\|_{\dot{B}^s_{p,q}(\mathscr{X})}.\end{aligned}$$

Hence, $f\in\mathscr{A}\dot{B}^s_{p,q}(\mathscr{X})$ and $\|f\|_{\mathscr{A}\dot{B}^s_{p,q}(\mathscr{X})} \lesssim \|f\|_{\dot{B}^s_{p,q}(\mathscr{X})}$.

The proof for $\dot{F}^s_{p,q}(\mathscr{X}) \subset \mathscr{A}\dot{F}^s_{p,q}(\mathscr{X})$ is similar. Let $f\in\dot{F}^s_{p,q}(\mathscr{X})$, $x\in\mathscr{X}$ and $\phi\in\mathscr{A}_l(x)$. For the case when $p<\infty$, using (2.3), Lemma 2.4(iv), the choice of $y^{k,\nu}_\tau$, and choosing $r\in(n/[n+(\beta'\wedge\gamma')], p\wedge q)$, we have

$$|\langle f,\phi\rangle| \lesssim \sum_{k\in\mathbf{Z}} 2^{-|k-l|\beta'+[(k\wedge l)-k]n(1-1/r)}\left[M\left(|D_k(f)|^r\right)(x)\right]^{1/r},$$

and hence

$$\begin{aligned}&\|f\|_{\mathscr{A}\dot{F}^s_{p,q}(\mathscr{X})}\\ &\lesssim \left\|\left\{\sum_{l\in\mathbf{Z}} 2^{(l-k)qs}\left(\sum_{k\in\mathbf{Z}} 2^{-|k-l|\beta'+[(k\wedge l)-k]n(1-1/r)}\left[M\left(|D_k(f)|^r\right)(x)\right]^{1/r}\right)^q\right\}^{1/q}\right\|_{L^p(\mathscr{X})},\end{aligned}$$

which together with the Hölder inequality when $q\in(1,\infty]$ or (2.4) when $q\in(p(s,\epsilon),1]$, and the Fefferman–Stein vector-valued maximal function inequality in [5] (see also [15, Lemma 3.14]) further implies that

$$\begin{aligned}\|f\|_{\mathscr{A}\dot{F}^s_{p,q}(\mathscr{X})} &\lesssim \left\|\left\{\sum_{k\in\mathbf{Z}}\left[M\left(2^{krs}|D_k(f)|^r\right)\right]^{q/r}\right\}^{r/q}\right\|^{1/r}_{L^{p/r}(\mathscr{X})}\\ &\lesssim \left\|\left\{\sum_{k\in\mathbf{Z}} 2^{ksq}|D_k(f)|^q\right\}^{1/q}\right\|_{L^p(\mathscr{X})} \sim \|f\|_{\dot{F}^s_{p,q}(\mathscr{X})}.\end{aligned}$$

For the case when $p=\infty$, notice that

$$\|f\|_{\mathscr{A}\dot{F}^s_{\infty,q}(\mathscr{X})} \sim \sup_{j\in\mathbf{Z}}\sup_{a\in I_j}\left\{\frac{1}{\mu(Q^j_a)}\int_{Q^j_a}\sum_{l=j}^\infty 2^{lsq}\sup_{\phi\in\mathscr{A}_l(x)}|\langle f,\phi\rangle|^q\,d\mu(x)\right\}^{1/q}.$$

If $q\in(0,\infty)$, for any $f\in\dot{F}^s_{\infty,q}(\mathscr{X})$, $j\in\mathbf{Z}$, $a\in I_j$, $x\in Q^j_a$ and $\phi\in\mathscr{A}_l(x)$, by (2.3) and the fact that $V_{2^{-(k\wedge l)}}(x)+V(x,y^{k,\tau}_\nu) \sim V_{2^{-(k\wedge l)}}(y^{l,\tau'}_{\nu'})+V(y^{l,\tau'}_{\nu'},y^{k,\tau}_\nu)$ and $2^{-(k\wedge l)}+d(x,y^{k,\tau}_\nu) \sim 2^{-(k\wedge l)}+d(y^{l,\tau'}_{\nu'},y^{k,\tau}_\nu)$ for all $x\in Q^{l,\tau'}_{\nu'}$ with $\tau'\in I_l$ and $\nu'\in N(l,\tau')$, we have

$$
\begin{aligned}
&\frac{1}{\mu(Q_a^j)}\int_{Q_a^j}\sum_{l=j}^{\infty}2^{lsq}\sup_{\phi\in\mathscr{A}_l(x)}|\langle f,\phi\rangle|^q\,d\mu(x)\\
&=\frac{1}{\mu(Q_a^j)}\sum_{l=j}^{\infty}\sum_{\tau'\in I_l}\sum_{\nu'=1}^{N(l,\tau')}2^{lsq}\chi_{\{(\tau',\nu'):Q_{\nu'}^{l,\tau'}\subset Q_a^j\}}(\tau',\nu')\int_{Q_{\nu'}^{l,\tau'}}\sup_{\phi\in\mathscr{A}_l(x)}|\langle f,\phi\rangle|^q\,d\mu(x)\\
&\lesssim\frac{1}{\mu(Q_a^j)}\sum_{l=j}^{\infty}\sum_{\tau'\in I_l}\sum_{\nu'=1}^{N(l,\tau')}2^{lsq}\chi_{\{(\tau',\nu'):Q_{\nu'}^{l,\tau'}\subset Q_a^j\}}(\tau',\nu')\int_{Q_{\nu'}^{l,\tau'}}\left[\sum_{k\in\mathbf{Z}}2^{-|k-l|\beta'}\right.\\
&\quad\left.\times\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\frac{\mu(Q_\nu^{k,\tau})|D_k(f)(y_\nu^{k,\tau})|}{V_{2^{-(k\wedge l)}}(x)+V(x,y_\nu^{k,\tau})}\left(\frac{2^{-(k\wedge l)}}{2^{-(k\wedge l)}+d(x,y_\nu^{k,\tau})}\right)^{\gamma'}\right]^q d\mu(x)\\
&\lesssim\frac{1}{\mu(Q_a^j)}\sum_{l=j}^{\infty}\sum_{\tau'\in I_l}\sum_{\nu'=1}^{N(l,\tau')}2^{lsq}\mu(Q_{\nu'}^{l,\tau'})\chi_{\{(\tau',\nu'):Q_{\nu'}^{l,\tau'}\subset Q_a^j\}}(\tau',\nu')\left[\sum_{k\in\mathbf{Z}}2^{-|k-l|\beta'}\right.\\
&\quad\left.\times\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\frac{\mu(Q_\nu^{k,\tau})|D_k(f)(y_\nu^{k,\tau})|}{V_{2^{-(k\wedge l)}}(y_{\nu'}^{l,\tau'})+V(y_{\nu'}^{l,\tau'},y_\nu^{k,\tau})}\left(\frac{2^{-(k\wedge l)}}{2^{-(k\wedge l)}+d(y_{\nu'}^{l,\tau'},y_\nu^{k,\tau})}\right)^{\gamma'}\right]^q.
\end{aligned}
$$

Then, similarly to the proof of [15, Proposition 6.3], by the choice of $y_\nu^{k,\tau}$, we further obtain

$$
\begin{aligned}
&\frac{1}{\mu(Q_a^j)}\int_{Q_a^j}\sum_{l=j}^{\infty}2^{lsq}\sup_{\phi\in\mathscr{A}_l(x)}|\langle f,\phi\rangle|^q\,d\mu(x)\\
&\lesssim\sup_{j\in\mathbf{Z}}\sup_{a\in I_j}\frac{1}{\mu(Q_a^j)}\sum_{k=j}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}2^{ksq}\mu(Q_\nu^{k,\tau})\chi_{\{(\tau,\nu):Q_\nu^{k,\tau}\subset Q_a^j\}}(\tau,\nu)\left[\inf_{x\in Q_\nu^{k,\tau}}|D_k(f)(x)|\right]^q\\
&\lesssim\sup_{j\in\mathbf{Z}}\sup_{a\in I_j}\frac{1}{\mu(Q_a^j)}\int_{Q_a^j}\sum_{k=j}^{\infty}2^{ksq}|D_k(f)(x)|^q\,d\mu(x)\sim\|f\|^q_{\dot F^s_{\infty,q}(\mathscr{X})}.
\end{aligned}
$$

The proof for the case when $q=\infty$ is similar, which completes the proof of Theorem 1.1. $\square$

We now turn to the proof of Theorem 1.2.

Proof of Theorem 1.2. We show this theorem by following a procedure used in the proof of [15, Theorem 8.8] with some modifications.

To verify (i), by Remark 1.2(i), we take $\mathscr{H}=(\mathring{\mathscr{G}}_0^\epsilon(\beta,\gamma))'$ with ϵ, β and γ as in (1.2). Let us first prove that

$$
(\mathscr{A}\dot B^1_{p,\infty}(\mathscr{X})\cap(\mathring{\mathscr{G}}_0^\epsilon(\beta,\gamma))',\mathscr{A}\dot B^{-1}_{p,\infty}(\mathscr{X})\cap(\mathring{\mathscr{G}}_0^\epsilon(\beta,\gamma))')_{\sigma,q}\subset\dot B^s_{p,q}(\mathscr{X}).
$$

Assume that $f\in(\mathscr{A}\dot B^1_{p,\infty}(\mathscr{X})\cap(\mathring{\mathscr{G}}_0^\epsilon(\beta,\gamma))',\mathscr{A}\dot B^{-1}_{p,\infty}(\mathscr{X})\cap(\mathring{\mathscr{G}}_0^\epsilon(\beta,\gamma))')_{\sigma,q}$ and $f=f_0+f_1$ with $f_0\in\mathscr{A}\dot B^1_{p,\infty}(\mathscr{X})\cap(\mathring{\mathscr{G}}_0^\epsilon(\beta,\gamma))'$ and $f_1\in\mathscr{A}\dot B^{-1}_{p,\infty}(\mathscr{X})\cap(\mathring{\mathscr{G}}_0^\epsilon(\beta,\gamma))'$.

Let $\{D_k\}_{k\in\mathbf{Z}}$ be as in Definition 1.4. Then

$$\begin{aligned}
2^k\|D_k(f)\|_{L^p(\mathscr{X})} &\lesssim 2^k\left\|\sup_{\phi\in\mathscr{A}_k(\cdot)}|\langle f,\phi\rangle|\right\|_{L^p(\mathscr{X})}\\
&\lesssim 2^k\left\|\sup_{\phi\in\mathscr{A}_k(\cdot)}|\langle f_0,\phi\rangle|\right\|_{L^p(\mathscr{X})}+2^{2k-k}\left\|\sup_{\phi\in\mathscr{A}_k(\cdot)}|\langle f_1,\phi\rangle|\right\|_{L^p(\mathscr{X})}\\
&\lesssim \|f_0\|_{\mathscr{A}\dot{B}^1_{p,\infty}(\mathscr{X})}+2^{2k}\|f_1\|_{\mathscr{A}\dot{B}^{-1}_{p,\infty}(\mathscr{X})}.
\end{aligned}$$

Taking the infimum over all representations $f=f_0+f_1$ yields that

$$2^k\|D_k(f)\|_{L^p(\mathscr{X})}\lesssim K\left(2^{2k},f;\mathscr{A}\dot{B}^1_{p,\infty}(\mathscr{X}),\mathscr{A}\dot{B}^{-1}_{p,\infty}(\mathscr{X})\right). \tag{2.5}$$

If $q\in(0,\infty)$, from (2.5), it follows that

$$\begin{aligned}
&\|f\|^q_{\left(\mathscr{A}\dot{B}^1_{p,\infty}(\mathscr{X}),\mathscr{A}\dot{B}^{-1}_{p,\infty}(\mathscr{X})\right)_{\sigma,q}}\\
&=\int_0^\infty t^{-\sigma q}\left[K\left(t,f;\mathscr{A}\dot{B}^1_{p,\infty}(\mathscr{X}),\mathscr{A}\dot{B}^{-1}_{p,\infty}(\mathscr{X})\right)\right]^q\frac{dt}{t}\\
&=\sum_{k\in\mathbf{Z}}\int_{2^{2(k-1)}}^{2^{2k}} t^{-\sigma q}\left[K\left(t,f;\mathscr{A}\dot{B}^1_{p,\infty}(\mathscr{X}),\mathscr{A}\dot{B}^{-1}_{p,\infty}(\mathscr{X})\right)\right]^q\frac{dt}{t}\\
&\gtrsim\sum_{k\in\mathbf{Z}}2^{-2\sigma qk}\left[K\left(2^{2k},f;\mathscr{A}\dot{B}^1_{p,\infty}(\mathscr{X}),\mathscr{A}\dot{B}^{-1}_{p,\infty}(\mathscr{X})\right)\right]^q\\
&\gtrsim\sum_{k\in\mathbf{Z}}2^{kqs}\|D_k(f)\|^q_{L^p(\mathscr{X})}\sim\|f\|^q_{\dot{B}^s_{p,q}(\mathscr{X})};
\end{aligned}$$

if $q=\infty$, by (2.5), we then have

$$\begin{aligned}
\|f\|_{\dot{B}^s_{p,\infty}(\mathscr{X})}&=\sup_{k\in\mathbf{Z}}2^{ks}\|D_k(f)\|_{L^p(\mathscr{X})}\\
&\lesssim\sup_{k\in\mathbf{Z}}2^{k(s-1)}K\left(2^{2k},f;\mathscr{A}\dot{B}^1_{p,\infty}(\mathscr{X}),\mathscr{A}\dot{B}^{-1}_{p,\infty}(\mathscr{X})\right)\\
&\lesssim\sup_{t\in(0,\infty)}t^{-\sigma}K\left(t,f;\mathscr{A}\dot{B}^1_{p,\infty}(\mathscr{X}),\mathscr{A}\dot{B}^{-1}_{p,\infty}(\mathscr{X})\right)\\
&\lesssim\|f\|_{\left(\mathscr{A}\dot{B}^1_{p,\infty}(\mathscr{X}),\mathscr{A}\dot{B}^{-1}_{p,\infty}(\mathscr{X})\right)_{\sigma,\infty}}.
\end{aligned}$$

Thus, $(\mathscr{A}\dot{B}^1_{p,\infty}(\mathscr{X})\cap(\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))',\mathscr{A}\dot{B}^{-1}_{p,\infty}(\mathscr{X})\cap(\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))')_{\sigma,q}\subset\dot{B}^s_{p,q}(\mathscr{X})$.

Notice that $\mathscr{A}\dot{F}^1_{p,\infty}(\mathscr{X})\subset\mathscr{A}\dot{B}^1_{p,\infty}(\mathscr{X})$ and $\mathscr{A}\dot{F}^{-1}_{p,\infty}(\mathscr{X})\subset\mathscr{A}\dot{B}^{-1}_{p,\infty}(\mathscr{X})$. We then have

$$\begin{aligned}
&(\mathscr{A}\dot{F}^1_{p,\infty}(\mathscr{X})\cap(\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))',\mathscr{A}\dot{F}^{-1}_{p,\infty}(\mathscr{X})\cap(\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))')_{\sigma,q}\\
&\subset(\mathscr{A}\dot{B}^1_{p,\infty}(\mathscr{X})\cap(\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))',\mathscr{A}\dot{B}^{-1}_{p,\infty}(\mathscr{X})\cap(\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))')_{\sigma,q}\subset\dot{B}^s_{p,q}(\mathscr{X}).
\end{aligned}$$

Thus, to complete the proof of (i), it suffices to show that

$$\dot{B}^s_{p,q}(\mathscr{X})\subset(\mathscr{A}\dot{F}^1_{p,\infty}(\mathscr{X})\cap(\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))',\mathscr{A}\dot{F}^{-1}_{p,\infty}(\mathscr{X})\cap(\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))')_{\sigma,q}.$$

We only consider the case $q\in(0,\infty)$ and we omit the details for the case $q=\infty$ by similarity and simplicity. Let now $f\in \dot B^s_{p,q}(\mathscr X)$. We then write

$$\begin{aligned}\|f\|^q_{(\mathscr A\dot F^1_{p,\infty}(\mathscr X),\mathscr A\dot F^{-1}_{p,\infty}(\mathscr X))_{\sigma,q}}&=\int_0^\infty t^{-\sigma q}\left[K\left(t,f;\mathscr A\dot F^1_{p,\infty}(\mathscr X),\mathscr A\dot F^{-1}_{p,\infty}(\mathscr X)\right)\right]^q\frac{dt}{t}\\&\lesssim\sum_{j\in\mathbf Z}2^{-2j\sigma q}\left[K\left(2^{2j},f;\mathscr A\dot F^1_{p,\infty}(\mathscr X),\mathscr A\dot F^{-1}_{p,\infty}(\mathscr X)\right)\right]^q.\end{aligned}$$

Let all notation be as in Lemma 2.3. For any $j\in\mathbf Z$, we write

$$\begin{aligned}f(z)&=\sum_{k=-\infty}^{j}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\mu(Q^{k,\nu}_\tau)D_k(z,y^{k,\nu}_\tau)\overline D_k(f)(y^{k,\nu}_\tau)+\sum_{k=j+1}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\cdots\\&\equiv f^j_0(z)+f^j_1(z).\end{aligned}\tag{2.6}$$

From this and the definition of the K-functional, it follows that

$$\begin{aligned}\|f\|_{(\mathscr A\dot F^1_{p,\infty}(\mathscr X),\mathscr A\dot F^{-1}_{p,\infty}(\mathscr X))_{\sigma,q}}&\lesssim\left\{\sum_{j\in\mathbf Z}2^{-2qj\sigma}\left(\|f^j_0\|^q_{\mathscr A\dot F^1_{p,\infty}(\mathscr X)}+2^{2jq}\|f^j_1\|^q_{\mathscr A\dot F^{-1}_{p,\infty}(\mathscr X)}\right)\right\}^{1/q}\\&\lesssim\left\{\sum_{j\in\mathbf Z}2^{-qj(1-s)}\left\|\sup_{l\in\mathbf Z}2^l\sup_{\phi\in\mathscr A_l(\cdot)}|\langle f^j_0,\phi\rangle|\right\|^q_{L^p(\mathscr X)}\right\}^{1/q}\\&\quad+\left\{\sum_{j\in\mathbf Z}2^{-qj(-1-s)}\left\|\sup_{l\in\mathbf Z}2^{-l}\sup_{\phi\in\mathscr A_l(\cdot)}|\langle f^j_1,\phi\rangle|\right\|^q_{L^p(\mathscr X)}\right\}^{1/q}\\&\equiv\mathrm I+\mathrm J.\end{aligned}$$

For $\phi\in\mathscr A_l(x)$, by the fact that $D_k=S_k-S_{k-1}$ and the estimates of S_k in Definition 1.3, similarly to the proof of (2.2), we have

$$\left|\int_{\mathscr X}D_k(z,y^{k,\nu}_\tau)\phi(z)\,d\mu(z)\right|\lesssim\frac{2^{-|k-l|}}{V_{2^{-(k\wedge l)}}(x)+V(x,y^{k,\nu}_\tau)}\left(\frac{2^{-(k\wedge l)}}{2^{-(k\wedge l)}+d(x,y^{k,\nu}_\tau)}\right)^2.\tag{2.7}$$

From (2.7), Lemma 2.4(iv) and the choices of $y^{k,v}_\tau$, we deduce that

$$\begin{aligned}|\langle f^j_0,\phi\rangle|&\lesssim\sum_{k=-\infty}^{j}2^{-|k-l|}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\frac{\mu(Q^{k,\nu}_\tau)|\overline D_k(f)(y^{k,\nu}_\tau)|}{V_{2^{-(k\wedge l)}}(x)+V(x,y^{k,\nu}_\tau)}\left(\frac{2^{-(k\wedge l)}}{2^{-(k\wedge l)}+d(x,y^{k,\nu}_\tau)}\right)^2\\&\lesssim\sum_{k=-\infty}^{j}2^{-|k-l|}2^{[(k\wedge l)-k]n(1-1/r)}\left[M(|\overline D_k(f)|^r)(x)\right]^{1/r},\end{aligned}$$

where we chose $r \in (n/(n+2), p)$. From this, the Minkowski inequality, the Fefferman–Stein vector-valued inequality and [15, Remark 5.5], we deduce that

$$\mathrm{I} \lesssim \left\{ \sum_{j\in\mathbf{Z}} 2^{-qj(1-s)} \left\| \sup_{l\in\mathbf{Z}} 2^l \sum_{k=-\infty}^{j} 2^{-|k-l|} 2^{[(k\wedge l)-k]n(1-1/r)} \left[M(|\overline{D}_k(f)|^r)(x) \right]^{1/r} \right\|_{L^p(\mathscr{X})}^q \right\}^{1/q}$$
$$\lesssim \left\{ \sum_{j\in\mathbf{Z}} 2^{-qj(1-s)} \left\{ \sum_{k=-\infty}^{j} 2^k \left\| \overline{D}_k(f) \right\|_{L^p(\mathscr{X})} \right\}^q \right\}^{1/q} \sim \|f\|_{\dot{B}^s_{p,q}(\mathscr{X})}.$$

Similarly, we obtain that $\mathrm{J} \lesssim \|f\|_{\dot{B}^s_{p,q}(\mathscr{X})}$. Hence, we have

$$\dot{B}^s_{p,q}(\mathscr{X}) \subset (\mathscr{A}\dot{F}^1_{p,\infty}(\mathscr{X}) \cap (\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))', \mathscr{A}\dot{F}^{-1}_{p,\infty}(\mathscr{X}) \cap (\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))')_{\sigma,q}$$

and then complete the proof of (i).

The proofs of (ii), (iii) and (iv) are similar and we only give the proof of (ii). First, following the procedure used in the proof of (i), we have

$$\left(\mathscr{A}\dot{B}^1_{p,\infty}(\mathscr{X}) \cap (\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))', \mathscr{A}\dot{B}^{s_1}_{p,\infty}(\mathscr{X}) \cap (\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))'\right)_{\sigma,q} \subset \dot{B}^s_{p,q}(\mathscr{X}).$$

Conversely, let $q_2 \in (0, p \wedge q_1)$. Notice that for $p \in (p(s,\epsilon), \infty]$ and $q_1 \in (0,\infty]$,

$$\begin{aligned}&\left(\mathscr{A}\dot{B}^1_{p,\infty}(\mathscr{X}) \cap (\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))', \mathscr{A}\dot{B}^{s_1}_{p,q_2}(\mathscr{X}) \cap (\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))'\right)_{\sigma,q}\\
&\quad\subset \left(\mathscr{A}\dot{B}^1_{p,\infty}(\mathscr{X}) \cap (\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))', \mathscr{A}\dot{B}^{s_1}_{p,q_1}(\mathscr{X}) \cap (\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))'\right)_{\sigma,q}\\
&\quad\subset \left(\mathscr{A}\dot{B}^1_{p,\infty}(\mathscr{X}) \cap (\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))', \mathscr{A}\dot{B}^{s_1}_{p,\infty}(\mathscr{X}) \cap (\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))'\right)_{\sigma,q},\end{aligned}$$

and for $p, q \in (p(s,\epsilon), \infty]$,

$$\begin{aligned}&\left(\mathscr{A}\dot{B}^1_{p,\infty}(\mathscr{X}) \cap (\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))', \mathscr{A}\dot{B}^{s_1}_{p,q_2}(\mathscr{X}) \cap (\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))'\right)_{\sigma,q}\\
&\quad\subset \left(\mathscr{A}\dot{F}^1_{p,\infty}(\mathscr{X}) \cap (\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))', \mathscr{A}\dot{F}^{s_1}_{p,q_1}(\mathscr{X}) \cap (\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))'\right)_{\sigma,q}\\
&\quad\subset \left(\mathscr{A}\dot{B}^1_{p,\infty}(\mathscr{X}) \cap (\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))', \mathscr{A}\dot{B}^{s_1}_{p,\infty}(\mathscr{X}) \cap (\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))'\right)_{\sigma,q}.\end{aligned}$$

To complete the proof of (ii), it suffices to prove that for $p \in (p(s,\epsilon), \infty]$ and $q \in (0,\infty]$,

$$\dot{B}^s_{p,q}(\mathscr{X}) \subset \left(\mathscr{A}\dot{B}^1_{p,\infty}(\mathscr{X}) \cap (\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))', \mathscr{A}\dot{B}^{s_1}_{p,q_2}(\mathscr{X}) \cap (\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))'\right)_{\sigma,q}.$$

To this end, we write $f \equiv f_0^j + f_1^j$ for any $j \in \mathbf{Z}$ as in (2.6). It follows from the definition of the K-functional that

$$\begin{aligned}
&\|f\|_{\left(\mathscr{A}\dot{B}^1_{p,\infty}(\mathscr{X}),\mathscr{A}\dot{B}^{s_1}_{p,q_2}(\mathscr{X})\right)_{\sigma,q}} \\
&\lesssim \left\{\sum_{j\in\mathbf{Z}} 2^{-qj(1-s)}\left\{\sup_{l\in\mathbf{Z}} 2^l \left\|\sup_{\phi\in\mathscr{A}_l(\cdot)} |\langle f_0^j,\phi\rangle|\right\|_{L^p(\mathscr{X})}\right\}^q\right\}^{1/q} \\
&\quad + \left\{\sum_{j\in\mathbf{Z}} 2^{-qj(s_1-s)}\left\{\sum_{l\in\mathbf{Z}} 2^{lq_2s_1}\left\|\sup_{\phi\in\mathscr{A}_l(\cdot)} |\langle f_1^j,\phi\rangle|\right\|^{q_2}_{L^p(\mathscr{X})}\right\}^{q/q_2}\right\}^{1/q} \equiv \mathrm{K}+\mathrm{L}.
\end{aligned}$$

The estimate for K is exactly the same as in the proof of (i) and we now turn to the estimate for L. For $\phi \in \mathscr{A}_l(x)$, by (2.7) and Lemma 2.4(iv), we have

$$|\langle f_1^j,\phi\rangle| \lesssim \sum_{k=j+1}^{\infty} 2^{-|k-l|}2^{[(k\wedge l)-k]n(1-1/r)}\left[M(|\overline{D}_k(f)|^r)(x)\right]^{1/r},$$

where we chose $r \in (p(s_1,\epsilon),p)$. Using the Minkowski inequality, the $L^{p/r}(\mathscr{X})$-boundedness of M and [15, Remark 5.5], we have

$$\mathrm{L} \lesssim \left\{\sum_{k\in\mathbf{Z}} 2^{kqs}\left\|\overline{D}_k(f)\right\|^q_{L^p(\mathscr{X})}\right\}^{1/q} \sim \|f\|_{\dot{B}^s_{p,q}(\mathscr{X})}.$$

Thus, (ii) holds, which completes the proof of Theorem 1.2. □

3. Inhomogeneous case

In this section, $\mu(\mathscr{X})$ can be *finite* or *infinite*, since in both cases the inhomogeneous Calderón reproducing formulae are available; see [15, Theorem 4.16]. For any Christ dyadic cube Q, we set $m_Q(f) \equiv \frac{1}{\mu(Q)}\int_Q f(x)d\mu(x)$. We recall the following notions; see, for example, [15].

Definition 3.1. Let $\epsilon \in (0,1)$ and $\{S_k\}_{k\in\mathbf{Z}_+}$ be an ϵ-AOTI with bounded support as in Definition 1.3. Set $D_0 \equiv S_0$ and $D_k \equiv S_k - S_{k-1}$ for $k \in \mathbf{N}$. Let $\{Q_\tau^{0,\nu}\colon \tau \in I_0,\ \nu = 1,\cdots,N(0,\tau)\}$ with a fixed large $j \in \mathbf{N}$ be dyadic cubes as in Section 2.

(i) Let $|s| < \epsilon$, $p \in (p(s,\epsilon),\infty]$ and $q \in (0,\infty]$. The *inhomogeneous Besov space* $B^s_{p,q}(\mathscr{X})$ is defined to be the space of all $f \in (\mathscr{G}_0^\epsilon(\beta,\gamma))'$, with some β, γ satisfying

$$\max\{s,0,-s+n(1/p-1)_+\} < \beta < \epsilon \text{ and } n(1/p-1)_+ < \gamma < \epsilon, \tag{3.1}$$

such that

$$\begin{aligned}
\|f\|_{B^s_{p,q}(\mathscr{X})} \equiv &\left\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{0,\nu})\left[m_{Q_\tau^{0,\nu}}(|D_0(f)|)\right]^p\right\}^{1/p} + \left\{\sum_{k\in\mathbf{N}} 2^{ksq}\|D_k(f)\|^q_{L^p(\mathscr{X})}\right\}^{1/q} \\
&< \infty
\end{aligned}$$

with the usual modifications made when $p = \infty$ or $q = \infty$.

(ii) Let $|s| < \epsilon$ and $p, q \in (p(s,\epsilon), \infty]$. The *inhomogeneous Triebel–Lizorkin space* $F^s_{p,q}(\mathscr{X})$ is defined to be the space of all $f \in (\mathscr{G}^\epsilon_0(\beta,\gamma))'$ with some β, γ satisfying (3.1) such that $\|f\|_{F^s_{p,q}(\mathscr{X})} < \infty$, where when $p < \infty$,

$$\|f\|_{F^s_{p,q}(\mathscr{X})} \equiv \left\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(k,\tau)} \mu(Q^{0,\nu}_\tau)\left[m_{Q^{0,\nu}_\tau}(|D_0(f)|)\right]^p\right\}^{1/p} + \left\|\left\{\sum_{k\in\mathbf{N}} 2^{ksq}|D_k(f)|^q\right\}^{1/q}\right\|_{L^p(\mathscr{X})},$$

and when $p = \infty$,

$$\|f\|_{\mathscr{A}F^s_{\infty,q}(\mathscr{X})} \equiv \max\left\{\sup_{\substack{\tau\in I_0\\ \nu\in\{1,\cdots,N(0,\tau)\}}} m_{Q^{0,\tau}_\nu}(|D_0(f)|),\right. \left.\sup_{l\in\mathbf{N}}\sup_{a\in I_l}\left[\frac{1}{\mu(Q^l_a)}\int_{Q^l_a}\sum_{k=l}^\infty 2^{ksq}|D_k(f)(x)|^q\,d\mu(x)\right]^{1/q}\right\}$$

with the usual modification made when $q = \infty$.

It was showed in [15] that the spaces $B^s_{p,q}(\mathscr{X})$ and $F^s_{p,q}(\mathscr{X})$ are independent of the choices of the approximations of the identity and the distribution spaces $(\mathscr{G}^\epsilon_0(\beta,\gamma))'$ with ϵ, β and γ as in (3.1). Also, when $s \in (0,1)$, the difference characterizations of $B^s_{p,q}(\mathscr{X})$ with $p \in [1,\infty]$ and $q \in (0,\infty]$ and $F^s_{p,q}(\mathscr{X})$ with $p \in (1,\infty)$ and $q \in (1,\infty]$ were presented in [20].

Definition 3.2. Let $s \in [-1,1]$, $p, q \in (0,\infty]$, $\mathscr{A} \equiv \{\mathscr{A}_k(x)\}_{k\in\mathbf{Z}_+, x\in\mathscr{X}}$ with $\mathscr{A}_0(x) \equiv \{\phi \in \mathscr{G}(1,2) : \|\phi\|_{\mathscr{G}(x,1,1,2)} \le 1\}$ for all $x \in \mathscr{X}$ and $\mathscr{A}_k(x)$ for all $k \in \mathbf{N}$ and all $x \in \mathscr{X}$ being as in Definition 1.5.

(i) The *inhomogeneous grand Besov space* $\mathscr{A}B^s_{p,q}(\mathscr{X})$ is defined to be the space of all $f \in (\mathscr{G}(1,2))'$ such that

$$\|f\|_{\mathscr{A}B^s_{p,q}(\mathscr{X})} \equiv \left\{\sum_{k\in\mathbf{Z}_+} 2^{kqs}\left\|\sup_{\phi\in\mathscr{A}_k(\cdot)}|\langle f,\phi\rangle|\right\|^q_{L^p(\mathscr{X})}\right\}^{1/q} < \infty$$

with the usual modifications made when $p = \infty$ or $q = \infty$.

(ii) The *inhomogeneous grand Triebel–Lizorkin space* $\mathscr{A}F^s_{p,q}(\mathscr{X})$ is defined to be the space of all $f \in (\mathscr{G}(1,2))'$ such that $\|f\|_{\mathscr{A}F^s_{p,q}(\mathscr{X})} < \infty$, where when $p \in (0,\infty)$,

$$\|f\|_{\mathscr{A}F^s_{p,q}(\mathscr{X})} \equiv \left\|\left\{\sum_{k\in\mathbf{Z}_+} 2^{kqs}\sup_{\phi\in\mathscr{A}_k(\cdot)}|\langle f,\phi\rangle|^q\right\}^{1/q}\right\|_{L^p(\mathscr{X})},$$

and when $p = \infty$,

$$\|f\|_{\mathscr{A}F^s_{\infty,q}(\mathscr{X})} \equiv \sup_{l\in\mathbf{Z}_+}\sup_{a\in I_l}\left\{\frac{1}{\mu(Q^l_a)}\int_{Q^l_a}\sum_{k=l}^\infty 2^{ksq}\sup_{\phi\in\mathscr{A}_k(x)}|\langle f,\phi\rangle|^q\,d\mu(x)\right\}^{1/q}$$

with the usual modification made when $q = \infty$.

Theorem 3.1. *Let all notation be as in Definition 3.1. Then*

$$\mathscr{A}B^s_{p,q}(\mathscr{X})\cap(\mathscr{G}^\epsilon_0(\beta,\gamma))' = B^s_{p,q}(\mathscr{X}) \text{ and } \mathscr{A}F^s_{p,q}(\mathscr{X})\cap(\mathscr{G}^\epsilon_0(\beta,\gamma))' = F^s_{p,q}(\mathscr{X})$$

with equivalent quasi-norms.

Similarly to the homogeneous case, it was proved in [18, 19] that $\mathscr{A}B^s_{p,q}(\mathscr{X}) = B^s_{p,q}(\mathscr{X})$ with equivalent quasi-norms, if $s\in(0,1)$ and $p\in(p(s,\epsilon),\infty]$ and $q\in(0,\infty]$; $\mathscr{A}F^s_{p,q}(\mathscr{X}) = F^s_{p,q}(\mathscr{X})$ with equivalent quasi-norms, if $s\in(0,1)$ and $p,q\in(p(s,\epsilon),\infty]$. Thus, Theorem 3.1 generalizes the corresponding results in [18, 19].

Theorem 3.2. *Let ϵ, β and γ be as in Definition 3.1, $\sigma\in(0,1)$ and $q_0,q_1,q\in(0,\infty]$. Then all the claims of Theorem 1.2 hold with $\dot{B}$ replaced by B, $\dot{F}$ by F and $(\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))'$ by $(\mathscr{G}^\epsilon_0(\beta,\gamma))'$, respectively.*

Similarly to the homogeneous case, when $s_0\in(0,1)$ (resp. $s_1\in(0,1)$), the space $\mathscr{A}B^{s_0}_{p,q_0}(\mathscr{X})\cap(\mathscr{G}^\epsilon_0(\beta,\gamma))'$ (resp. $\mathscr{A}B^{s_1}_{p,q_1}(\mathscr{X})\cap(\mathscr{G}^\epsilon_0(\beta,\gamma))'$) in Theorem 3.2 can be replaced by $\mathscr{A}B^{s_0}_{p,q_0}(\mathscr{X})$ (resp. $\mathscr{A}B^{s_1}_{p,q_1}(\mathscr{X})$).

The proofs of Theorems 3.1 and 3.2 are similar to those of the homogeneous cases. We point out that instead of the homogeneous Calderón reproducing formulae, in the proof of Theorems 3.1 and 3.2, we need the inhomogeneous ones established in [15]. We omit the details.

From the proofs of Theorems 1.1, 1.2, 3.1 and 3.2, it is easy to see that the following remark holds.

Remark 3.1. Let $s\in(-1,1)$ and $\epsilon\in(0,s)$.

(i) Let β,γ be as in (1.2). If in Definition 1.5, we let $\mathscr{A}_k(x)\equiv\{\phi\in\mathring{\mathscr{G}}^\epsilon_0(x,2^{-k},\beta,\gamma):\|\phi\|_{\mathring{\mathscr{G}}^\epsilon_0(x,2^{-k},\beta,\gamma)}\le 1\}$ for all $x\in\mathscr{X}$ and $k\in\mathbf{Z}$, then $\mathscr{A}\dot{B}^s_{p,q}(\mathscr{X})\cap(\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))'$ and $\mathscr{A}\dot{F}^s_{p,q}(\mathscr{X})\cap(\mathring{\mathscr{G}}^\epsilon_0(\beta,\gamma))'$ in Theorems 1.1 and 1.2 can be replaced, respectively, just by $\mathscr{A}\dot{B}^s_{p,q}(\mathscr{X})$ and $\mathscr{A}\dot{F}^s_{p,q}(\mathscr{X})$.

(ii) Let β,γ be as in (3.1). If in Definition 3.2, we let $\mathscr{A}_0(x)\equiv\{\phi\in\mathscr{G}^\epsilon_0(x,1,\beta,\gamma):\|\phi\|_{\mathscr{G}^\epsilon_0(x,1,\beta,\gamma)}\le 1\}$ for all $x\in\mathscr{X}$ and $\mathscr{A}_k(x)$ for all $k\in\mathbf{N}$ and all $x\in\mathscr{X}$ be as in (i), then $\mathscr{A}B^s_{p,q}(\mathscr{X})\cap(\mathscr{G}^\epsilon_0(\beta,\gamma))'$ and $\mathscr{A}F^s_{p,q}(\mathscr{X})\cap(\mathscr{G}^\epsilon_0(\beta,\gamma))'$ in Theorems 3.1 and 3.2 can be replaced, respectively, just by $\mathscr{A}B^s_{p,q}(\mathscr{X})$ and $\mathscr{A}F^s_{p,q}(\mathscr{X})$.

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