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A NOTE ON MAPPINGS OF FINITE DISTORTION: EXAMPLES FOR THE SHARP MODULUS OF CONTINUITY

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Abstract. We construct a mapping with exponentially integrable distortion which attains a modulus of continuity by Onninen and Zhong, showing that it is sharp.

1. Introduction

Let $\Omega \subset \mathbf{R}^n$, $n \geq 2$, be a connected and open set. A Sobolev mapping $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$ is said to have finite distortion if the Jacobian $J_f(x)$, i.e., the determinant of the matrix of derivatives $Df(x)$ is locally integrable and there is a measurable function $K(x) \geq 1$ finite almost everywhere such that

$$|Df(x)|^n \leq K(x) J_f(x) \quad \text{a.e. } x \in \Omega.$$

Here we have used the operator norm of the differential matrix with respect to the Euclidean distance.

If we, moreover, require that $K(x) \in L^\infty(\Omega)$, we arrive at mappings of bounded distortion also called quasiregular mappings. In [7] Reshetnyak proved among many other things that quasiregular mappings are Hölder continuous with the exponent $1/K$, where K is the L^∞ norm of the distortion. It has been shown recently that mappings of finite distortion with exponentially integrable distortion

$$\exp(\lambda K(x)) \in L^1(\Omega)$$

share many nice properties of mappings of bounded distortion. We would like to point the reader's attention to the monographs [1] and [3] for the motivation, applications and the history of the subject.

Our aim is to study the modulus of continuity of the mappings of finite distortion with $\exp(\lambda K) \in L^1(\Omega)$. Let us first recall the history of such estimates. First, it was shown by Iwaniec, Koskela and Onninen [2] that mappings in this class are continuous and satisfy

$$|f(x) - f(y)| \leq \frac{C}{\log^{1/n} \log(e^e + 1/|x-y|)}.$$

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This was later improved by Koskela and Onninen [5] to

$$|f(x)-f(y)| \leq \frac{C}{\log^{\lambda/n-\varepsilon}(1/|x-y|)}$$

and finally using very delicate arguments it has been shown by Onninen and Zhong [6] that

$$|f(x)-f(y)| \leq \frac{C}{\log^{\lambda/n}(1/|x-y|)}. \tag{1.1}$$

Extremal mappings for continuity of mappings of finite distortion are usually radial maps and therefore the natural candidate for the extremal map is

$$f_0(x) = \frac{x}{|x|}\frac{1}{\log^{\lambda/n}(1/|x|)}.$$

Standard computations (see (2.2) below) give us

$$K(x) = \frac{n}{\lambda}\log\frac{1}{|x|}$$

and hence

$$\int_{B(0,\frac{1}{2})} \exp(\lambda K(x))\,dx = \int_{B(0,\frac{1}{2})} \frac{1}{|x|^n}\,dx = \infty.$$

This elementary computation suggests that there is some room for improvement in the estimate (1.1) and maybe we can add some supplementary term like $\log\log 1/|x-y|$ to some negative power to our estimate. We show that, surprisingly, this is not the case and the modulus of continuity (1.1) is already sharp.

Theorem 1.1. *Given $\lambda > 0$, there is a mapping of finite distortion $f\colon B(0,\frac{1}{2}) \to \mathbf{R}^n$ such that*

$$\int_{B(0,\frac{1}{2})} \exp(\lambda K(x))\,dx < \infty$$

and

$$|f(x)-f(0)| \geq \frac{C}{\log^{\lambda/n}(1/|x|)} \quad \textit{for all } x \in B(0,\tfrac{1}{2}).$$

There have also been studies on mappings of subexponentially integrable distortion (see e.g. [4]). One requires that

$$\int_B \exp\bigl(\mathscr{A}(K(x))\bigr)\,dx < \infty \tag{1.2}$$

for some Orlicz function $\mathscr{A}$ and the above mentioned example corresponds to the case $\mathscr{A}(t) = \lambda t$. We call an infinitely differentiable and strictly increasing function $\mathscr{A}\colon [0,\infty) \to [0,\infty)$ with $\mathscr{A}(0) = 0$ and $\lim_{t\to\infty}\mathscr{A}(t) = \infty$ an Orlicz function. As usual we impose additional condition

$$\int_1^\infty \frac{\mathscr{A}'(s)}{s} = \infty. \tag{1.3}$$

It is easy to see that the critical functions for this condition are

$$\begin{aligned}\mathscr{A}_1(t) &= \lambda t, \quad \mathscr{A}_2(t) = \lambda \frac{t}{\log(e+t)},\\ \mathscr{A}_3(t) &= \lambda \frac{t}{\log(e+t)\log(e+\log(e+t))} \quad \text{and so on.}\end{aligned} \tag{1.4}$$

We will also require that

$$\begin{aligned}&\text{(i)} \quad \exists t_0 > 0 \ \forall t > t_0 \ \ \mathscr{A}^{-1}(nt) < t^{\frac{3}{2}},\\ &\text{(ii)} \quad \mathscr{A}'(t) \text{ is non-increasing},\\ &\text{(iii)} \quad b'(t) \text{ is non-increasing for } b(t) := \frac{t}{\mathscr{A}(t)},\\ &\text{(iv)} \quad b(0) := \lim_{t\to 0+} b(t) \text{ is finite and positive.}\end{aligned} \tag{1.5}$$

Let us note that the critical functions from (1.4) satisfy these conditions and therefore these assumptions are not substantially restrictive. It has been shown in [4] that a mapping f is continuous under the assumptions (1.2) and (1.3) and that the assumption (1.3) is sharp.

It was proved in [6] that under the assumptions (1.2) and (1.3) we have

$$|f(x) - f(y)| \le C \exp\Big(-\int_{|x-y|}^{R} \frac{dt}{t\mathscr{A}^{-1}(n\log C/t)}\Big)$$

for $|x-y|$ sufficiently small and $B(x, 80R) \subset \Omega$. Note further that if we put $\mathscr{A}_1(t) = \lambda t$ we arrive at the modulus given in (1.1). Our result shows the sharpness of this estimate.

Theorem 1.2. *Suppose that an Orlicz function $\mathscr{A}$ satisfies* (1.3) *and* (1.5). *Then there is a ball $B := B(0, r)$ and a mapping of finite distortion $f\colon B \to \mathbf{R}^n$ such that*

$$\int_B \exp\big(\mathscr{A}(K(x))\big)\, dx < \infty$$

and

$$|f(x) - f(0)| \ge C \exp\Big(-\int_{|x|}^{1/2} \frac{dt}{t\mathscr{A}^{-1}(n\log 1/t)}\Big) \quad \textit{for all } x \in B. \tag{1.6}$$

2. Proofs of the theorems

To prove Theorem 1.1 we simply set

$$f(x) = \frac{x}{|x|} \frac{(\log 1/|x|)^{\frac{a}{\log 1/|x|}}}{\log^{\lambda/n}(1/|x|)},$$

where $a > 0$. The additional term clearly satisfies

$$\lim_{|x|\to 0} (\log 1/|x|)^{\frac{a}{\log 1/|x|}} = 1$$

and thus the modulus of continuity of our f is exactly as required in (1.1). On the other hand, the additional term slightly affects the distortion and the standard computation (see the general case below for details) will give us

$$K(x) \sim \frac{n}{\lambda} \log \frac{1}{|x|} - \frac{n^2 a}{\lambda^2} \log \log \frac{1}{|x|},$$

and hence

$$\int_{B(0,\frac{1}{2})} \exp(\lambda K(x))\, dx < \infty$$

for sufficiently large a.

To prove Theorem 1.2, let us put $B := B(0, \min\{\exp(-t_0), e^{-e}\})$ and choose $\alpha > b(0)^{-1} n^{-2}$. Without loss of generality we can assume that t_0 is big enough such that

$$t^{\frac{3}{2}} < \frac{1}{\alpha(\alpha+1)} \frac{t^2}{\log t} \quad \text{for all } \; t > t_0. \tag{2.1}$$

We define the function f as,

$$f(x) := \frac{x}{|x|} \exp\Big(- \int_{|x|}^{\frac{1}{2}} \frac{1}{t \mathscr{A}^{-1}(n \log \frac{1}{t})} dt\Big) (\log |x|^{-1})^{\frac{\alpha+2}{\log |x|^{-1}}} \quad \text{for } \; x \neq 0.$$

Note that

$$\lim_{|x| \to 0} (\log |x|^{-1})^{\frac{\alpha+2}{\log |x|^{-1}}} = \lim_{|x| \to 0} \exp\Big(\frac{(\alpha+2) \log \log |x|^{-1}}{\log |x|^{-1}} \Big) = 1,$$

which easily gives that f satisfies the condition given in (1.6).

Let $\rho\colon (0,\infty) \to (0,\infty)$ be a strictly monotone, differentiable function and let us consider the mapping

$$f(x) = \frac{x}{|x|} \rho(|x|), \quad x \neq 0.$$

It can be verified by an elementary computation (see e.g. [1, Chapter 2.6.]) that

$$\begin{aligned} |Df(x)| &= \max\Big\{ \frac{\rho(|x|)}{|x|}, |\rho'(|x|)| \Big\}, \quad \text{and thus} \\ K(x) &= \max\Big\{ \frac{\rho(|x|)}{|x|\ |\rho'(|x|)|}, \frac{|x|\ |\rho'(|x|)|}{\rho(|x|)} \Big\}. \end{aligned} \tag{2.2}$$

It follows that for our mapping we obtain

$$|Df(x)| = \frac{|f(x)|}{|x|} \max\Big\{ 1, \Big(\frac{1}{\mathscr{A}^{-1}(n \log |x|^{-1})} + (\alpha+2) \frac{\log\log |x|^{-1} - 1}{\log^2 |x|^{-1}} \Big) \Big\}.$$

Clearly,

$$\lim_{x \to 0} \Big(\frac{1}{\mathscr{A}^{-1}(n \log |x|^{-1})} + (\alpha+2) \frac{\log\log |x|^{-1} - 1}{\log^2 |x|^{-1}} \Big) = 0,$$

and therefore the greater element is the first one. From (1.5) (i) and (2.1) we obtain

$$\mathscr{A}^{-1}(nt) < t^{\frac{3}{2}} < \frac{1}{\alpha(\alpha+1)} \frac{t^2}{\log t} \quad \text{for all } \; t > t_0.$$

This, however, implies that

$$\alpha(\alpha+1) \frac{\mathscr{A}^{-1}(nt) \log t}{t^2} < 1 \quad \text{for all } \; t > t_0.$$

Now, by multiplying both sides by $\frac{\mathscr{A}^{-1}(nt)\log t}{t^2}$ and by substituting $t = \log|x^{-1}|$ we get that

$$\mathscr{A}^{-1}(n\log|x|^{-1})\frac{\log\log|x|^{-1}}{\log^2|x|^{-1}} > \alpha(\alpha+1)\Big(\mathscr{A}^{-1}(n\log|x|^{-1})\frac{\log\log|x|^{-1}}{\log^2|x|^{-1}}\Big)^2$$

for all $x \in B$. Using this fact and because $\log\log|x|^{-1} > 1$ for all $x \in B$, we deduce that

$$\begin{aligned} K(x) &= \frac{1}{\Big(\frac{1}{\mathscr{A}^{-1}(n\log|x|^{-1})} + (\alpha+2)\frac{\log\log|x|^{-1}-1}{\log^2|x|^{-1}}\Big)} \\ &\le \frac{\mathscr{A}^{-1}(n\log|x|^{-1})}{1+(\alpha+1)\mathscr{A}^{-1}(n\log|x|^{-1})\frac{\log\log|x|^{-1}}{\log^2|x|^{-1}}} \\ &\le \mathscr{A}^{-1}(n\log|x|^{-1})\Big(1-\alpha\mathscr{A}^{-1}(n\log|x|^{-1})\frac{\log\log|x|^{-1}}{\log^2|x|^{-1}}\Big) =: \tilde{K}(x). \end{aligned}$$

Note that

$$\mathscr{A}^{-1}(n\log|x|^{-1}) - \tilde{K}(x) = \alpha n^2\Big(\frac{\mathscr{A}^{-1}(n\log|x|^{-1})}{n\log|x|^{-1}}\Big)^2\log\log|x|^{-1}. \tag{2.3}$$

By (1.5) (iii) we obtain that

$$b(s) - b(0) = b'(\xi)s \ge b'(s)s$$

and therefore

$$\mathscr{A}'(s)\Big(\frac{s}{\mathscr{A}(s)}\Big)^2 = \frac{b(s)-sb'(s)}{b^2(s)}b^2(s) \ge b(0). \tag{2.4}$$

From (1.5) (ii) we know that $\mathscr{A}'(t)$ is a non-increasing function and hence

$$\mathscr{A}(a-d) = \mathscr{A}(a) - \mathscr{A}'(\xi)d \le \mathscr{A}(a) - \mathscr{A}'(a)d \tag{2.5}$$

for some $\xi \in (a-d, a)$. We now use (2.5) putting

$$a := \mathscr{A}^{-1}\Big(n\log\frac{1}{|x|}\Big), \quad d := \mathscr{A}^{-1}\Big(n\log\frac{1}{|x|}\Big) - \tilde{K}(x)$$

using (2.3) and then (2.4) (where we put $s := \mathscr{A}^{-1}(n\log|x|^{-1})$) to get that

$$\begin{aligned} &\mathscr{A}(K(x)) \le \mathscr{A}(\tilde{K}(x)) \\ &\le \mathscr{A}\big(\mathscr{A}^{-1}(n\log|x|^{-1})\big) - \mathscr{A}'\big(\mathscr{A}^{-1}(n\log|x|^{-1})\big)\Big[\mathscr{A}^{-1}\big(n\log|x|^{-1}\big) - \tilde{K}(x)\Big] \\ &\le n\log|x|^{-1} - \alpha n^2\mathscr{A}'\big(\mathscr{A}^{-1}(n\log|x|^{-1})\big)\Big(\frac{\mathscr{A}^{-1}(n\log|x|^{-1})}{n\log|x|^{-1}}\Big)^2\log\log|x|^{-1} \\ &\le n\log|x|^{-1} - b(0)\alpha n^2\log\log|x|^{-1}. \end{aligned}$$

But this, for $\alpha > b(0)^{-1}n^{-2}$, yields

$$\begin{aligned} \int_B \exp(\mathscr{A}(K(x)))\,dx &\le \int_B \exp\Big(n\log\frac{1}{|x|} - b(0)\alpha n^2\log\log\frac{1}{|x|}\Big)\,dx \\ &\le \int_B \frac{1}{|x|^n\log^{b(0)\alpha n^2}\frac{1}{|x|}}\,dx < \infty. \end{aligned}$$

□

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