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INTEGRAL MEANS AND COEFFICIENT ESTIMATES ON PLANAR HARMONIC MAPPINGS

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Abstract. In this paper, we investigate some properties of planar harmonic mappings in Hardy spaces. First, we discuss the integral means of harmonic mappings and those of their derivatives, and as a consequence, we solve the open problem of Girela and Peláez in the setting of harmonic mappings. In addition, we establish coefficient estimates and a distortion theorem for harmonic mappings in Hardy spaces.

1. Introduction and preliminaries

For each $r \in (0,1]$, we denote by $\mathbf{D}_r$ the open disk $\{z \in \mathbf{C} \colon |z| < r\}$ and by $\mathbf{D}$, the open unit disk $\mathbf{D}_1$. A complex-valued function f defined on $\mathbf{D}$ is called a *harmonic mapping* in $\mathbf{D}$ if and only if it is twice continuously differentiable and $\Delta f = 0$, i.e. the real and imaginary parts are real harmonic in $\mathbf{D}$, where Δ represents the usual complex Laplacian operator

$$\Delta = 4\frac{\partial^2}{\partial z \partial \overline{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

An obvious fact is that every harmonic mapping f defined in $\mathbf{D}$ admits the canonical decomposition $f = h + \overline{g}$, where h and g are analytic in $\mathbf{D}$ with $g(0) = 0$. We refer to [5, 9] for the theory of harmonic mappings.

A classical result of Hardy and Littlewood asserts that if $p \in (0,\infty]$, $\alpha \in (1,\infty)$ and f is an analytic function in $\mathbf{D}$, then

$$M_p(r, f') = O\left(\left(\frac{1}{1-r}\right)^{\alpha}\right) \quad \text{as } r \to 1$$

if and only if

$$M_p(r, f) = O\left(\left(\frac{1}{1-r}\right)^{\alpha-1}\right) \quad \text{as } r \to 1,$$

where

$$M_p(r,f) = \begin{cases} \left(\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right)^{1/p} & \text{if } p \in (0,\infty), \\ \displaystyle\max_{|z|=r} |f(z)| & \text{if } p = \infty. \end{cases}$$

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We refer to [4, 8, 9, 11, 12, 13, 14, 15] for many other related discussions concerning the case of analytic functions. Indeed the above result of Hardy and Littlewood provides a close relationship between the integral means of analytic functions and those of their derivatives [8, 12, 13]. In [11, Theorem 1(a)], Girela and Peláez refined the above result for the case $\alpha = 1$ as follows.

Theorem A. *If $p \in (2, \infty)$ and f is an analytic function in $\mathbf{D}$ such that*

$$M_p(r, f') = O\left(\left(\frac{1}{1-r}\right)\right) \quad \text{as } r \to 1,$$

then for all $\beta > \frac{1}{2}$,

$$M_p(r, f) = O\left(\left(\log \frac{1}{1-r}\right)^{\beta}\right) \quad \text{as } r \to 1. \tag{1}$$

In [11, p. 464, Equation (26)], Girela and Peláez asked whether or not β in (1) can be substituted by 1/2. This problem was affirmatively settled by Girela, Pavlovic and Peláez in [10]. In this paper, we first prove that the answer to this problem is affirmative in the setting of harmonic mappings in the unit disk. Then coefficient estimates and a distortion theorem for harmonic mappings in Hardy spaces are also obtained. In order to state our results, we need to introduce some notations. For $p \in (0, \infty]$, the *harmonic Hardy space* $\mathcal{H}_h^p$ consists of those functions f, harmonic in $\mathbf{D}$, such that $\|f\|_p < \infty$, where

$$\|f\|_p = \begin{cases} \sup\limits_{0<r<1} M_p(r, f) & \text{if } p \in (0, \infty), \\ \sup\limits_{z \in \mathbf{D}} |f(z)| & \text{if } p = \infty. \end{cases}$$

In addition, we let

$$\nabla f = (f_z, f_{\overline{z}}) \quad \text{and} \quad |\nabla f| = (|f_z|^2 + |f_{\overline{z}}|^2)^{1/2}.$$

We now state our first result which generalizes Theorem A. In the case of analytic function f, ∇f equals $f'(z)$ and therefore, the following theorem contains another solution to the open problem of Girela and Peláez [11, p. 464, Equation (26)] provided later by Girela, Pavlović and Peláez in [10].

Theorem 1. *If $p \in (2, \infty)$ and f is a harmonic mapping in $\mathbf{D}$ such that*

$$M_p(r, \nabla f) = O\left(\left(\frac{1}{1-r}\right)\right) \quad \text{as } r \to 1, \tag{2}$$

then we have the following:

(a) $M_p(r, f) = O\left(\left(\log \frac{1}{1-r}\right)^{1/2}\right)$ *as* $r \to 1$,

(b) $M_\infty(r, f) = O\left(\left(\frac{1}{1-r}\right)^{1/p}\right)$ *as* $r \to 1$.

The following result is a generalization of [11, Theorem 2] for the harmonic case.

Theorem 2. *If $p \in (0,2]$ and f is a harmonic mapping in* **D** *satisfying the condition* (2), *then*

$$M_p(r,f) = O\left(\left(\log \frac{1}{1-r}\right)^{1/p}\right) \quad \text{as } r \to 1.$$

Moreover, the result is sharp and one of the extreme functions is $f(z) = 1/(1-z)^{1/p}$, where $p \in (0,2]$.

We now state our next two theorems which provide coefficient estimates and a distortion theorem for harmonic Hardy mappings.

Theorem 3. *Let f be a harmonic mapping in* **D** *such that*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b}_n \overline{z}^n,$$

and $f \in \mathcal{H}_h^p$ for some $p \in [1,\infty]$. Then we have the following:

(a) $|a_0| \le \|f\|_p$;

(b) *for $p \in [1,\infty)$,*

$$|a_n| + |b_n| \le \frac{2^{(1/p)+2}(1+np)^{n+(1/p)}}{\pi (pn)^n}\|f\|_p \quad \text{for each } n \ge 1;$$

(c) *for $p = \infty$,*

$$|a_n| + |b_n| \le \frac{4}{\pi}\|f\|_\infty \quad \text{for each } n \ge 1.$$

The estimate in this case is sharp and the only extremal functions are

$$f_n(z) = \frac{2\alpha}{\pi}\|f\|_\infty \arg\left(\frac{1+\beta z^n}{1-\beta z^n}\right),$$

where $|\alpha| = |\beta| = 1$.

Theorem 4. *Let f be a harmonic mapping in* **D** *with $f \in \mathcal{H}_h^p$ for some $p \in [1,\infty]$. Then for $p \in [1,\infty)$*

$$|f_z(z)| + |f_{\overline{z}}(z)| \le \frac{2^{(1/p)+2}(1+p)^{1+(1/p)}}{\pi p(1-|z|^2)}\|f\|_p,$$

and for $p = \infty$,

$$|f_z(z)| + |f_{\overline{z}}(z)| \le \frac{4}{\pi(1-|z|^2)}\|f\|_\infty. \tag{3}$$

When $p = \infty$, the estimate (3) *is sharp and the only extremal functions are*

$$f(z) = \frac{2\alpha}{\pi}\|f\|_\infty \arg\left(\frac{1+\phi(z)}{1-\phi(z)}\right),$$

where $|\alpha| = 1$ and ϕ is a conformal automorphism of **D**.

We remark that Theorem 4 is a generalization of [6, Theorems 3 and 4].

The proofs of Theorems 1–4 are presented in the following section.

2. Proofs

It seems that the standard technologies from the theory of analytic functions are not useful to prove Theorem 1, and therefore, we use some ideas from Green's theorem in their proofs. Green's theorem (cf. [3, 16]) states that if $g \in C^2(\mathbf{D})$, i.e., twice continuously differentiable on $\mathbf{D}$, then

$$\frac{1}{2\pi}\int_0^{2\pi} g(re^{i\theta})\,d\theta = g(0) + \frac{1}{2}\int_{\mathbf{D}_r} \Delta g(z)\log\frac{r}{|z|}\,dA(z) \tag{4}$$

for $r \in (0,1)$, where $dA(z)$ denotes the normalized area measure in $\mathbf{D}$.

Moreover, the proof of Theorem 1(b) relies on the following lemma.

Lemma B. [8, Lemma 3, p. 84] *If $a \in (1,\infty)$ and $\rho = (1+r)/2$, then*

$$\int_0^{2\pi} |\rho e^{it} - r|^{-a}\,dt = O\big((1-r)^{1-a}\big) \quad \text{as } r \to 1.$$

Proof of Theorem 1. (a) Let

$$A(r,f) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^{p-2}|\nabla f(re^{i\theta})|^2\,d\theta.$$

Then Hölder's inequality yields

$$A(r,f) \le M_p^2(r,\nabla f)\cdot M_p^{p-2}(r,f). \tag{5}$$

By the identity (4), the inequality (5), the subharmonicity of $|f|$ and the fact "$M_p(t,f)$ being an increasing function of t", we have

$$\begin{aligned}
M_p^p(r,f) &= |f(0)|^p + \frac{1}{2}\int_{\mathbf{D}_r} \Delta(|f(z)|^p)\log\frac{r}{|z|}\,dA(z)\\
&= |f(0)|^p + p\int_{\mathbf{D}_r}\Big[(p/2-1)|f(z)|^{p-4}|f_z(z)\overline{f(z)} + f(z)\overline{f_{\overline{z}}(z)}|^2\\
&\quad + |f(z)|^{p-2}|\nabla f(z)|^2\Big]\log\frac{r}{|z|}\,dA(z)\\
&\le |f(0)|^p + p(p-1)\int_{\mathbf{D}_r} |f(z)|^{p-2}|\nabla f(z)|^2\log\frac{r}{|z|}\,dA(z)\\
&= |f(0)|^p + \frac{p(p-1)}{\pi}\int_0^{2\pi}\int_0^r |f(te^{i\theta})|^{p-2}|\nabla f(te^{i\theta})|^2 t\log\frac{r}{t}\,dt\,d\theta\\
&\le |f(0)|^p + 2p(p-1)\int_0^r M_p^2(t,\nabla f)M_p^{p-2}(t,f)t\log\frac{r}{t}\,dt
\end{aligned}$$

which in particular implies that

$$\begin{aligned}
M_p^2(r,f) &\le |f(0)|^2 + 2p(p-1)\int_0^r M_p^2(t,\nabla f)t\log\frac{r}{t}\,dt\\
&\le |f(0)|^2 + 2p(p-1)\int_0^r M_p^2(t,\nabla f)(r-t)\,dt\\
&= |f(0)|^2 + 2p(p-1)r^2\int_0^1 M_p^2(rt,\nabla f)(1-t)\,dt \le |f(0)|^2 + C\log\frac{1}{1-r}
\end{aligned}$$

and hence,

$$M_p^2(r,f) = O\left(\left(\log\frac{1}{1-r}\right)\right) \quad \text{as } r\to 1,$$

or equivalently

$$M_p(r,f) = O\left(\left(\log\frac{1}{1-r}\right)^{1/2}\right) \quad \text{as } r\to 1,$$

where C is a positive constant. The proof of Part (a) in Theorem 1 is complete.

(b) Let $z\in \mathbf{D}_\rho$ be arbitrary with $\rho\in(0,1)$. Then, by the Poisson integral formula and Hölder's inequality, there is a positive constant C such that

$$\begin{aligned}|f(z)| &\le \frac{1}{2\pi}\int_0^{2\pi}\frac{\rho^2-|z|^2}{|z-\rho e^{i\theta}|^2}|f(\rho e^{i\theta})|\,d\theta \le \frac{1}{2\pi}\int_0^{2\pi}\frac{\rho+|z|}{|z-\rho e^{i\theta}|}|f(\rho e^{i\theta})|\,d\theta\\ &\le CM_p(\tau,f)\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{|z-\rho e^{i\theta}|^{\frac{p}{p-1}}}\right)^{\frac{p-1}{p}}.\end{aligned}$$

Now, we set $\rho=(1+|z|)/2$ and apply Lemma B. Obviously, there exists a positive constant C such that

$$|f(z)| \le CM_p\left(\frac{1+|z|}{2},f\right)\left[(1-|z|)^{1-\frac{p}{p-1}}\right]^{\frac{p-1}{p}} = \frac{CM_p(\frac{1+|z|}{2},f)}{(1-|z|)^{1/p}}. \tag{6}$$

By computations, we have

$$|f(\rho e^{i\theta})| \le |f(0)| + \left|\int_0^\rho \frac{df(te^{i\theta})}{dt}\,dt\right| \le |f(0)| + \sqrt{2}\int_0^\rho |\nabla f(te^{i\theta})|\,dt. \tag{7}$$

By the well-known Minkowski inequality and (7), we have

$$\begin{aligned}M_p(\rho,f) &= \left(\frac{1}{2\pi}\int_0^{2\pi}|f(\rho e^{i\theta})|^p\,d\theta\right)^{1/p}\\ &\le \left\{\frac{1}{2\pi}\int_0^{2\pi}\left[|f(0)|+\sqrt{2}\int_0^\rho|\nabla f(te^{i\theta})|\,dt\right]^p d\theta\right\}^{1/p}\\ &\le |f(0)| + C\int_0^\rho\left(\frac{1}{2\pi}\int_0^{2\pi}|\nabla f(te^{i\theta})|^p\,d\theta\right)^{1/p}dt = |f(0)|+C\int_0^\rho M_p(t,\nabla f)\,dt,\end{aligned}$$

where C is a positive constant. In particular,

$$M_{2p}(\rho,f)\le |f(0)| + C\int_0^\rho M_{2p}(t,\nabla f)\,dt.$$

By (6) (with $2p$ in place of p) and letting $\rho=(1+r)/2$, we get

$$M_\infty(r,f) \le \frac{C}{(1-r)^{1/(2p)}}\left[|f(0)| + C_1\int_0^{(1+r)/2} M_{2p}(t,\nabla f)\,dt\right], \tag{8}$$

where C and C_1 are positive constants.

On the other hand,

$$
\begin{aligned}
M_{2p}(t,\nabla f) &= \left(\frac{1}{2\pi}\int_0^{2\pi}|\nabla f(te^{i\theta})|^{2p}\,d\theta\right)^{1/(2p)} \\
&= \left(\frac{1}{2\pi}\int_0^{2\pi}|\nabla f(te^{i\theta})|^{p}\cdot|\nabla f(te^{i\theta})|^{p}\,d\theta\right)^{1/(2p)} \leq M_\infty^{1/2}(t,\nabla f)\,M_p^{1/2}(t,\nabla f).
\end{aligned} \tag{9}
$$

For $\alpha\in[0,2\pi]$ and $z\in\mathbf{D}$, let $F_\alpha(z)=f_z(z)+e^{i\alpha}f_{\overline{z}}(z)$. Then for each $z\in\mathbf{D}_r$,

$$
|F_\alpha(z)|\leq\frac{1}{2\pi}\int_0^{2\pi}\frac{r^2-|z|^2}{|z-re^{i\theta}|^2}|F_\alpha(re^{i\theta})|\,d\theta\leq\frac{\sqrt{2}}{2\pi}\int_0^{2\pi}\frac{r^2-|z|^2}{|z-re^{i\theta}|^2}|\nabla f(re^{i\theta})|\,d\theta,
$$

which gives

$$
|\nabla f(z)|\leq\max_{\alpha\in[0,2\pi]}|F_\alpha(z)|\leq\frac{\sqrt{2}}{2\pi}\int_0^{2\pi}\frac{r^2-|z|^2}{|z-re^{i\theta}|^2}|\nabla f(re^{i\theta})|\,d\theta.
$$

A procedure similar to the proof of (6) shows that

$$
|\nabla f(z)|\leq\frac{CM_p(\frac{1+|z|}{2},\nabla f)}{(1-|z|)^{1/p}} \tag{10}
$$

for some constant $C>0$, and (9) implies

$$
M_{2p}(s,\nabla f)\leq\frac{CM_p(\frac{1+s}{2},\nabla f)}{(1-s)^{\frac{1}{2p}}}\leq\frac{C}{(1-s)^{1+\frac{1}{2p}}}, \tag{11}
$$

where $s\in[0,\frac{1+r}{2}]$. Therefore, by (8), (10) and (11), we get

$$
M_\infty(r,f)\leq O\left(\left(\frac{1}{(1-r)^{\frac{1}{2p}}}\int_0^{\frac{1+r}{2}}(1-s)^{-1-\frac{1}{2p}}\,ds\right)\right)=O\left(\left(\frac{1}{1-r}\right)^{1/p}\right),
$$

and the proof of Theorem 1(a) follows. □

We now recall the following well-known result which we need to prove Lemma 1 (and hence Theorem 2).

Lemma C. [17, Lemma 2.29] *Let $a,b\in[0,\infty)$ and $p\in[1,\infty)$. Then we have*

$$
a^p+b^p\leq(a+b)^p\leq 2^{p-1}(a^p+b^p).
$$

Lemma 1. *Let f be harmonic in $\mathbf{D}$ and $p\in(0,2]$. If*

$$
\int_0^1(1-r)^{p-1}M_p^p(r,\nabla f)\,dr<\infty,
$$

then

$$
\|f\|_p^p\leq C(p)\left(|f(0)|^p+\int_0^1(1-r)^{p-1}M_p^p(r,\nabla f)\,dr\right), \tag{12}
$$

where $C(p)$ is a positive constant which depends only on p.

Proof. If $p \in [1,2]$, then the proof of the inequality (12) follows from Lemma C, [11, Theorem B] and the fact that f has the canonical decomposition $f = h + \overline{g}$, where h and g are analytic in $\mathbf{D}$. Therefore, it suffices to prove the theorem for the case $p \in (0,1)$. Let $0 \le r_1 < r_2 < 1$. Then we see that

$$|f(r_2 e^{i\theta}) - f(r_1 e^{i\theta})| = \left| \int_{r_1}^{r_2} \frac{d}{dt} f(te^{i\theta})\, dt \right| \le \sqrt{2} \int_{r_1}^{r_2} |\nabla f(te^{i\theta})|\, dt,$$

and therefore,

$$|f(r_2 e^{i\theta}) - f(r_1 e^{i\theta})|^p \le 2^{p/2}(r_2 - r_1)^p \max_{r_1 \le t \le r_2} |\nabla f(te^{i\theta})|^p,$$

which gives

$$\begin{aligned} M_p^p(r_2, f) - M_p^p(r_1, f) &\le \frac{1}{2\pi} \int_0^{2\pi} |f(r_2 e^{i\theta}) - f(r_1 e^{i\theta})|^p\, d\theta \\ &\le 2^{p/2}(r_2 - r_1)^p \max_{r_1 \le t \le r_2} |\nabla f(te^{i\theta})|^p. \end{aligned} \tag{13}$$

For $\alpha \in [0, 2\pi]$ and $z \in \mathbf{D}$, let $K_\alpha(z) = f_z(z) + e^{i\alpha}\overline{f_{\overline{z}}(z)}$. By using Hardy–Littlewood maximal theorem to K_α, there is a positive constant C such that

$$\max_{r_1 \le t \le r_2} |K_\alpha(te^{i\theta})|^p \le \frac{C}{2\pi} \int_0^{2\pi} |K_\alpha(r_2 e^{i\theta})|^p\, d\theta \le \frac{2^{p/2} C}{2\pi} \int_0^{2\pi} |\nabla f(r_2 e^{i\theta})|^p\, d\theta.$$

The arbitrariness of α in $[0, 2\pi]$ implies that

$$\max_{r_1 \le t \le r_2} |\nabla f(te^{i\theta})|^p \le \frac{2^{p/2} C}{2\pi} \int_0^{2\pi} |\nabla f(r_2 e^{i\theta})|^p\, d\theta.$$

Without loss of generality, we assume $f(0) = 0$ so that $M_p(0, f) = 0$. For any $n \in \{0, 1, 2, \ldots\}$, let $r_n = 1 - 2^{-n}$. Applying (13), we deduce that

$$\begin{aligned} M_p^p(r_{n+1}, f) &= \sum_{k=1}^{n+1} \left[M_p^p(r_k, f) - M_p^p(r_{k-1}, f) \right] \le C \sum_{k=1}^{n+1} (r_k - r_{k-1})^p M_p^p(r_k, \nabla f) \\ &\le C \sum_{k=1}^{n} 2^{-k(p-1)} \int_{r_k}^{r_{k+1}} M_p^p(r, \nabla f)\, dr \le C \int_0^{r_{n+1}} (1-r)^{p-1} M_p^p(r, \nabla f)\, dr, \end{aligned}$$

where the positive constant C is not necessarily the same at each occurrence in the above three inequalities. Then the inequality (12) follows if we let $n \to \infty$. □

Proof of Theorem 2. Without loss of generality, we assume that $f(0) = 0$. For $r \in (0,1]$, let $f_r(z) = f(rz)$, $z \in \mathbf{D}$. By using (2) and Lemma 1, there is a positive constant C such that

$$\begin{aligned} M_p^p(r, f) &\le C \int_0^1 (1-t)^{p-1} M_p^p(rt, \nabla f)\, dt \\ &\le C \left(\int_0^r \frac{(1-t)^{p-1}}{(1-rt)^p}\, dt + \int_r^1 \frac{(1-t)^{p-1}}{(1-rt)^p}\, dt \right) \\ &\le C \left(\int_0^r \frac{1}{1-t}\, dt + \frac{1}{(1-r)^p} \int_r^1 (1-t)^{p-1}\, dt \right) = O\left(\left(\log \frac{1}{1-r} \right) \right). \end{aligned}$$

The proof of this theorem is complete. □

We can now prove our final two results concerning the coefficient estimates and a distortion theorem for harmonic Hardy mappings. Let's recall the following result which is referred to as Jensen's inequality (cf. [19]).

Lemma D. *Let (Ω, A, μ) be a measure space such that $\mu(\Omega) = 1$. If g is a real-valued function that is μ-integrable, and if φ is a convex function on the real line, then*

$$\varphi\left(\int_\Omega g\, d\mu\right) \le \int_\Omega \varphi \circ g\, d\mu.$$

Proof of Theorem 3. For $0 \le s < 1$, by the Poisson integral formula, we have

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{s^2 - |z|^2}{|z - se^{i\theta}|^2} f(se^{i\theta})\, d\theta$$

for $z \in \mathbf{D}_s$. Using Jensen's inequality (see Lemma D), we have

$$|f(z)|^p \le \frac{1}{2\pi}\int_0^{2\pi} \frac{s^2 - |z|^2}{|z - se^{i\theta}|^2} |f(se^{i\theta})|^p\, d\theta \le \frac{2s}{s - |z|} M_p^p(s, f).$$

This in particular gives

$$|f(z)| < \frac{2^{1/p}}{(1 - |z|)^{1/p}} \|f\|_p \quad \text{for } z \in \mathbf{D}.$$

For $\zeta \in \partial\mathbf{D}$ and a fixed $r \in (0, 1)$, let $F(\zeta) = f(r\zeta)/r = H(\zeta) + \overline{G(\zeta)}$ so that

$$H(\zeta) = \frac{a_0}{r} + \sum_{n=1}^{\infty} A_n \zeta^n \quad \text{and} \quad (\zeta) = \sum_{n=1}^{\infty} \overline{B}_n \overline{\zeta}^n,$$

where $A_n = a_n r^{n-1}$ and $B_n = b_n r^{n-1}$. It is not difficult to see that

$$|a_0| = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(0)|^p\, d\theta\right)^{1/p} \le \|f\|_p$$

and

$$|F(\zeta)| < \frac{2^{1/p}}{r(1 - r)^{1/p}} \|f\|_p = M(r) \quad \text{for } \zeta \in \mathbf{D}.$$

In particular, the proof of Theorem 3(a) follows.

For the proof of the remaining two cases, we let

$$T(\zeta) = \frac{F(\zeta)}{M(r)} = H_1(\zeta) + \overline{G_1(\zeta)},$$

where $H(\zeta) = H_1(\zeta)M(r)$ and $G(\zeta) = G_1(\zeta)M(r)$. Then for any $\zeta \in \mathbf{D}$, $|T(\zeta)| < 1$. As in [2] (see also [1]), it follows easily that

$$|A_n| + |B_n| \le \frac{4M(r)}{\pi} \quad \text{for } n = 1, 2, \ldots. \tag{14}$$

For the sake of completeness, we include the details from [2]. For $\theta \in [0, 2\pi)$, let

$$v_\theta(\zeta) = \operatorname{Im}(e^{i\theta} T(\zeta))$$

and observe that

$$v_\theta(\zeta) = \operatorname{Im}(e^{i\theta} H_1(\zeta) + \overline{e^{-i\theta} G_1(\zeta)}) = \operatorname{Im}(e^{i\theta} H_1(\zeta) - e^{-i\theta} G_1(\zeta)).$$

Because $|v_\theta(\zeta)| < 1$, it follows that

$$e^{i\theta}H_1(\zeta) - e^{-i\theta}G_1(\zeta) \prec K(\zeta) = \lambda + \frac{2}{\pi}\log\left(\frac{1+\zeta\xi}{1-\zeta}\right), \tag{15}$$

where $\xi = e^{-i\pi \mathrm{Im}(\lambda)}$, $\lambda = e^{i\theta}H_1(0) - e^{-i\theta}G_1(0)$ and "$\prec$" denotes the subordination [9, p. 27]. The superordinate function $K(\zeta)$ maps $\mathbf{D}$ onto a convex domain with $K(0) = \lambda$ and $K'(0) = \frac{2}{\pi}(1+\xi)$, and therefore, by a theorem of Rogosinski [18, Theorem 2.3] it follows that

$$|A_n - e^{-2i\theta}B_n| \leq \frac{2M(r)}{\pi}|1+\xi| \leq \frac{4M(r)}{\pi} \quad \text{for } n = 1, 2, \ldots.$$

By the arbitrariness of θ in $[0, 2\pi)$, we have (14), whence we deduce that

$$\begin{aligned} |a_n| + |b_n| &\leq \frac{2^{2+(1/p)}\|f\|_p}{\pi} \inf_{0<r<1}\left[\frac{1}{r^n(1-r)^{1/p}}\right] \\ &= \frac{2^{2+(1/p)}\|f\|_p}{\pi} \frac{1}{\max\limits_{0<r<1}\left[r^n(1-r)^{1/p}\right]} = \frac{2^{2+(1/p)}\|f\|_p}{\pi}\left[\frac{(1+pn)^{n+(1/p)}}{(pn)^n}\right]. \end{aligned}$$

Thus, the proof of Theorem 3(b) follows.

Finally, by a simple calculation, it can be easily seen that

$$\lim_{p\to\infty} \frac{2^{1/p}(1+pn)^{n+(1/p)}}{(pn)^n} = 1$$

and thus, the proof of (c) follows from (b) as a limiting case. Indeed, by letting $p \to \infty$, we conclude that

$$|a_n| + |b_n| \leq \frac{4\|f\|_\infty}{\pi} \quad \text{for } n = 1, 2, \ldots \tag{16}$$

for $f \in \mathcal{H}_h^\infty$. The estimates of (16) are sharp. By subordination, the equality sign occurs in (16) if and only if

$$f_n(z) = \frac{2\alpha\|f\|_\infty}{\pi}\,\mathrm{Im}\left(\log\frac{1+\beta z^n}{1-\beta z^n}\right), \quad |\alpha| = |\beta| = 1,$$

and the images of $\mathbf{D}$ under f_n are confined to a diametral segment of the disk $\mathbf{D}_{\|f\|_\infty} = \{z\colon |z| < \|f\|_\infty\}$. The proof of this theorem is complete. □

Proof of Theorem 4. For a fixed $z \in \mathbf{D}$, let $F(w) = f(\phi(w))$, where $\phi(w) = \frac{z+w}{1+\bar{z}w}$ is a conformal automorphism of $\mathbf{D}$. By calculations, we have

$$F_w(w) = f_\zeta(\phi(w))\phi'(w) \quad \text{and} \quad F_{\overline{w}}(w) = f_{\overline{\zeta}}(\phi(w))\overline{\phi'(w)},$$

where $\zeta = \phi(w)$. This gives

$$|F_w(w)| + |F_{\overline{w}}(w)| = (|f_\zeta(\phi(w))| + |f_{\overline{\zeta}}(\phi(w))|)|\phi'(w)|.$$

By Theorem 3, we have

$$|F_w(0)| + |F_{\overline{w}}(0)| = (|f_\zeta(z)| + |f_{\overline{\zeta}}(z)|)(1-|z|^2) \leq \frac{2^{(1/p)+2}(1+p)^{1+(1/p)}\|f\|_p}{\pi p}.$$

Then

$$|f_\zeta(z)| + |f_{\overline{\zeta}}(z)| \leq \frac{2^{(1/p)+2}(1+p)^{1+(1/p)}\|f\|_p}{\pi p(1-|z|^2)}.$$

If we allow $p \to \infty$, then the last inequality implies that

$$|f_\zeta(z)| + |f_{\overline{\zeta}}(z)| \leq \frac{4}{\pi(1-|z|^2)}\|f\|_\infty \tag{17}$$

The estimates of (17) are sharp. The only extremal functions are

$$f(z) = \frac{2\alpha\|f\|_\infty}{\pi} \arg\left(\frac{1+\phi(z)}{1-\phi(z)}\right),$$

where $|\alpha| = 1$ and ϕ is a conformal automorphism of $\mathbf{D}$. □

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