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THE WEAKLY COMPACT APPROXIMATION OF THE PROJECTIVE TENSOR PRODUCT OF BANACH SPACES

Qingying Bu

University of Mississippi, Department of Mathematics
University, MS 38677, U.S.A.; qbu@olemiss.edu

Abstract. We show that $X\hat{\otimes}Y$, the projective tensor product of Banach spaces X and Y, has the (bounded) compact approximation property if and only if both X and Y have the same property. We also show that $X\hat{\otimes}Y$ has the weakly compact approximation property (W.A.P.) if both X and Y has the W.A.P. and either (i) every bounded linear operator from X (resp. from Y) to Y^* (resp. to X^*) is completely continuous, or (ii) one of X and Y has the Dunford–Pettis property. As a consequence, we show that if K is scattered and Y has the W.A.P., then $C(K)^*\hat{\otimes}Y$ has the W.A.P.

A Banach space X is said to have the *weakly compact approximation property* (W.A.P. for short) if there is a finite constant $C \geq 1$ so that for every weakly compact subset D of X and every $\varepsilon > 0$ there is a weakly compact operator $T\colon X \to X$ satisfying $\|T\| \leq C$ and $\|T(x) - x\| < \varepsilon$ for all $x \in D$. This weakly compact approximation property was introduced by Astala and Tylli in [1] and systematically studied by Odell and Tylli in [7] from the perspective of Banach space theory. Some applications of this property can be found in [1, 10]. All reflexive Banach spaces clearly have the W.A.P. There are few examples of non-reflexive Banach spaces with (or without) the W.A.P. For instance, the classical Banach space ℓ^1 has the W.A.P. while the classical Banach spaces $c_0, C[0,1], L^1[0,1]$ and the Hardy space H^1 fail to have the W.A.P. (see [1, 9]). The quasi-reflexive James' space J and its dual J^* have the W.A.P. while the related James' tree space JT fails to have the W.A.P. (see [7]). Saksman and Tylli [9] showed that the projective tensor product $\ell^p\hat{\otimes}\ell^q$ has the W.A.P. whenever $1 < p, q < \infty$ (note that $\ell^p\hat{\otimes}\ell^q$ is not reflexive whenever $1 < p \leq q'$, here $1/q + 1/q' = 1$). In this paper we show that the projective tensor product $X\hat{\otimes}Y$ of Banach spaces X and Y inherits the W.A.P. from both X and Y if either (i) every bounded linear operator from X (resp. from Y) to Y^* (resp. to X^*) is completely continuous, or (ii) one of X and Y has the Dunford–Pettis property. As a consequence, we give new examples of non-reflexive Banach spaces with the W.A.P.

Diestel, Fourie and Swart in [4] mentioned that the (bounded) approximation property is inherited by the projective tensor product $X\hat{\otimes}Y$ from both X and Y. In this paper we show that the (bounded) compact approximation property is also inherited by $X\hat{\otimes}Y$ from both X and Y.

For a Banach space X, let X^* denote its dual space and B_X denote its closed unit ball. For vector spaces X and Y, let $X \otimes Y$ denote the algebraic tensor product of X and Y, and for $A \subseteq X$ and $B \subseteq Y$, let $A \otimes B$ denote the set $\{x \otimes y\colon x \in A, y \in B\}$.

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Moreover, if X and Y are Banach spaces, let $X\hat{\otimes}Y$ denote the *projective tensor product* of X and Y, that is, the completion of $X \otimes Y$ in the projective tensor norm

$$\|u\|_\wedge = \inf\Big\{\sum_{i=1}^n \|x_i\|\cdot\|y_i\| \colon u = \sum_{i=1}^n x_i \otimes y_i\Big\}, \quad u \in X \otimes Y,$$

where the infimum is taken over all possible representations of u. Grothendieck [6] showed that an element $u \in X\hat{\otimes}Y$ if and only if, for every $\varepsilon > 0$, there are $x_n \in X$ and $y_n \in Y$ with $\sum_{n=1}^\infty \|x_n\|\cdot\|y_n\| < \infty$ such that

$$u = \sum_{n=1}^\infty x_n \otimes y_n \text{ (converges absolutely in } X\hat{\otimes}Y)$$

and

$$\|u\|_\wedge \le \sum_{n=1}^\infty \|x_n\|\cdot\|y_n\| \le \|u\|_\wedge + \varepsilon.$$

From this representation, it is easy to see that

$$B_{X\hat{\otimes}Y} = \overline{co}(B_X \otimes B_Y).$$

For Banach spaces X_1, X_2, Y_1, Y_2, and bounded linear operators $T_1\colon X_1 \to Y_1$ and $T_2\colon X_2 \to Y_2$, one can define a bounded linear operator $T_1 \otimes T_2\colon (X_1 \otimes X_2, \|\cdot\|_\wedge) \to (Y_1 \otimes Y_2, \|\cdot\|_\wedge)$ by

$$(T_1 \otimes T_2)(x_1 \otimes x_2) = T_1(x_1) \otimes T_2(x_2), \quad x_1 \in X_1, x_2 \in X_2.$$

Then $\|T_1 \otimes T_2\| \le \|T_1\|\cdot\|T_2\|$. Thus $T_1 \otimes T_2$ can be extended from $X_1\hat{\otimes}X_2$ to $Y_1\hat{\otimes}Y_2$, still denoted by $T_1 \otimes T_2$.

Recall that for Banach spaces X and Y, a bounded linear operator $T\colon X \to Y$ is called *completely continuous* if T maps weakly convergent sequences in X to norm convergent sequences in Y. A Banach space X is said to have the *Dunford–Pettis property* if, for every Banach space Y, every weakly compact operator from X to Y is completely continuous.

In the projective tensor product $X\hat{\otimes}Y$, if A and B are, respectively, weakly compact subsets of X and Y then $A \otimes B$ may not be a weakly compact subset of $X\hat{\otimes}Y$. Indeed, if both X and Y are reflexive then B_X and B_Y are weakly compact. Since $B_{X\hat{\otimes}Y} = \overline{co}(B_X \otimes B_Y)$, $B_X \otimes B_Y$ is weakly compact in $X\hat{\otimes}Y$ only in the case that $X\hat{\otimes}Y$ is reflexive, which requires that every bounded linear operator from X to Y^* is compact. Diestel and Puglisi in [5, Proposition 2.5] showed that if either X or Y has the Dunford–Pettis property then $A \otimes B$ is a weakly compact subset of $X\hat{\otimes}Y$ whenever A and B are, respectively, weakly compact subsets of X and Y. Using the same idea we have the following.

Proposition 1. *Let X and Y be Banach spaces such that every bounded linear operator $T\colon X \to Y^*$ (respectively, $T\colon Y \to X^*$) is completely continuous. If A and B are, respectively, weakly compact subsets of X and Y then $A \otimes B$ is a weakly compact subset of $X\hat{\otimes}Y$.*

Proof. By the Eberleiň–Smuliàn theorem, it suffices to show that $A \otimes B$ is weakly sequentially compact. Take a sequence $(x_n \otimes y_n)_n$ in $A \otimes B$. Then there are a subsequence $(x_{n_k})_k$ of $(x_n)_n$ and a subsequence $(y_{n_k})_k$ of $(y_n)_n$ such that

$$\text{weak-}\lim_k x_{n_k} = x_0 \in A \quad\text{and}\quad \text{weak-}\lim_k y_{n_k} = y_0 \in B.$$

Next we show that $x_{n_k} \otimes y_{n_k} \to x_0 \otimes y_0$ weakly in $X \hat{\otimes} Y$.

Take any $\phi \in (X \hat{\otimes} Y)^*$. Since $(X \hat{\otimes} Y)^* = L(X, Y^*)$, the space of all bounded linear operators from X to Y^*, let $T \in L(X, Y^*)$ be corresponding to ϕ. That is,

$$\langle x \otimes y, \phi \rangle = \langle T(x), y \rangle, \quad \forall x \in X, \ \forall y \in Y.$$

Thus

$$\begin{aligned} |\langle x_{n_k} \otimes y_{n_k} - x_0 \otimes y_0, \phi \rangle &= |\langle T(x_{n_k}), y_{n_k} \rangle - \langle T(x_0), y_0 \rangle| \\ &\le |\langle T(x_{n_k}) - T(x_0), y_{n_k} \rangle| + |\langle T(x_0), y_{n_k} - y_0 \rangle| \\ &\le M \cdot \|T(x_{n_k}) - T(x_0)\| + |\langle T(x_0), y_{n_k} - y_0 \rangle|, \end{aligned}$$

where $M = \sup\{\|y\| \colon y \in B\} < \infty$. By hypothesis, T is completely continuous and hence, $T(x_{n_k}) \to T(x_0)$ norm in Y^*. Note that $y_{n_k} \to y_0$ weakly in Y. Therefore, $x_{n_k} \otimes y_{n_k} \to x_0 \otimes y_0$ weakly in $X \hat{\otimes} Y$. □

Saksman and Tylli [9] showed that the W.A.P. is inherited by the projective tensor product $\ell^p \hat{\otimes} \ell^q$ $(1 < p, q < \infty)$. Next we give a sufficient condition for the inheritance of the W.A.P. by the projective tensor product $X \hat{\otimes} Y$.

Theorem 2. (i) *If X and Y are Banach spaces such that every bounded linear operator $T \colon X \to Y^*$ (respectively, $T \colon Y \to X^*$) is completely continuous, then $X \hat{\otimes} Y$ has the W.A.P. if and only if both X and Y have the W.A.P.* (ii) *If X and Y are Banach spaces such that either X or Y has the Dunford–Pettis property, then $X \hat{\otimes} Y$ has the W.A.P. if and only if both X and Y have the W.A.P.*

Proof. Take any weakly compact subset D of $X \hat{\otimes} Y$ and any $\varepsilon > 0$. By [5, Theorem 3.1], there exist weakly compact subsets A_i of X and weakly compact subsets B_i of Y such that $D \subseteq \bigcup_{i=1}^n \overline{co}(A_i \otimes B_i)$. Let $A = \bigcup_{i=1}^n A_i$ and $B = \bigcup_{i=1}^n B_i$. Then A and B are, respectively, weakly compact subsets of X and Y. Moreover, it is easy to see that $D \subseteq \overline{co}(A \otimes B)$. Let

$$M_1 = \sup\{\|x\| \colon x \in A\} \quad \text{and} \quad M_2 = \sup\{\|y\| \colon y \in B\}.$$

Since X has the W.A.P., there are $C_1 > 0$ and a weakly compact operator $T_1 \colon X \to X$ such that

$$\|T_1\| \le C_1 \quad \text{and} \quad \|T_1(x) - x\| \le \frac{\varepsilon}{4M_2}, \quad \forall\, x \in A.$$

Since Y has the W.A.P., there are $C_2 > 0$ and a weakly compact operator $T_2 \colon Y \to Y$ such that

$$\|T_2\| \le C_2 \quad \text{and} \quad \|T_2(y) - y\| \le \frac{\varepsilon}{4M_1C_1}, \quad \forall\, y \in B.$$

Thus for every $x \otimes y \in A \otimes B$, one has

$$\begin{aligned} \|(T_1 \otimes T_2)(x \otimes y) - (x \otimes y)\| &= \|T_1(x) \otimes T_2(y) - x \otimes y\| \\ &\le \|T_1(x) \otimes (T_2(y) - y)\| + \|(T_1(x) - x) \otimes y\| \\ &\le C_1 M_1 \cdot \frac{\varepsilon}{4M_1C_1} + M_2 \cdot \frac{\varepsilon}{4M_2} = \varepsilon/2. \end{aligned}$$

Now for every $u \in D \subseteq \overline{co}(A \otimes B)$ there is $v \in co(A \otimes B)$ such that

$$\|u - v\| \le \frac{\varepsilon}{2(1 + C_1C_2)}.$$

Write $v = \sum_{i=1}^n t_i(x_i \otimes y_i)$ where $x_i \otimes y_i \in A \otimes B$ and $\sum_{i=1}^n |t_i| \leq 1$. Then

$$\|(T_1 \otimes T_2)(v) - v\| \leq \sum_{i=1}^n |t_i| \cdot \|(T_1 \otimes T_2)(x_i \otimes y_i) - (x_i \otimes y_i)\| \leq \frac{\varepsilon}{2},$$

which implies that

$$\begin{aligned}\|(T_1 \otimes T_2)(u) - u\| &\leq \|(T_1 \otimes T_2)(u - v)\| + \|(T_1 \otimes T_2)(v) - v\| + \|v - u\| \\ &\leq (\|T_1\| \cdot \|T_2\| + 1) \cdot \|u - v\| + \frac{\varepsilon}{2} \leq \varepsilon.\end{aligned}$$

Moreover, $\|T_1 \otimes T_2\| \leq \|T_1\| \cdot \|T_2\| \leq C_1 \cdot C_2$. Next we need only to show that $T_1 \otimes T_2 \colon X\hat{\otimes}Y \to X\hat{\otimes}Y$ is weakly compact.

Since $B_{X\hat{\otimes}Y} = \overline{co}(B_X \otimes B_Y)$, one has

$$\begin{aligned}(T_1 \otimes T_2)[B_{X\hat{\otimes}Y}] &= (T_1 \otimes T_2)[\overline{co}(B_X \otimes B_Y)] \\ &\subseteq \overline{co}((T_1 \otimes T_2)[B_X \otimes B_Y]) \\ &= \overline{co}(T_1[B_X] \otimes T_2[B_Y]).\end{aligned}$$

Note that $T_1[B_X]$ and $T_2[B_Y]$ are, respectively, relatively weakly compact subsets of X and Y. By Proposition 1 and [5, Proposition 2.5], $T_1[B_X] \otimes T_2[B_Y]$ is a relatively weakly compact subset of $X\hat{\otimes}Y$. It follows from the Kreĭn–Smuliàn theorem that $\overline{co}(T_1[B_X] \otimes T_2[B_Y])$ is a weakly compact subset of $X\hat{\otimes}Y$ and hence, $(T_1 \otimes T_2)[B_{X\hat{\otimes}Y}]$ is a relatively weakly compact subset of $X\hat{\otimes}Y$. This implies that $T_1 \otimes T_2 \colon X\hat{\otimes}Y \to X\hat{\otimes}Y$ is weakly compact and thus $X\hat{\otimes}Y$ has the W.A.P. $\square$

Remark. By applying the construction in [2] to ℓ^1 one obtains a separable $\mathcal{L}^\infty$-space X so that $\ell^1 \subset X$ isometrically and X has the Schur property (also see [9]). It follows from [1, Corollary 3] that X has the W.A.P. Saksman and Tylli in [9] asked a question whether $X\hat{\otimes}X$ has the W.A.P.? Note that the Schur property implies the Dunford–Pettis property. Thus the previous Theorem 2 gives an affirmative answer to this question.

Every reflexive Banach space clearly has the W.A.P. There are a few examples of non-reflexive Banach spaces with the W.A.P. (see [7, 9]). If K is scattered then $C(K)^*$ has the Schur property (see [8], also see [4]) and hence, has the Dunford–Pettis property. It follows from [1, Corollary 3] that $C(K)^*$ also has the W.A.P. Thus we have the following corollary that gives us new examples of non-reflexive Banach spaces with the W.A.P.

Corollary 3. *Let K be scattered and Y be a Banach space with the W.A.P. Then $C(K)^*\hat{\otimes}Y$ has the W.A.P. In particular, $C(K)^*\hat{\otimes}C(K)^*$ has the W.A.P.*

Recall that a Banach space X is said to have the *compact approximation property* (see [3, p. 308]) if for every compact subset K of X and every $\varepsilon > 0$ there is a compact operator $T \colon X \to X$ such that $\|T(x) - x\| \leq \varepsilon$ for all $x \in K$. A Banach space X is said to have the *bounded compact approximation property* (see [3, p. 308]) if there exists $\lambda \geq 1$ so that for every compact subset K of X and every $\varepsilon > 0$ there is a compact operator $T \colon X \to X$ satisfying $\|T\| \leq \lambda$ and $\|T(x) - x\| \leq \varepsilon$ for all $x \in K$. It is well known that the (bounded) approximation property implies the (bounded) compact approximation property, but the converse is not true (see [11] or see [3, p. 309]).

Diestel, Fourie and Swart [4] showed that if Banach spaces X and Y have the approximation property (respectively, the bounded approximation property) then the projective tensor product $X\hat{\otimes}Y$ has the same property. To obtain the compact approximation property and the bounded compact approximation property for the projective tensor product $X\hat{\otimes}Y$, we need the following characterization of relatively norm compact subsets in $X\hat{\otimes}Y$ which is due to Grothendieck [6] (also see [5]).

Proposition 4. *A subset K of $X\hat{\otimes}Y$ is relatively norm compact if and only if there are compact subsets A of X and B of Y such that $K \subseteq \overline{co}(A \otimes B)$.*

By the previous proposition and following the last part in the proof of Theorem 2, we have the following corollary.

Corollary 5. *Let X_1, X_2, Y_1, Y_2 be Banach spaces. If $T_1\colon X_1 \to Y_1$ and $T_2\colon X_2 \to Y_2$ are compact, then $T_1 \otimes T_2\colon X_1\hat{\otimes}X_2 \to Y_1\hat{\otimes}Y_2$ is compact.*

Similar to the proof of Theorem 2 we have the following.

Theorem 6. *If X and Y are Banach spaces with the compact approximation property (respectively, the bounded compact approximation property), then $X\hat{\otimes}Y$ has the compact approximation property (respectively, the bounded compact approximation property).*

References

[1] Astala, K., and H.-O. Tylli: Seminorms related to weak compactness and to Tauberian operators. - Math. Proc. Cambridge Philos. Soc. 107, 1990, 367–375.

[2] Bourgain, J., and G. Pisier: A construction of $\mathcal{L}^\infty$-spaces and related Banach spaces. - Bol. Soc. Brasil. Mat. 14, 1983, 109–123.

[3] Casazza, P. G.: Approximation properties. - In: Handbook of the geometry of Banach spaces, vol. I, edited by Johnson and Lindenstrauss, North-Holland, Amsterdam, 2001, 271–316.

[4] Diestel, J., J. Fourie, and J. Swart: The projective tensor product I. - Contemp. Math. 321, 2003, 37–65.

[5] Diestel, J., and D. Puglisi: A note on weakly compact subsets in the projective tensor product of Banach spaces. - J. Math. Anal. Appl. 350, 2009, 508–513.

[6] Grothendieck, A.: Produits tensoriels topologiques et espaces nucleaires. - Mem. Amer. Math. Soc. 16, 1955.

[7] Odell, E., and H.-O. Tylli: Weakly compact approximation in Banach spaces. - Trans. Amer. Math. Soc. 357, 2005, 1125–1159.

[8] Pelczynski, A., and Z. Semadeni: Spaces of continuous functions, III. Spaces $C(\Omega)$ for Ω without perfect subsetes. - Studia Math. 18, 1959, 211–222.

[9] Saksman, E., and H.-O. Tylli: New examples of weakly compact approximation in Banach spaces. - Ann. Acad. Sci. Fenn. Math. 33, 2008, 429–438.

[10] Tylli, H.-O.: Duality of the weak essential norm. - Proc. Amer. Math. Soc. 129, 2001, 1437–1442.

[11] Willis, G.: The compact approximation property does not imply the approximation property. - Studia Math. 103, 1992, 99–108.

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