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ANISOTROPIC SOBOLEV HOMEOMORPHISMS

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Abstract. Let $\Omega \subset \mathbf{R}^2$ be a domain. Suppose that $f \in \mathscr{W}^{1,1}_{\rm loc}(\Omega;\mathbf{R}^2)$ is a homeomorphism. Then the components $x(w)$, $y(w)$ of the inverse $f^{-1} = (x,y)\colon \Omega' \to \Omega$ have total variations given by

$$|\nabla y|(\Omega') = \int_\Omega \left|\frac{\partial f}{\partial x}\right| dz, \quad |\nabla x|(\Omega') = \int_\Omega \left|\frac{\partial f}{\partial y}\right| dz.$$

1. Introduction

Let $\Omega \subseteq \mathbf{R}^2$ and $\Omega' \subseteq \mathbf{R}^2$ be domains. Recently, homeomorphisms $f = (u,v)\colon \Omega \xrightarrow{\rm onto} \Omega'$ which are a.e. differentiable together with their inverses $f^{-1} = (x,y)\colon \Omega' \xrightarrow{\rm onto} \Omega$ have been intensively studied (see [9], [11]).

A homeomorphism $f\colon \Omega \xrightarrow{\rm onto} \Omega'$ which belongs to the Sobolev space $\mathscr{W}^{1,1}_{\rm loc}(\Omega;\mathbf{R}^2)$ is called a $\mathscr{W}^{1,1}$-homeomorphism. If also f^{-1} is a $\mathscr{W}^{1,1}$-homeomorphism, we say that f is a bi-Sobolev map (see [13]). We recall that a $\mathscr{W}^{1,1}$-homeomorphism is differentiable a.e. thanks to the well known Gehring–Lehto Theorem (see [6], Theorem 2).

If we adopt the following notations:

$$f(x,y) = (u(x,y), v(x,y)) \quad \text{for } (x,y) \in \Omega,$$
$$f^{-1}(u,v) = (x(u,v), y(u,v)) \quad \text{for } (u,v) \in \Omega',$$

then the bi-Sobolev condition for f and f^{-1} can be precisely expressed by

$$u_x, u_y, v_x, v_y \in L^1_{\rm loc}(\Omega) \tag{1.1}$$

and

$$x_u, x_v, y_u, y_v \in L^1_{\rm loc}(\Omega'). \tag{1.2}$$

The following result derives from [3],[9] and [13].

Theorem 1.1. *If $f\colon \Omega \xrightarrow{\rm onto} \Omega'$ is a bi-Sobolev map, then*

$$\int_\Omega |Df|\, dz = \int_{\Omega'} \left|Df^{-1}\right| dw. \tag{1.3}$$

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If f is an a.e. differentiable homeomorphism, then the Jacobian determinant J_f satisfies either the inequality $J_f \geq 0$ or $J_f \leq 0$ a.e. ([2], [12]). For simplicity let us assume $J_f(z) \geq 0$ for a.e. $z \in \Omega$.

Let us point out that if the Jacobians J_f of f and $J_{f^{-1}}$ of f^{-1} are strictly positive a.e., it is possible to prove (1.3) by mean of the area formula (see Sections 2 and 3). On the other hand, bi-Sobolev mappings do not enjoy such a property; it may happen that their Jacobian vanishes on a set of positive measure ([19], [20], [14]).

The bi-Sobolev assumption rules out the Lipschitz homeomorphism

$$f_0\colon (0,2) \times (0,1) \to (0,1) \times (0,1), \quad f_0(x,y) = (h(x), y)\,, \tag{1.4}$$

where $h^{-1}(t) = t + c(t)$ and $c\colon (0,1) \to (0,1)$ is the usual Cantor ternary function because f_0^{-1} does not belong to $\mathscr{W}^{1,1}_{\mathrm{loc}}$. On the contrary, our first results deal with $\mathscr{W}^{1,1}$-homeomorphisms which include f_0 as well (Theorem 1.3). Another interesting property of a bi-Sobolev map $f = (u,v)$ in the plane is that u and v have the same critical points ([13], [17]).

Theorem 1.2. *Let $f\colon \Omega \overset{\text{onto}}{\longrightarrow} \Omega'$ be a bi-Sobolev map $f = (u,v)$. Then u and v have the same critical points:*

$$\{z \in \Omega\colon\ |\nabla u(z)| = 0\} = \{z \in \Omega\colon\ |\nabla v(z)| = 0\} \quad \textit{a.e.} \tag{1.5}$$

The same result holds also for the inverse f^{-1}. The analogue of this Theorem is not valid in more than two dimensions (see [13]).

Let us point out that we only assume that f and f^{-1} are in $\mathscr{W}^{1,1}_{\mathrm{loc}}$. In the category of $\mathscr{W}^{1,p}$-bi-Sobolev maps, that is, f belongs to $\mathscr{W}^{1,p}_{\mathrm{loc}}(\Omega; \mathbf{R}^2)$ and f^{-1} belongs to $\mathscr{W}^{1,p}_{\mathrm{loc}}(\Omega'; \mathbf{R}^2)$, the case $1 \leq p < 2$ (see [20]) is critical with respect to the so-called N property of Lusin, i.e., that a function maps every set of measure zero to a set of measure zero. Let us mention that for $\mathscr{W}^{1,2}$-bi-Sobolev mappings the statement of Theorem 1.2 is obviously satisfied. In fact (see [16], p. 150), for homeomorphisms in $\mathscr{W}^{1,2}_{\mathrm{loc}}$ we have the N property. Clearly

$$\{z \in \Omega\colon\ |\nabla u(z)| = 0\} \subset \{z \in \Omega\colon J_f(z) = 0\} \quad \text{a.e.}$$

We can decompose the set $\{J_f = 0\}$ into a null set Z and countably many sets on which we can use the Sard's Lemma (see [4], Theorem 3.1.8). It follows that

$$|f\,(\{J_f = 0\} \setminus Z)| = 0 \ \text{ and hence } \ |f\,(\{\nabla u = 0\} \setminus Z)| = 0.$$

Since f^{-1} satisfies the N property, we obtain $|\{\nabla u = 0\}| = 0$ and analogously $|\{\nabla v = 0\}| = 0$ as well.

We observe that the following identity

$$\left\{z \in \Omega\colon\ \left|\frac{\partial f}{\partial x}(z)\right| = 0\right\} = \left\{z \in \Omega\colon\ \left|\frac{\partial f}{\partial y}(z)\right| = 0\right\} \qquad \text{a.e.}$$

where $\left|\frac{\partial f}{\partial x}(z)\right|^2 = u_x^2(z) + v_x^2(z)$ and $\left|\frac{\partial f}{\partial y}(z)\right|^2 = u_y^2(z) + v_y^2(z)$, is true for bi-Sobolev maps and parallels (1.5). This is a consequence of the following characteristic property of a bi-Sobolev map which was proved in [3], [13], [9]:

$$J_f(z) = 0 \quad \Longrightarrow \quad |Df(z)| = 0 \quad \text{a.e.} \tag{1.6}$$

Our first result is the following, in which we give some identities for $\mathscr{W}^{1,1}$-homeomorphism. Notice that the symbol $|\nabla\varphi|(\Omega')$ denotes the total variation of the real function φ belonging to the space $\mathrm{BV}(\Omega')$ of functions of bounded variation on Ω' (see Section 2).

Theorem 1.3. *Let $f = (u,v)\colon \Omega \subset \mathbf{R}^2 \xrightarrow{\text{onto}} \Omega' \subset \mathbf{R}^2$ be a homeomorphism whose inverse is $f^{-1} = (x,y)$. If we assume $u, v \in \mathscr{W}^{1,1}_{\mathrm{loc}}(\Omega)$, then $x, y \in \mathrm{BV}_{\mathrm{loc}}(\Omega')$ and*

$$|\nabla y|\,(\Omega') = \int_\Omega \left|\frac{\partial f}{\partial x}(z)\right| dz, \tag{1.7}$$

$$|\nabla x|\,(\Omega') = \int_\Omega \left|\frac{\partial f}{\partial y}(z)\right| dz. \tag{1.8}$$

In [11] it was proved that if $f\colon \Omega \subset \mathbf{R}^2 \xrightarrow{\text{onto}} \Omega' \subset \mathbf{R}^2$ has bounded variation, $f \in \mathrm{BV}_{\mathrm{loc}}(\Omega; \mathbf{R}^2)$, then $f^{-1} \in \mathrm{BV}_{\mathrm{loc}}(\Omega'; \mathbf{R}^2)$ and both f and f^{-1} are differentiable a.e. We notice that our identities (1.7) and (1.8) represent an improvement of such a result when f is $\mathscr{W}^{1,1}$-homeomorphism; in particular the following estimate

$$\left|Df^{-1}\right|(\Omega') \le 2 \int_\Omega |Df|\, dz$$

holds (Corollary 3.4). A $\mathscr{W}^{1,p}_{\mathrm{loc}}$-homeomorphism in the plane, $1 \le p < 2$ whose Jacobian vanishes a.e., has been recently constructed by Hencl [8]; such a mapping satisfies the assumptions of Theorem 1.3. If in Theorem 1.3 we add the hypothesis $J_f > 0$ a.e., we obtain the identities (1.7) and (1.8) using the area formula (see Sections 2 and 3).

Condition (1.6) makes it possible, for a given bi-Sobolev mapping f, to consider the *distortion quotient*

$$\frac{|Df(z)|^2}{J_f(z)} \quad \text{for a.e. } z \in \Omega. \tag{1.9}$$

Hereafter the undetermined ratio $\frac{0}{0}$ is understood to be equal to 1 for z in the zero set of the Jacobian. The Borel function

$$K_f(z) := \begin{cases} \dfrac{|Df(z)|^2}{J_f(z)} & \text{if } J_f(z) > 0, \\ 1 & \text{otherwise,} \end{cases} \tag{1.10}$$

is the *distortion function* of f and has relevant properties: it is the smallest function $K(z)$ greater or equal to 1 for which the distortion inequality:

$$|Df(z)|^2 \le K(z) J_f(z) \quad \text{a.e. } z \in \Omega \tag{1.11}$$

holds true. Moreover, there are interesting interplay between the integrability of the distortions K_f and $K_{f^{-1}}$ and the regularity of f and f^{-1} (see [13], Theorem 5).

In our general context of $\mathscr{W}^{1,1}$-homeomorphisms there are different distortion functions which play a significant role (see Section 4). We obtain conditions under which one of these functions is finite a.e. or integrable.

2. Preliminaries

We denote by $|A|$ the Lebesgue measure of a set $A \subset \mathbf{R}^2$. We say that two sets $A, B \subseteq \mathbf{R}^2$ satisfy $A = B$ a.e. if their symmetrical difference has measure zero, i.e.,

$$|(A \setminus B) \cup (B \setminus A)| = 0.$$

A homeomorphic mapping $f\colon \Omega \subset \mathbf{R}^2 \xrightarrow{\text{onto}} \Omega' \subset \mathbf{R}^2$ is said to satisfy the N property of Lusin on the domain Ω if for every $A \subset \Omega$ such that $|A| = 0$ we have $|f(A)| = 0$.

A function $u \in \mathscr{L}^1(\Omega)$ is of bounded variation, $u \in \mathrm{BV}(\Omega)$ if the distributional partial derivatives of u are measures with finite total variation in Ω: there exist Radon signed measures $D_1 u$, $D_2 u$ in Ω such that for $i = 1, 2$, $|D_i u|(\Omega) < \infty$ and

$$\int_\Omega u D_i \phi(z)\, dz = -\int_\Omega \phi(z)\, dD_i u(z) \quad \forall \phi \in C_0^1(\Omega).$$

The gradient of u is then a vector-valued measure with finite total variation

$$|\nabla u|(\Omega) = \sup\left\{\int_\Omega u \,\mathrm{div}\varphi(z)\, dz\colon \varphi \in C_0^1(\Omega, \mathbf{R}^2),\ \|\varphi\|_\infty \le 1\right\} < \infty.$$

By $|\nabla u|$ we denote the total variation of the signed measure Du.

The Sobolev space $\mathscr{W}^{1,1}(\Omega)$ is contained in $\mathrm{BV}(\Omega)$; indeed for any $u \in \mathscr{W}^{1,1}(\Omega)$ the total variation is given by $\int_\Omega |\nabla u| = |\nabla u|(\Omega)$. We say that $f = (u, v) \in \mathscr{L}^1(\Omega; \mathbf{R}^2)$ belongs to $\mathrm{BV}(\Omega; \mathbf{R}^2)$ if $u, v \in \mathrm{BV}(\Omega)$. Finally we say that $f \in \mathrm{BV}_{\mathrm{loc}}(\Omega; \mathbf{R}^2)$ if $f \in \mathrm{BV}(A; \mathbf{R}^2)$ for every open $A \subset\subset \Omega$. In the following, for $f \in \mathrm{BV}_{\mathrm{loc}}(\Omega; \mathbf{R}^2)$ we will denote the total variation of f by:

$$|Df|(\Omega) = \sup\left\{\int_\Omega u \,\mathrm{div}\, \varphi_1(z)\, dz + \int_\Omega v \,\mathrm{div}\, \varphi_2(z)\, dz : \varphi_i \in C_0^1(\Omega; \mathbf{R}^2), \|\varphi_i\|_\infty \le 1, i = 1,2\right\}.$$

We will need the definition of sets of finite perimeter (see [1]).

Definition 2.1. Let E be a Lebesgue measurable subset of $\mathbf{R}^2$. For any open set $\Omega \subset \mathbf{R}^2$ the perimeter of E in Ω, denoted by $P(E, \Omega)$, is the total variation of χ_E in Ω, i.e.,

$$P(E, \Omega) = \sup\left\{\int_E \mathrm{div}\, \varphi\, dz\colon \varphi \in C_0^1(\Omega; \mathbf{R}^2),\ \|\varphi\|_\infty \le 1\right\}.$$

We say that E is a set of finite perimeter in Ω if $P(E, \Omega) < \infty$.

We say that $f = (u, v) \in \mathscr{W}^{1,p}_{\mathrm{loc}}(\Omega; \mathbf{R}^2)$, $1 \le p \le \infty$, if for each open $A \subset\subset \Omega$, f belongs to the Sobolev space $\mathscr{W}^{1,p}(A; \mathbf{R}^2)$, i.e., if $u \in \mathscr{L}^p(A)$ and $v \in \mathscr{L}^p(A)$ have distributional derivatives in $\mathscr{L}^p(A)$.

We are interested in the area formula for a homeomorphism $f \in \mathscr{W}^{1,1}_{\mathrm{loc}}(\Omega; \mathbf{R}^2)$ with $\Omega \subset \mathbf{R}^2$. In this case we have

$$\int_\Omega \eta(f(z)) J_f(z)\, dz \le \int_{\mathbf{R}^2} \eta(w)\, dw \tag{2.1}$$

for any non negative Borel function η on $\mathbf{R}^2$. This follows from the area formula for Lipschitz mappings (see [4], Theorem 3.2.3), and from a general property of a.e. differentiable functions (see [4], Theorem 3.1.8), namely that Ω can be exhausted up to a set of measure zero by sets the restriction to which of f is Lipschitz continuous.

Moreover, the area formula

$$\int_E \eta(f(z)) J_f(z)\, dz = \int_{\mathbf{R}^2} \eta(w)\, dw \tag{2.2}$$

holds on each set $E \subset \Omega$ on which the N property of Lusin is satisfied.

3. The identities for $\mathscr{W}^{1,1}$-homeomorphisms

Before proving Theorem 1.3 in its full generality we give now a partial proof under the following additional assumptions:

$$\{w\colon J_{f^{-1}}(w)=0\}=\{w\colon |\nabla y(w)|=0\} \quad \text{a.e.,} \tag{3.1}$$

$$\{z\colon J_f(z)=0\}=\left\{z\colon \left|\frac{\partial f}{\partial x}(z)\right|=0\right\} \quad \text{a.e.,} \tag{3.2}$$

where $J_{f^{-1}}$ denotes the determinant of the absolutely continuous part of Df^{-1}; moreover, we suppose f^{-1} differentiable a.e. in the classical sense. Therefore, we have

$$\int_{\Omega'} |\nabla y(w)|\, dw = \int_{A'} |\nabla y(w)|\, dw,$$

where A' is a Borel subset of the set E' where f^{-1} is differentiable with $J_{f^{-1}}>0$ such that $|A'|=|E'|$.

Applying (2.1), (3.1) and basic linear algebra, we arrive at:

$$\begin{aligned}\int_{A'} |\nabla y(w)|\; dw &= \int_{A'} \frac{|\nabla y(w)|}{J_{f^{-1}}(w)} J_{f^{-1}}(w)\, dw \le \int_{f^{-1}(A')} \frac{|\nabla y(f(z))|}{J_{f^{-1}}(f(z))}\, dz \\ &= \int_{f^{-1}(A')} \left|\frac{\partial f}{\partial x}(z)\right| dz \le \int_{\Omega} \left|\frac{\partial f}{\partial x}(z)\right| dz.\end{aligned}$$

Here we are using the identity $D\,\mathrm{adj}\,D = I\ \det D$ and the fact that $J_f(z)J_{f^{-1}}(f(z))=1$ at the points of differentiability with nonzero Jacobian. We have used as well the expression of the inverse matrix to the differential 2×2 matrix $Df=\begin{pmatrix} u_x & u_y\\ v_x & v_y\end{pmatrix}$ in terms of $Df^{-1}=\begin{pmatrix} x_u & x_v\\ y_u & y_v\end{pmatrix}$, namely

$$y_u(f(z))=-v_x(z)J_{f^{-1}}(f(z)),\quad y_v(f(z))=u_x(z)J_{f^{-1}}(f(z)) \quad \forall z\in f^{-1}(A'),$$

and the identity

$$|\nabla y(f(z))|^2=\left[v_x(z)^2+u_x(z)^2\right] J_{f^{-1}}(f(z))^2 \quad \forall z\in f^{-1}(A').$$

The opposite inequality follows by a symmetric procedure which relies on (3.2).

Notice that (3.1) and (3.2) are certainly satisfied if $J_f>0$ a.e. and $J_{f^{-1}}>0$ a.e. We observe that Theorem 1.1 can be proved using the same technique under the additional assumptions that $J_f>0$ and $J_{f^{-1}}>0$ a.e. In the general case the proof of Theorem 1.3 is completely different; to prove the Theorem we need some preliminary results. The next Lemma is known as Coarea Formula (see [1], Theorem 3.40):

Lemma 3.1. *For any open set $\Omega'\subset \mathbf{R}^2$ and $y\in \mathscr{L}^1_{\mathrm{loc}}(\Omega')$ we have*

$$|\nabla y|(\Omega')=\int_{-\infty}^{+\infty} P\left(\{w\in\Omega'\colon y(w)>t\},\Omega'\right) dt. \tag{3.3}$$

We understand the left-hand side of (3.3) to be infinity if $y\notin \mathrm{BV}$.

The following Lemma is the main step towards the equality in the area formula (see Theorem 1.3 of [3] and also [13], where the case f ACL, i.e., absolutely continuous on lines, is treated).

Lemma 3.2. *Let $f \in \mathscr{W}^{1,1}_{\text{loc}}((-1,1)^2;\mathbf{R}^2)$ be a homeomorphism. Then for almost every $t \in (-1,1)$ the mapping $f_{|(-1,1)\times\{t\}}$ satisfies the N property of Lusin, i.e., for every $A \subset (-1,1)\times\{t\}$, $\mathscr{H}^1(A) = 0$ implies $\mathscr{H}^1(f(A)) = 0$.*

Proof of Theorem 1.3. Without loss of generality we take $\Omega = (-1,1)\times(-1,1)$. Let us apply Lemma 3.2 to the homeomorphism f. Then, the mapping

$$f(\cdot,t) : x \in (-1,1) \mapsto (u(x,t), v(x,t)) \in \Omega'$$

belongs to $\mathscr{W}^{1,1}((-1,1),\mathbf{R}^2)$ for a.e. t and satisfies the N property. In particular, the area formula holds for $f(\cdot,t)$ on $(-1,1)$:

$$\int_{-1}^{1} \left|\frac{\partial f}{\partial x}(x,t)\right| dx = \mathscr{H}^1\big(f((-1,1)\times\{t\})\big). \tag{3.4}$$

Integrating with respect to t we obtain:

$$\int_\Omega \left|\frac{\partial f}{\partial x}(z)\right| dz = \int_{-1}^{1} \mathscr{H}^1\big(f((-1,1)\times\{t\})\big)\, dt. \tag{3.5}$$

Since it is clear that

$$f((-1,1)\times\{t\}) = \{w \in \Omega' \colon y(w) = t\},$$

then

$$\int_\Omega \left|\frac{\partial f}{\partial x}(z)\right| dz = \int_{-1}^{1} \mathscr{H}^1(\{w \in \Omega' \colon y(w) = t\})\, dt.$$

As y is continuous, then the set $\{w \in \Omega' \colon y(w) = t\}$ is the boundary of the level set $\{w \in \Omega' \colon y(w) > t\}$. By assumptions we know that for a.e. t, $\mathscr{H}^1(\{w \in \Omega' \colon y(w) = t\}) < \infty$ and from [1] (p. 209) we have

$$\mathscr{H}^1(\{w \in \Omega' \colon y(w) = t\}) = P(\{w \in \Omega' \colon y(w) > t\}, \Omega') \quad \text{a.e. } t \in (-1,1).$$

Using Coarea Formula from Lemma 3.1, we obtain

$$|\nabla y|(\Omega') = \int_\Omega \left|\frac{\partial f}{\partial x}(z)\right| dz$$

and we deduce that $y \in \mathrm{BV}_{\text{loc}}(\Omega')$.

The equality (1.8) is proved using the same technique. □

Remark 3.3. From the above proof it is clear that if f is a homeomorphism in $\mathrm{BV}_{\text{loc}}(\Omega;\mathbf{R}^2)$ such that $\dfrac{\partial f}{\partial x} \in \mathscr{L}^1(\Omega;\mathbf{R}^2)$, then (1.7) holds true.

Since the total variation of a map is less or equal than the sum of total variation of the components, by Theorem 1.3 we immediately get

Corollary 3.4. *Let $f = (u,v)\colon \Omega \subset \mathbf{R}^2 \xrightarrow{\text{onto}} \Omega' \subset \mathbf{R}^2$ be a homeomorphism whose inverse is $f^{-1} = (x,y)$. If we assume $u, v \in \mathscr{W}^{1,1}_{\text{loc}}(\Omega)$, then*

$$\left|Df^{-1}\right|(\Omega') \le 2\int_\Omega |Df|. \tag{3.6}$$

4. The distortions of anisotropic Sobolev maps

In Section 1 we have already mentioned the known fact that, if $f\colon \Omega \subset \mathbf{R}^2 \xrightarrow{\text{onto}} \Omega' \subset \mathbf{R}^2$ is *bi-Sobolev* , then we have

$$\{z\colon J_f(z) = 0\} = \{z\colon |Df(z)| = 0\} \quad \text{a.e.}$$

and this makes it possible to consider the distortion function

$$K_f(z) := \begin{cases} \dfrac{|Df(z)|^2}{J_f(z)} & \text{if } J_f(z) > 0, \\ 1 & \text{otherwise.} \end{cases} \tag{4.1}$$

Moreover, the distortion inequality

$$|Df(z)|^2 \le K_f(z) J_f(z)$$

holds for a.e. $z \in \Omega$. According to a well established terminology, we say that f has finite distortion K_f.

For a *Sobolev homeomorphism*, under suitable assumptions, it is possible to introduce different distortion functions (see [21]). Namely, if $f = (u, v)$ satisfies the condition

$$\{z\colon J_f(z) = 0\} = \{z\colon |\nabla u(z)| = 0\} \quad \text{a.e.},$$

then we are allowed to define the Borel function

$$K_f^{(1)}(z) := \begin{cases} \dfrac{|\nabla u(z)|^2}{J_f(z)} & \text{if } J_f(z) > 0, \\ 1 & \text{otherwise.} \end{cases} \tag{4.2}$$

Similarly, if $f = (u, v)$ satisfies the condition

$$\{z\colon J_f(z) = 0\} = \{z\colon |\nabla v(z)| = 0\} \quad \text{a.e.},$$

then the Borel function

$$K_f^{(2)}(z) := \begin{cases} \dfrac{|\nabla v(z)|^2}{J_f(z)} & \text{if } J_f(z) > 0, \\ 1 & \text{otherwise,} \end{cases} \tag{4.3}$$

is well defined. On the other hand, if $f = (u, v)$ satisfies the condition

$$\{z\colon J_f(z) = 0\} = \left\{z\colon \left|\frac{\partial f}{\partial x}(z)\right| = 0\right\} \quad \text{a.e.},$$

then we can define the Borel function

$$H_f^{(1)}(z) := \begin{cases} \dfrac{\left|\frac{\partial f}{\partial x}(z)\right|^2}{J_f(z)} & \text{if } J_f(z) > 0, \\ 1 & \text{otherwise.} \end{cases} \tag{4.4}$$

Finally, for f satisfying

$$\{z\colon J_f(z) = 0\} = \left\{z\colon \left|\frac{\partial f}{\partial y}(z)\right| = 0\right\} \quad \text{a.e.}$$

we define

$$(4.5)\qquad H_f^{(2)}(z) := \begin{cases} \dfrac{\left|\frac{\partial f}{\partial y}(z)\right|^2}{J_f(z)} & \text{if } J_f(z) > 0, \\ 1 & \text{otherwise.} \end{cases}$$

In the following, given a $\mathscr{W}^{1,1}$-homeomorphism f, we establish conditions which guarantee that one of its distortions is finite a.e. or $\mathscr{L}^1$. Let us begin with the following

Theorem 4.1. *Let $f = (u, v) \colon \Omega \subset \mathbf{R}^2 \xrightarrow{\text{onto}} \Omega' \subset \mathbf{R}^2$ be a $\mathscr{W}^{1,1}$- homeomorphism whose inverse is $f^{-1} = (x, y)$. If $x \in \mathscr{W}^{1,1}_{\text{loc}}(\Omega')$ and $v_y \neq 0$ on a positive measure set $P \subset \Omega$, then*

$$(4.6)\qquad \{z \in P \colon J_f(z) = 0\} = \left\{z \in P \colon \left|\frac{\partial f}{\partial y}(z)\right| = 0\right\} \quad \text{a.e.}$$

and the distortion $H_f^{(2)}(z)$ is finite a.e. Moreover, we have the following identities

$$(4.7)\qquad \int_{\Omega'} |\nabla x(w)|\, dw = \int_{\Omega} \left|\frac{\partial f}{\partial y}(z)\right| dz$$

$$(4.8)\qquad |\nabla y(w)|\,(\Omega') = \int_{\Omega} \left|\frac{\partial f}{\partial x}(z)\right| dz.$$

Proof. By contradiction we suppose that there exists a set $A \subset P$ with positive Lebesgue measure such that f is differentiable in A and

$$J_f(z) = 0 \quad \text{and} \quad \left|\frac{\partial f}{\partial y}(z)\right| > 0 \quad \forall z \in A.$$

We can assume that f is Lipschitz on A and use the area formula (2.2) to get

$$|f(A)| = 0 \quad \text{since} \quad \int_A J_f(z)\, dz = 0.$$

We denote by

$$p_2 \colon (x_1, x_2) \in \mathbf{R}^2 \to \mathbf{H}_2 = \left\{x \in \mathbf{R}^2 \colon x_2 = 0\right\}$$

the orthogonal projection and by

$$p^{(2)} \colon (x_1, x_2) \in \mathbf{R}^2 \to x_2 \in \mathbf{R}$$

the second coordinate function.

We observe that

$$\{\omega \in \Omega' \colon x(\omega) = t\} = \left(p_2 \circ f^{-1}\right)^{-1} \{(t, 0)\} \quad \forall t \in \mathbf{R}.$$

By assumptions we know that

$$\mathscr{H}^1 \left(\{w \in f(A) \colon x(w) = t\}\right) < \infty$$

and from [1] (p. 209)

$$\mathscr{H}^1 \left(\{w \in f(A) \colon x(\omega) = t\}\right) = P\left(\{w \in f(A) \colon x(w) > t\}, \Omega'\right).$$

By Lemma 3.1 and the assumption that x belongs to $\mathscr{W}^{1,1}_{\text{loc}}(\Omega')$, we have

$$\int_{\mathbf{R}} \mathscr{H}^1 \left(\{w \in f(A) \colon x(w) = t\}\right) = \int_{f(A)} |\nabla x(\omega)|\, dw = 0.$$

Thus the curve $\{w\in f(A)\colon x(w)=t\}$ has zero one dimensional measure for a.e. $t\in\mathbf{R}$ and in particular its second projection to the axis have zero one-dimensional measure as well:

$$\mathscr{H}^1\big(p^{(2)}\left(\{w\in f(A)\colon x(\omega)=t\}\right)\big)=0\quad \text{a.e. } t\in\mathbf{R}. \tag{4.9}$$

On the other hand, using Fubini Theorem, we have

$$|A|=\int_{\mathbf{R}}\left|A\cap p_2^{-1}\{(t,0)\}\right|dt>0.$$

Hence, there exists $t_0\in\mathbf{R}$ such that

$$\mathscr{H}^1\left(A\cap p_2^{-1}\{(t_0,0)\}\right)>0.$$

Applying the area formula to the differentiable function $v(t_0,\cdot)\colon \tau\in p^{(2)}(A)\to v(t_0,\tau)$, we have

$$\begin{aligned}0&<\int_{A\cap p_2^{-1}(t_0)}|v_y(t_0,\tau)|\,d\mathscr{H}^1(\tau)\le\int_{\mathbf{R}}N(v,A\cap p_2^{-1}(t_0),\sigma)\,d\sigma\\&=\int_{p^{(2)}(f(A)\cap(p_2\circ f^{-1})^{-1}(t_0))}N(v,A\cap p_2^{-1}(t_0),\sigma)\,d\sigma,\end{aligned}\tag{4.10}$$

where $N(v,A\cap p_2^{-1}(t_0),\sigma)$ is the number of preimages of σ under v in $A\cap p_2^{-1}(t_0)$. The last integral is zero by (4.9) and this is a contradiction. ☐

The following result shows that if the distortion $K_f^{(2)}$ is $\mathscr{L}^1$, then f^{-1} has better Sobolev regularity.

Theorem 4.2. *Let $f=(u,v)\colon\Omega\subset\mathbf{R}^2\xrightarrow{\text{onto}}\Omega'\subset\mathbf{R}^2$ be a $\mathscr{W}^{1,1}$-homeomorphism and denote by $f^{-1}=(x,y)$ its inverse. If we assume*

$$\{w\in\Omega'\colon J_{f^{-1}}(w)=0\}=\left\{w\in\Omega'\colon\left|\frac{\partial f^{-1}}{\partial u}(w)\right|=0\right\},\tag{4.11}$$

$$\{z\in\Omega\colon J_f(z)=0\}=\{z\in\Omega\colon|\nabla v(z)|=0\}\tag{4.12}$$

and $K_f^{(2)}\in\mathscr{L}^1$, then

$$\left|\frac{\partial f^{-1}}{\partial u}\right|\in\mathscr{L}^2(\Omega)\tag{4.13}$$

and

$$\int_{\Omega'}\left|\frac{\partial f^{-1}}{\partial u}(w)\right|^2dw\le\int_\Omega K_f^{(2)}(z)\,dz.\tag{4.14}$$

Proof. Let A' be the Borel subset of the set E' where f^{-1} is differentiable with $J_{f^{-1}}>0$, such that $|A'|=|E'|$. Applying the area formula, we obtain

$$\begin{aligned}\int_{\Omega'}\left|\frac{\partial f^{-1}}{\partial u}(w)\right|^2dw&=\int_{A'}\left|\frac{\partial f^{-1}}{\partial u}(w)\right|^2dw=\int_{A'}\frac{\left|\frac{\partial f^{-1}}{\partial u}(w)\right|^2}{J_{f^{-1}}(w)}J_{f^{-1}}(w)\,dw\\&\le\int_{f^{-1}(A')}\frac{\left|\frac{\partial f^{-1}}{\partial u}(f(z))\right|^2}{J_{f^{-1}}(f(z))}dz=\int_{f^{-1}(A')}\frac{\frac{1}{J_f(z)^2}|\nabla v(z)|^2}{J_{f^{-1}}(f(z))}dz\end{aligned}$$

$$= \int_{f^{-1}(A')} \frac{|\nabla v(z)|^2}{J_f(z)}\, dz \le \int_{\Omega'} K_f^{(2)}(z)\, dz. \qquad \square$$

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